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Abstract

We show, in the context of school choice, that when the lists of preferences and priority orderings are acyclic in the sense of Gutin et al. (2023), the outcomes of the deferred acceptance and top trading cycle algorithms coincide. This implies that the student-optimal stable matching is efficient. Furthermore, we show that if schools' priority orderings of students are based on the sum of school-independent basic points and school-dependent additional points, and if students' preferences align with these additional points, then the lists are acyclic. Additionally, if students can and do decline the addition of points that their preferences do not align with, then the lists become acyclic, regardless of the preference list.

Keywords: School choice, Stability and efficiency of matching, Top trading cycle algorithm, Acyclicity

JEL Classification: C78 (Matching Theory)

1 Introduction

Stability and efficiency are two desirable but often incompatible properties of matchings in school choice. The student-proposing *deferred acceptance* (DA) algorithm, used in college admissions (Gale and Shapley, 1962), produces a stable matching—specifically, the student-optimal stable matching. In contrast, the *top trading cycle* (TTC) algorithm, adapted to school choice, finds an efficient matching (Abdulkadiroğlu and Sönmez, 2003). However, their outcomes do not always coincide. In the school choice model, where students are assigned to schools with priority orderings and capacities, Ergin (2002) shows that DA's outcome is

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efficient under any student preferences if and only if priority orderings and capacities are acyclic. Kesten (2006) establishes that DA and TTC produce the same outcome under any student preferences if and only if priority orderings and capacities are acyclic in his sense. While important, these conditions are quite strong, as they require efficiency or equivalence to hold *under any student preferences.*

In this paper we take a more modest approach, examining conditions under which DA and TTC produce the same outcome *given specific lists of student prefernces and school priority orderings.* We show that if these lists are *acyclic* in the sense of Gutin et al. (2023), then DA and TTC produce the same outcome (Theorem 3.1). This acyclicity is equivalent to the absence of *simultaneous cycle* as defined by Romero-Medina and Triossi (2013), which ensures the uniqueness of stable matchings. Gutin et al. (2023) demonstrate its necessity in their normal form of the matching model. In the following, we call a cycle in the sense of Gutin et al. (2023) simply a *cycle*, and its absence *acyclicity.* A closely related but different "cycle" that appears in the TTC algorithm will be called a *TT cycle.* A natural question is how strong the acyclicity condition is. It is known that if all schools share a common priority ordering based on *points* (e.g., exam scores), the lists are acyclic (Romero-Medina and Triossi, 2013). However, priority orderings often differ when schools adjust points by adding school-specific factors (e.g., proximity or sibling presence). We show that if student preferences *align with these additional points* (i.e., students prefer schools with higher additional points), then the lists are acyclic (Theorem 3.2). Moreover, if students can and do decline the addition of points that their preferences do not align with, then the lists become acyclic regardless of their preference list (Corollary to Theorem 3.2). Thus, acyclicity can be achieved by appropriately reflecting students' preferences in the priority orderings.

The rest of the paper is organized as follows. Section 2 presents the model and provides some preliminaries. Section 3 shows our main results.

2 The model

A school choice model is a tuple $(I, S, \succ^I, \succ^S, q)$, where I is a finite set of students, S is a finite set of schools, $\succ^I := (\succ_i)_{i \in I}$ is a list of total strict orders $\succ_i$ of i over $S \cup \{\phi\}$, $\succ^S := (\succ_s)_{s \in S}$ is a list of total strict orders $\succ_s$ of s over I, and $q := (q_s)_{s \in S}$ is a list of capacities $q_s \geq 1$ of s. We call $\succ_i$ the *preference* of i and $\succ_s$ the *priority ordering* of s. $s \succ_i \phi$ means that "i prefers school s to remaining unassigned." We write $s \succeq_i s'$ when $s \succ_i s'$ or $s = s'$. Matching is a function $\mu \colon I \to S \cup \{\phi\}$. $\mu(i) = \phi$ means that "i is not assigned to a school." We consider many-to-one matching μ such that $|\mu^{-1}(s)| \leq q_s$ for any $s \in S$. Let

$(i,s) \in I \times S$. A matching μ is *individually rational* if $s = \mu(i)$ implies $s \succ_i \phi$. A pair (i,s) *blocks* μ if $s \neq \mu(i)$ and if (a$^\bullet$) $s \succ_i \mu(i)$ and (b$^\bullet$) $i \succ_s j$ for some $j \in \mu^{-1}(s)$ or $|\mu^{-1}(s)| < q_s$. Matching μ is *stable* if it is individually rational and no pair blocks μ. Thus, if μ is stable, then for any pair (i,s) such that $s \neq \mu(i)$, (a) $\mu(i) \succ_i s$ or (b) $j \succ_s i$ for every $j \in \mu^{-1}(s)$ and $|\mu(s)| = q_s$. Let L be the set of stable matchings. Defining a binary relation $\succeq_I$ on L by $\mu \succeq_I \mu'$ iff $\mu(i) \succeq_i \mu'(i)$ for all $i \in I$, $(L, \succeq_I)$ is a lattice. The greatest element $\bar{\mu}$ of $(L, \succeq_I)$ is said to be *student-optimal*. Although students agree that $\bar{\mu}$ is the best among the stable matchings, there could be a matching, not in L, that Pareto-dominates $\bar{\mu}$, i.e., a matching μ such that $\mu(i) \succeq_i \bar{\mu}(i)$ for all $i \in I$ and $\mu(i) \succ_i \bar{\mu}(i)$ for at least one $i \in I$. A matching $\hat{\mu}$ is *efficient* if there is no matching that Pareto-dominates $\hat{\mu}$. DA finds the student-optimal stable matching. TTC finds an efficient matching. The outcomes of the two may differ depending on the lists $(\succ^I, \succ^S)$. Meanwhile, Romero-Medina and Triossi (2013) show that stable matching is unique if the lists $(\succ^I, \succ^S)$ are acyclic in the following sense.

Definition 2.1 (Gutin et al. (2023))**.** The lists $(\succ^I, \succ^S)$ have a *cycle* if there exist a subset of I of size k and a subset of S of size k, where $k \geq 2$, and an enumeration and an ordering of the agents $(s_1, i_1, s_2, i_2, \ldots, s_k, i_k)$ such that

$$i_h \succ_{s_h} i_{h-1} \quad \forall h = 1, \ldots, k \text{ (modulo } k)$$
$$s_{h+1} \succ_{i_h} s_h \quad \forall h = 1, \ldots, k \text{ (modulo } k).$$

The lists $(\succ^I, \succ^S)$ are *acyclic* if they have no cycle.

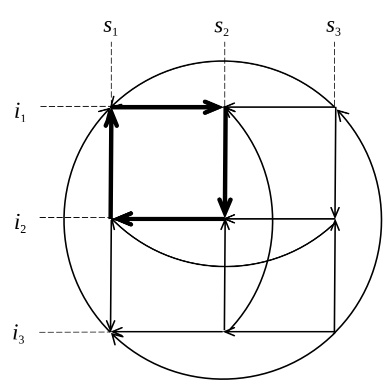
In the following, we will alternatively write cycle $(s_1, i_1, s_2, i_2, \ldots, s_k, i_k)$ as

$$(i_k, s_1) \to (i_1, s_1) \to (i_1, s_2) \to \cdots \to (i_{k-1}, s_k) \to (i_k, s_k) \to (i_k, s_1).$$

Example 2.1. Let $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, $q_1 = q_2 = q_3 = 1$, and, denoting $s_2 \succ_{i_1} s_1 \succ_{i_1} s_3$ by $i_1\colon s_2\ s_1\ s_3$ and so on,

$$i_1\colon s_2\ s_1\ s_3 \quad i_2\colon s_1\ s_2\ s_3, \quad i_3\colon s_1\ s_2\ s_3, \qquad s_1\colon i_1\ i_3\ i_2, \quad s_2\colon i_2\ i_1\ i_3, \quad s_3\colon i_2\ i_1\ i_3.$$

The *matching digraph* (Gutin et al., 2023) of this example looks like this:



(The horizontal arrows indicate the directions of preferences and the vertical arrows indicate those of priority orderings.) There is a cycle (thick arrows) of length 4 (s_1, i_1, s_2, i_2), or, $(i_2, s_1) \to (i_1, s_1) \to (i_1, s_2) \to (i_2, s_2) \to (i_2, s_1)$ such that

$$i_1 \succ_{s_1} i_2,\ s_2 \succ_{i_1} s_1,\ i_2 \succ_{s_2} i_1,\ s_1 \succ_{i_2} s_2.$$

Note that 4 is the smallest length of the cycle of Definition 2.1 (where $k = 2$). It can be checked that there is a unique stable matching $\{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$ dominated by the efficient matching $\{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$. DA finds the former and TTC finds the latter. The latter is blocked by (i_3, s_1). This example suggests that acyclicity is only a sufficient condition for the uniqueness of stable matching[1] and that the uniqueness of stable matching is not sufficient for its efficiency. □

In the next section, we will show that, given $(\succ^I, \succ^S)$, the outcomes of DA and TTC coincide if $(\succ^I, \succ^S)$ is acyclic.[2]

3 Main results

3.1 Acyclicity implies the coincidence of DA and TTC outcomes

Let $(I, S, \succ^I, \succ^S, q)$ be a school choice model. Our first result rests on the following uniqueness result. We provide a proof in the context of school choice in the Appendix.

Lemma 3.1 (Romero-Medina and Triossi (2013))**.** *If $(\succ^I, \succ^S)$ is acyclic, then stable matching is unique.*

Now, the TTC algorithm of Abdulkadiloğlu and Sönmez (2003) is summarized as follows:
Step 1: Initialize each school's counter to q_s. All students and schools are considered *remaining* at this step.
Step $t \geq 1$: Each remaining school points to its highest-priority remaining student. Each remaining student points to their most preferred remaining school or is removed if they prefer to remain unassigned. If a student points to a school, there is at least one *TT cycle* $(s_1, i_1, s_2, i_2, \ldots, s_k, i_k)$ such that s_1 points to i_1, i_i points to s_2, …, s_k points to i_k, and i_k points to s_1. Every student in a TT cycle is assigned the school they are pointing to and is then removed. Every school in a TT cycle reduces its counter by one and is removed if the counter reaches zero. The algorithm terminates when there are no remaining student or schools. Any remaining students at this point remain unassigned.

[1]However, it is necessary when the model is reduced to the normal form (Gutin et al., 2023).

[2]A condition on $(\succ^S, q)$ that is necessary and sufficient for the uniqueness given any $\succ^I$ is given in Kesten (2006).

Note that 2 is the smallest TT cycle length, the length of the TT cycle (s_1, i_1) such that s_1 points to i_1 and i_1 points to s_1 (the case $k = 1$). Also, any TT cycle $(s_1, i_1, s_2, i_2, \ldots, s_k, i_k)$ of $k \geq 2$ can be seen as a cycle $(i_k, s_1) \to (i_1, s_1) \to (i_1, s_2) \to \cdots \to (i_{k-1}, s_k) \to (i_k, s_k) \to (i_k, s_1)$ such that $i_1 \succ_{s_1} i_k$, $s_2 \succ_{i_1} s_1$, $\ldots$, $i_k \succ_{s_k} i_{k-1}$, and $s_1 \succ_{i_k} s_k$.

Theorem 3.1. *If $(\succ^I, \succ^S)$ is acyclic then the outcomes of DA and TTC coincide.*

Proof. Let $\mu\colon I \to S \cup \{\phi\}$ be the outcome of TTC. Then, μ is individually rational, i.e., $s = \mu(i)$ implies $s \succ_i \phi$, as s is what i has pointed to in a TT cycle. If $(\succ^I, \succ^S)$ is acyclic, then any TT cycle is of the form (s, i) such that s points to i and i points to s. Thus, $s \neq \mu(i)$ implies (a) $\mu(i) \succ_i s$ or, otherwise, (b) $j \succ_s i$ for every $j \in \mu^{-1}(s)$ and $|\mu(s)| = q_s$ (i has never been the first-priority student of s). Hence, no pair (i, s) such that $s \neq \mu(i)$ blocks μ, and μ is stable. By Lemma 3.1, the stable matching is unique. Hence, μ coincides with the outcome of DA. $\square$

3.2 Point-aligned preferences imply acyclicity

Hereafter, we make our model closer to actual school choice by making a couple of assumptions as follows.

Assumption 3.1. (i) Each student i has *basic points* $\hat{r}_i \in \mathbb{R}_+$ and knows *possible additional points* at school s, $\hat{p}_{i,s} \in \mathbb{R}_+$, for each $s \in S$. Students apply to schools by putting desired schools s (such that $s \succ_i \phi$) in the application form in the order of $\succ_i$.

(ii) The admission office collects the forms and determines the set I of applicant students. For each $i \in I$ and $s \in S$, it decides i's *additional points* at s, $p_{i,s} \in \mathbb{R}_+$, as $p_{i,s} = \hat{p}_{i,s}$ if s appears in the form of i (which occurs if $s \succ_i \phi$); otherwise $p_{i,s} = 0$. The *total points* $r_i + p_{i,s}$ of i at s are determined by tie-breaking $\hat{r}_i + p_{i,s}$ in such a way that $r_i + p_{i,s} \neq r_j + p_{j,s}$, and $r_i + p_{i,s} > r_j + p_{j,s}$ if $\hat{r}_i + p_{i,s} > \hat{r}_j + p_{j,s}$, for any $i, j \in I$ and any $s \in S$.[3] The priority ordering of $s \in S$ is determined by

$$i \succ_s j \iff r_i + p_{i,s} > r_j + p_{j,s}, \ \forall i, j \in I. \tag{1}$$

Here, $\succ_s$ is a total strict order because the "greater than" relation $>$ of real numbers is a total strict order. The following property of preferences is crucial to our second result.

Definition 3.1. *A preference $\succ_i$ aligns with additional points* if for any $s, s' \in S$,

$$p_{i,s} > p_{i,s'} \implies s \succ_i s'. \tag{2}$$

[3]Such a tie-breaking is always possible: Let $\epsilon := \min_{s \in S} \min_{\hat{r}_i + p_{i,s} > \hat{r}_j + p_{j,s},\ i \neq j} \big((\hat{r}_i + p_i, s) - (\hat{r}_j + p_{j,s})\big)$ and randomly choose $r_i \in (\hat{r}_i - \frac{\epsilon}{2}, \hat{r}_i + \frac{\epsilon}{2})$ for each $i \in I$.

Note that the antecedent is $p_{i,s} > p_{i,s'}$, not $\hat{p}_{i,s} > \hat{p}_{i,s'}$. That is, Eq. (2) claims that if $\hat{p}_{i,s} > \hat{p}_{i,s'}$ *and* $s \succ_i \phi$ then $s \succ_i s'$ (see Assumption 3.1 (ii) regarding how $\hat{p}_{i,s}$ is translated to $p_{i,s}$). It does not require that an otherwise undesired s be preferred to a desired s' just because possible additional points at s are higher than at s'.

Theorem 3.2. *Under Assumption 3.1, if $\succ_i$ aligns with additional points for every $i \in I$, then $(\succ^I, \succ^S)$ is acyclic.*

Proof. Suppose to the contrary that there was a cycle

$$(i_k, s_1) \to (i_1, s_1) \to (i_1, s_2) \to \cdots \to (i_{k-1}, s_k) \to (i_k, s_k) \to (i_k, s_1)$$

such that

$$i_h \succ_{s_h} i_{h-1}, \ s_{h+1} \succ_{i_h} s_h \quad \forall h = 1, \ldots, k \text{ (modulo } k).$$

Let $f\colon I \times S \to \mathbb{R}$ be a function such that $f(i, s) = r_i + p_{i,s}$ (total points of i at s). Then, for each $h = 1, \ldots, k$,

$$i_h \succ_{s_h} i_{h-1} \implies f(i_h, s_h) > f(i_{h-1}, s_h)$$

by Eq. (1). Also, for each $h = 1, \ldots, k$,

$$s_{h+1} \succ_{i_h} s_h \implies f(i_h, s_{h+1}) \geq f(i_h, s_h)$$

because the contrapositive of Eq. (2) implies $p_{i_h,s_{h+1}} \geq p_{i_h,s_h}$. Thus, we have

$$f(i_1, s_1) \leq f(i_1, s_2) < f(i_2, s_2) \leq \cdots \leq f(i_k, s_1) < f(i_1, s_1),$$

a contradiction. Hence we must have that $(\succ^I, \succ^S)$ is acyclic. □

It should be stressed that by "the preference aligns with additional points," we do not mean that preferences are affected by the addition of points; we mean that preferences are "compatible" with the additional point system in the sense of Eq. (2). Thus, there could be a case such that preferences do not align with additional points, e.g., a case such that $s' \succ_i s \succ_i \phi$ and $\hat{p}_{i,s} > \hat{p}_{i,s'}$, which results in $p_{i,s} > p_{i,s'}$ and $s' \succ_i s$ under the rule $s \succ_i \phi \implies p_{i,s} = \hat{p}_{i,s}$ of Assumption 3.1. Giving a chance to decline otherwise favorable additional points to students will save the situation.

Assumption 3.2. The office sets $p_{i,s} = 0$ if student i declines adding points to school s. Every student i declines adding points to school s if there is another school s' such that $s' \succ_i s$ and $\hat{p}_{i,s} > \hat{p}_{i,s'}$.

Corollary 3.1. *Under Assumptions 3.1 and 3.2, $(\succ^I, \succ^S)$ is acyclic with any $\succ^I$.*

Proof. If $p_{i,s} > p_{i,s'}$ then $\hat{p}_{i,s} > \hat{p}_{i,s'}$. If $s'_i \succ_i s_i$ then, by Assumption 3.2, we have a contradiction $0 = p_{i,s} > p_{i,s'} \geq 0$. Thus, $p_{i,s} > p_{i,s'} \implies s \succ_i s'$. The rest are given by Theorem 3.2. $\square$

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A Appendix

A.1 Proof of Theorem 3.1

To prove the theorem, we use the result of Roth (1984), which states, in the context of school choice, the following: If all priority orderings are strict, the set of students assigned to schools is the same at every stable matching. Namely, if μ and ν are two stable matchings, we have $\bigcup_{s\in S}\mu^{-1}(s) = \bigcup_{s\in S}\nu^{-1}(s)$; thus, $\mu^{-1}(\phi) = \nu^{-1}(\phi)$ also.

Proof. Suppose, by way of contradiction, that the set of stable matchings is not a singleton. Then, by its lattice property, there are two matchings μ and ν such that $\nu \succeq_I \mu$ and $\mu \neq \nu$. Since $\mu^{-1}(\phi) = \nu^{-1}(\phi)$, if $\mu^{-1}(s) \subseteq \nu^{-1}(s)$ for all $s \in S$, then $\mu(i) = \nu(i)$ for all $i \in I$, so we must have $\mu^{-1}(s) \not\subseteq \nu^{-1}(s)$ for some $s \in S$, i.e., $\mu^{-1}(s) \setminus \nu^{-1}(s) \neq \emptyset$ for some $s \in S$. In the following, we assume that every student has an (unwritten) ID number and, when $\mu^{-1}(s) \setminus \nu^{-1}(s) \neq \emptyset$, we always pick the student with the smallest ID number from this set. Let $s_0 \in S$ be such that $\mu^{-1}(s_0) \setminus \nu^{-1}(s_0) \neq \emptyset$, and pick $i_0 \in \mu^{-1}(s_0) \setminus \nu^{-1}(s_0)$, i.e.,

$$i_0 \in \mu^{-1}(s_0),\ i_0 \notin \nu^{-1}(s_0).$$

The stability of ν implies (a) $\nu(i_0) \succ_{i_0} s_0$ or (b) $j \succ_{s_0} i_0$ for every $j \in \nu^{-1}(s_0)$ and $|\nu^{-1}(s_0)| = q_{s_0}$. If (b) is true, however, $|\nu^{-1}(s_0)| = q_{s_0}$ implies $\nu^{-1}(s_0)\setminus\mu^{-1}(s_0) \neq \emptyset$ (otherwise $\nu^{-1}(s_0) \subseteq \mu^{-1}(s_0)$ and $\nu^{-1}(s_0) = \mu^{-1}(s_0)$), and for $j \in \nu^{-1}(s_0)\setminus\mu^{-1}(s_0)$, we have $s_0 \neq \mu(j)$, $s_0 \succ_j \mu(j)$ (since $s_0 = \nu(j) \succeq_j \mu(j)$ and $s_0 \neq \mu(j)$), and $j \succ_{s_0} i_0$, with $i_0 \in \mu^{-1}(s_0)$, i.e., (j, s_0) blocks μ. Hence, (b) is impossible. In (a), letting $s_1 = \nu(i_0)$, $s_1 \succ_{i_0} s_0$ and $i_0 \in \mu^{-1}(s_0)$ imply $i \succ_{s_1} i_0$ for all $i \in \mu^{-1}(s_1)$ and $|\mu^{-1}(s_1)| = q_{s_1}$. Then $|\mu^{-1}(s_1)| = q_{s_1}$ implies $\mu^{-1}(s_1) \setminus \nu^{-1}(s_1) \neq \emptyset$ (otherwise $\mu^{-1}(s_1) \subseteq \nu^{-1}(s_1)$ and $\mu^{-1}(s_1) = \nu^{-1}(s_1)$). Pick $i_1 \in \mu^{-1}(s_1) \setminus \nu^{-1}(s_1)$. Then,

$$i_1 \in \mu^{-1}(s_1) \setminus \nu^{-1}(s_1),\ s_1 = \nu(i_0),\ \text{and}\ i_1 \succ_{s_1} i_0,\ s_1 \succ_{i_0} s_0. \tag{3}$$

Starting from s_0 such that $\mu^{-1}(s_0) \setminus \nu^{-1}(s_0) \neq \emptyset$, we have derived i_0, s_1, and i_1 satisfying Eq. (3). The iteration of this procedure yields a sequence of $(i_h, s_h) \in I \times S$ satisfying

$$i_h \in \mu^{-1}(s_h) \setminus \nu^{-1}(s_h),\ s_h = \nu(i_{h-1}),\ \text{and}\ i_h \succ_{s_h} i_{h-1},\ s_h \succ_{i_{h-1}} s_{h-1}\ \text{for all}\ h = 1, 2, \ldots.$$

However, since I and S are finite, this sequence includes a cycle of the form

$$i_h \succ_{s_h} i_{h-1},\ s_{h+1} \succ_{i_h} s_h \quad \forall h = 1, \ldots, k \text{ (modulo } k)$$

where k is such that $2 \leq k \leq \min\{|I|, |S|\}$, contradicting the acyclicity of $(\succ^I, \succ^S)$. $\square$

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