Research Paper Series

No. 54

Coordination Failure and Accuracy of Private Information in Global Games

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December, 2024

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December 26, 2024

Abstract

In 2×2 coordination games with Pareto-ranked equilibria, two types of coordination failures can occur. The first is an equilibrium selection problem, where both players coordinate their actions but reach a Pareto-inferior equilibrium. The second is miss-coordination, where players fail to coordinate and do not reach any pure strategy equilibrium. This paper examines the relationship between the probability of these two types of coordination failures and the accuracy of private information within the framework of 2×2 global games. Our findings indicate that as the accuracy of private information improves, the probability of miss-coordination decreases, whereas the probability of selecting a Pareto-inferior equilibrium may increase.

1 Introduction

While increased information, whether public or private, is generally benefits for a single decision maker facing uncertainty, this advantage does not always extend to multi-player games. Previous literature has highlighted that in coordination games, the higher accuracy of public information does not necessarily increase individual payoffs or social welfare.

Morris and Shin (2002, 2004) demonstrated that when players receive public information in addition to private information, fundamental uncertainty decreases, but their overall welfare does not necessarily improve due to

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coordination effects. Morris and Shin (2004) explained that while accurate private information can reduce fundamental uncertainty, it does not resolve the strategic uncertainty about others' decisions. Even with infinitely precise private information, fundamental uncertainty persists, leading to significant inefficiencies.

In this paper, we shift our focus from public information to a setting where only private information is available, examining whether greater accuracy of private information reduces coordination failures and enhances player payoffs.

In 2×2 coordination games with Pareto-ranked equilibria, two types of coordination failures can arise: equilibrium selection problem, where both players coordinate their actions but end up in a Pareto-inferior equilibrium, and miss-coordination, where both players fail to coordinate and do not achieve any pure strategy equilibrium. This paper investigates the relation between these two coordination failures and the accuracy of private information in the framework of global games. Our results show that as the accuracy of private information increases, the probability of miss-coordination decreases while the probability of selecting a Pareto-inferior equilibrium may increase. This means that player payoffs do not necessarily improve countering the common perception that more precise information inherently leads to better outcomes for players.

Our initial motivation was to investigate how more accurate private information influences both the probability of miss-coordination and the probability of reaching a Pareto superior (or inferior) equilibrium. However, in global games, equilibria are determined solely by deviation losses. Without explicitly specifying payoffs, either of the two equilibria, given these deviation losses, could be Pareto superior or inferior. Therefore, our analysis shifts to exploring how the accuracy of private information affects both the probability of miss-coordination and the probability of reaching one equilibrium or the other, considering deviation losses, rather than focusing on Pareto-ranking the equilibria.

We consider a global game on a class of 2×2 games where the state is uniformly distributed and determined by nature, and each player observes it with small noise, also uniformly distributed. In a Bayesian Nash equilibrium of the global game, both players use a "switching strategy" with some threshold. If the realized state is far from the threshold or the noise is small, both players receive signals that are either below or above the threshold. This leads them to choose the same action and the global game's outcome achieves an equilibrium of the (complete information) game at that state. However, if the realized state is near the threshold and the noise is somewhat larger, then the signals each player observes may differ, leading to different actions between players, resulting in miss-coordination. From this perspective, higher accuracy of private information is more likely to reduce miscoordination.

Conversely, decreased noise, or higher accuracy of individual information, can reduce the range in which players select an equilibrium that is "less risky" in the sense that the Nash product (the product of deviation losses) is larger. If this less risky equilibrium corresponds to the Pareto superior equilibrium, the reduction in this range results in a lower probability of selecting the Pareto superior equilibrium. This intuitively explains why, despite reducing miss-coordination, increased private information accuracy may still decrease the likelihood of selecting a Pareto superior equilibrium.

Our study is also related to several experiments on coordination games. Experiments involving 2×2 coordination games with Pareto-ranked equilibria, such as those by Cooper et al. (1990, 1994), are representative.[1] These experiments have shown that a Pareto inferior equilibrium is always chosen in one-shot games rather than the Pareto superior equilibrium to avoid miscoordination. In global games, which are games of incomplete information, increasing the accuracy of information makes the game more closely approximate a complete information game. As Morris and Shin (2004) noted, while higher information accuracy reduces fundamental uncertainty, it can simultaneously increase strategic uncertainty. Consequently, as information accuracy improves, the likelihood of miscoordination decreases, but the probability of resulting in a Pareto inferior equilibrium may increase. Therefore, the impact of more accurate private information depends on the trade-off between reducing miscoordination and the increased risk of Pareto inferior outcomes.

[1] There are several studies about global games. Cabrales et al. (2007) conducted experiments on incomplete information games equivalent to global games. Additionally, Heinemann et al. (2009) conducted experiments on coordination games and estimated global games by assuming that subjects have private information about the payoff functions. These results indicate that in symmetric games, like those addressed in this paper, global games provide good predictions. Heinemann (2024) pointed out that in asymmetric games, global games do not necessarily provide accurate predictions.

For our analysis to be meaningful, it is crucial that the global game possesses a unique equilibrium. By introducing a few additional assumptions to the standard global game model, we demonstrate that if the equilibrium in which both players adopt switching strategies is unique, then this uniqueness holds not only among switching strategies but across all possible mixed strategies. While this result has been previously proven through the iterated elimination of interim-dominated strategies by Carlsson and van Damme (1993) and Morris and Shin (2003), we offer a simplified proof by assuming that both the state and the noise are uniformly distributed, making the argument more straightforward.

The uniqueness of an equilibrium in global games is a central issue for equilibrium selection and has been extensively studied (see, for example, Carlsson and van Damme (1993), Frankel et al. (2003), Basteck and Daniëls (2011), and Hoffmann and Sabarwal (2019)). These studies focus on the limit case where noise approaches zero, considering it a method for selecting equilibria in complete information games. In contrast, we consider the case where the noise is sufficiently small but non-zero, treating global games as games of incomplete information reflecting significant economic phenomena. Applications of this framework include Morris and Shin (2004), who examined coordination in investment rollovers, and Goldstein and Pauzner (2005), who discussed bank runs. In such cases, multiple equilibria may exist. For scenarios where both the state and noise are normally distributed, the conditions for equilibrium uniqueness have been explored by Hellwig (2002), Morris and Shin (2003), and Morris and Shin (2004). Our modest contribution is to present a condition for equilibrium uniqueness when both the state and the noise are uniformly distributed, along with an explicit proof. While this result may seem intuitive in light of prior research on global games, the proof has not been explicitly documented in the literature.

The remainder of this paper is organized as follows. Section 2 introduces a model for global games with certain assumptions. Section 3 discusses properties related to the existence and uniqueness of equilibria under these assumptions. Section 4 examines the probability of miscoordination and equilibrium selection. We derive a necessary and sufficient condition that the increased accuracy of information reduces the probability of occurring one equilibrium. Section 5 considers two examples. The first example involves a class of linear payoff functions with respect to the state and the second example is a somewhat artificial payoff function. In the first example, the probability of achieving both equilibria always increases with the accuracy of noise whereas in the second example the probability of achieving a single equilibrium decreases as the accuracy of the noise increases.

2 Model

We consider a symmetric game with two players, denoted by 1 and 2, and two actions, denoted by A and B. The payoff of the player depends on a state θ and the payoff of player i is denoted by $u(a_i, a_j, \theta)$ when player i and the other player j choose actions a_i and a_j under state θ.

We assume that θ is uniformly distributed over $[-d, 1+d]$. Let h be the density function of θ: $h(\theta) = 1/(1+2d)$ if $\theta \in [-d, 1+d]$, otherwise $h(\theta) = 0$. Each player i observes noisy signal $x_i = \theta + e_i$ where the noise e_i is uniformly distributed over $[-\epsilon, \epsilon]$ for some $\epsilon > 0$. Let ϕ and Φ be the density function and the distribution function of e_i: $\phi(e_i) = 1/2\epsilon$ if $-\epsilon \leq e_i \leq \epsilon$, otherwise $\phi(e_i) = 0$ and note that $\Phi(k - \theta)$ is given by

$$\Phi(k-\theta) = \begin{cases} 1 & \theta \leq k - \epsilon \\ \frac{1}{2\epsilon}(k - \theta + \epsilon) & k - \epsilon \leq \theta \leq k + \epsilon \\ 0 & \theta \geq k + \epsilon. \end{cases} \tag{1}$$

We assume that $0 < 2\epsilon < d$: ϵ is small relative to d.

Let $g(A, \theta)$ and $g(B, \theta)$ be "deviation losses" from action profile (A, A) and (B, B), respectively under state θ. A deviation loss of an action profile is the loss that a player deviates unilaterally from the action profile, formally defined by

$$g(A, \theta) = u(A, A, \theta) - u(B, A, \theta), \quad g(B, \theta) = u(B, B, \theta) - u(B, A, \theta).$$

We make the following assumptions on g:

A1 $g(A, \theta)$ is strictly increasing and $g(B, \theta)$ is strictly decreasing in θ.

A2 $g(A, 0) = g(B, 1) = 0$.

A3 For any $\theta \in [-2\epsilon, 1 + 2\epsilon]$, $g(A, \theta) + g(B, \theta) > 0$.

To use a usual argument of global games, we require the existence of dominance regions of θ for both actions. A1 and A2 imply that $g(A, \theta) > 0$ and $g(B, \theta) < 0$ for any $\theta > 1$, and this means that A is strictly dominant for $\theta \in (1, 1+d]$. Similarly, B is strictly dominant for $\theta \in [-d, 0)$. When $\theta \in [0, 1]$, a complete information game at θ becomes a coordination game; that is, (A, A) and (B, B) are the Nash equilibria.

Assumption A3 is not so restrictive because $g(A, \theta) + g(B, \theta) > 0$ is always satisfied by A1 and A2 with respect to $\theta \in [0, 1]$. This also holds for $[-2\epsilon, 0)$ and $(1, 1+2\epsilon]$ when g is continuous and ϵ is sufficiently small.

3 Equilibrium

In this section, we show the existence and the uniqueness of Bayesian Nash equilibria under A1–A3. As in the usual global game literature, we consider equilibria for "switching (or monotone) strategies". A strategy of player i is called a switching strategy with switching point k if the player chooses action A if $x > k$, and chooses action B if $x < k$ when player i receive signal x. Let $\pi(x, k)$ be the gain of the expected payoff of player i choosing A rather than B when player i receive signal x and opponent player j plays a switching strategy with switching point k. $\pi(x, k)$ is given by

$$\begin{aligned}
&\pi(x,k) \\
&= \int_{-d}^{1+d} \Bigg[\left(u(A,A,\theta) \int_{k-\theta}^{\epsilon} \phi(e_j) de_j + u(A,B,\theta) \int_{-\epsilon}^{k-\theta} \phi(e_j) de_j \right) \\
&- \left(u(B,A,\theta) \int_{k-\theta}^{\epsilon} \phi(e_j) de_j + u(B,B,\theta) \int_{-\epsilon}^{k-\theta} \phi(e_j) de_j \right) \Bigg] h(\theta|x) d\theta \\
&= \int_{-d}^{1+d} \left\{ g(A,\theta)(1-\Phi(k-\theta)) - g(B,\theta)\Phi(k-\theta) \right\} h(\theta|x) d\theta,
\end{aligned}$$

where $h(\theta|x)$ is the conditional density function of θ observing signal x.

For $x \in [-d+\epsilon, 1+d-\epsilon]$, $h(\theta|x)$ is given by

$$h(\theta|x) = \begin{cases} \frac{1}{2\epsilon} & \text{if } x-\epsilon \le \theta \le x+\epsilon \\ 0 & \text{otherwise,} \end{cases}$$

and $\pi(x, k)$ is rewritten as

$$\pi(x, k) = \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} \{g(A, \theta) - (g(A, \theta) + g(B, \theta))\Phi(k - \theta)\}d\theta. \tag{2}$$

From assumptions A1 and A2, we immediately see that the following proposition holds.

Proposition 3.1. *For all* k, $\pi(x, k) < 0$ *for* $x \in [-d-\epsilon, -\epsilon]$ *and* $\pi(x, k) > 0$ *for* $x \in [1 + \epsilon, 1 + d + \epsilon]$.

Assumptions A1, A2, and A3 imply the following proposition.

Proposition 3.2. *For any* $x \in [-\epsilon, 1 + \epsilon]$, $\pi(x, k)$ *is strictly increasing in* x *and* $\pi(x, k)$ *is strictly decreasing in* k *.*

Proof. Since both $g(A, \theta)(1-\Phi(k-\theta))$ and $-g(B, \theta)\Phi(k-\theta)$ is increasing in θ and either of them is strictly increasing in θ, $\pi(x, k)$ is strictly increasing in x by (2). Since A3 implies that $g(A, \theta)+g(B, \theta) > 0$ for $x-\epsilon \leq \theta \leq x+\epsilon$ when $x \in [-\epsilon, 1 + \epsilon]$, the integrand in (2) is strictly decreasing in k. Hence, $\pi(x, k)$ is strictly decreasing in k. □

Proposition 3.1 imply that there exists k^* in $(-\epsilon, 1+\epsilon)$ such that $\pi(k^*, k^*) = 0$ if π is continuous on (x, k). By Proposition 3.2, a strategy profile that both players choose a switching strategy with switching point k^* satisfying $\pi(k^*, k^*) = 0$ is an equilibrium because $\pi(x, k^*) > 0$ for $x > k^*$ and $\pi(x, k^*) < 0$ for $x < k^*$.

Proposition 3.2 does not imply that monotonicity of $\pi(k, k)$ in k, so that k^* satisfying $\pi(k^*, k^*) = 0$ may not be unique. But, if such k^* is unique and $\pi(x, k)$ is continuous on (x, k), then a strategy profile that the switching strategy with switching point k^* is the essentially unique equilibrium not only among switching strategies, but among all mixed strategies by the argument of the iterated elimination of interim-dominated strategies shown by Carlsson and van Damme (1993) and Morris and Shin (2003). We prove this fact in Appendix. Our examples shown in Section 5 satisfies that $\pi(x, k)$ is continuous on (x, k) and k^* satisfying $\pi(k^*, k^*) = 0$ is unique so an equilibrium is unique in our examples.

4 Probability of Miss-coordination and each equilibrium

This section examines ex ante probability of occurring each action profile in a global game when both players choose the same switching strategy.

Suppose that both players choose a switching strategy with switching point k for $-d+\epsilon < k < 1+d-\epsilon$. Consider the partition of (θ, x_i, x_j)-region defined as follows:

$$
\begin{aligned}
I_A &= \{(\theta, x_i, x_j) | k+\epsilon < \theta \leq 1+d, \theta-\epsilon \leq x_i \leq \theta+\epsilon, \theta-\epsilon \leq x_j \leq \theta+\epsilon\} \\
I_B &= \{(\theta, x_i, x_j) | -d \leq \theta < k-\epsilon, \theta-\epsilon \leq x_i \leq \theta+\epsilon, \theta-\epsilon \leq x_j \leq \theta+\epsilon\} \\
I_{AA} &= \{(\theta, x_i, x_j) | k-\epsilon \leq \theta \leq k+\epsilon, k \leq x_i \leq \theta+\epsilon, k \leq x_j \leq \theta+\epsilon, \} \\
I_{BB} &= \{(\theta, x_i, x_j) | k-\epsilon \leq \theta \leq k+\epsilon, \theta-\epsilon \leq x_i < k, \theta-\epsilon \leq x_j < k, \} \\
I_{AB} &= \{(\theta, x_i, x_j) | k-\epsilon \leq \theta \leq k+\epsilon, k \leq x_i \leq \theta+\epsilon, \theta-\epsilon \leq x_j < k, \} \\
I_{BA} &= \{(\theta, x_i, x_j) | k-\epsilon \leq \theta \leq k+\epsilon, \theta-\epsilon \leq x_i < k, k \leq x_j \leq \theta+\epsilon, \}
\end{aligned}
$$

When θ belongs to $k+\epsilon < \theta \leq 1+d$, action profile (A, A) is always realized for any signal x_i and x_j if both players choose a switching strategy with switching point k. Similarly (B, B) must occur when $-d \leq \theta < k-\epsilon$. I_A and I_B correspond to these areas. Otherwise, when $k-\epsilon \leq \theta \leq k+\epsilon$, every strategy profile, (A, A), (B, B), (A, B) and (B, A) can be realized. I_{AA}, I_{BB}, I_{AB} and I_{BA} correspond to the areas in which action profiles (A, A), (A, B), (B, A) and (B, B) occur, respectively.

Let $P(I)$ be the probability of occurring area I. $P(I_A)$ is calculated by

$$
\begin{aligned}
P(I_A) &= \int_{k+\epsilon}^{1+d} \int_{\theta-\epsilon}^{\theta+\epsilon} \int_{\theta-\epsilon}^{\theta+\epsilon} \phi(x_i-\theta)\phi(x_j-\theta)h(\theta) dx_i dx_j d\theta \\
&= \int_{k+\epsilon}^{1+d} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon}\frac{1}{2\epsilon}\frac{1}{1+2d} de_i de_j d\theta \\
&= \frac{1}{4\epsilon^2(1+2d)} \int_{k+\epsilon}^{1+d} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} de_i de_j d\theta \\
&= \frac{1+d-k-\epsilon}{1+2d}.
\end{aligned}
$$

Similarly, we obtain

$$
P(I_B) = \frac{1}{4\epsilon^2(1+2d)} \int_{-d}^{k-\epsilon} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} de_i de_j d\theta = \frac{d+k-\epsilon}{1+2d}
$$

$$P(I_{AA}) = \frac{1}{4\epsilon^2(1+2d)} \int_{k-\epsilon}^{k+\epsilon} \int_{k-\theta}^{\epsilon} \int_{k-\theta}^{\epsilon} de_i de_j d\theta = \frac{2\epsilon}{3(1+2d)}$$

$$P(I_{BB}) = \frac{1}{4\epsilon^2(1+2d)} \int_{k-\epsilon}^{k+\epsilon} \int_{-\epsilon}^{k-\theta} \int_{-\epsilon}^{k-\theta} de_i de_j d\theta = \frac{2\epsilon}{3(1+2d)}$$

$$P(I_{AB}) = \frac{1}{4\epsilon^2(1+2d)} \int_{k-\epsilon}^{k+\epsilon} \int_{k-\theta}^{\epsilon} \int_{-\epsilon}^{k-\theta} de_i de_j d\theta = \frac{\epsilon}{3(1+2d)}$$

$$P(I_{BA}) = \frac{1}{4\epsilon^2(1+2d)} \int_{k-\epsilon}^{k+\epsilon} \int_{-\epsilon}^{k-\theta} \int_{k-\theta}^{\epsilon} de_i de_j d\theta = \frac{\epsilon}{3(1+2d)}.$$

Thus, as long as both players choose the same switching strategy, the probability of miscoordination is independent of the strategy because $P(I_{AB})$ and $P(I_{BA})$ do not depend on k. Let P_{miss} be the probability of miscoordination. P_{miss} is given by:

$$P_{miss} = P(I_{AB}) + P(I_{BA}) = \frac{2\epsilon}{3(1+2d)} \tag{3}$$

As ϵ increases, the probabilities of the occurrences of (A, B) and (B, A), which represent miss-coordination, increase, resulting in a decrease in the sum of the probabilities of the occurrences of (A, A) and (B, B), which are equilibria in the complete information game. However, the change in the switching point k might lead to an increase in the probability of either equilibrium occurring. Let us focus on the probability of (A, A) occurring and denote it by P_A. P_A is given by

$$P_A = P(I_A) + P(I_{AA}) = \frac{3(1+d-k)-\epsilon}{3(1+2d)}.$$

To simplify the discussion, let us assume that k is differentiable by ϵ. Differentiating P_A by ϵ implies

$$\frac{dP_A}{d\epsilon} = -\frac{1}{(1+2d)} \left(\frac{dk}{d\epsilon} + \frac{1}{3} \right).$$

Thus, the necessary and sufficient condition for the probability of (A, A) occurring to increase is as follows.

Proposition 4.1. $\frac{dP_A}{d\epsilon} > 0$ *if and only if* $\frac{dk}{d\epsilon} < -\frac{1}{3}$.

5 Examples

This section presents some examples of the function of deviation losses. We examine the probability of miscoordination and one of equilibria with respect to increasing accuracy of private information. Our original interest is the probability of a Pareto superior (or inferior) equilibrium, but equilibria of the global game depends only on the function of deviation losses. Even if the function of the deviation losses are the same, the relation of Pareto dominance between equilibria changes with the specific payoffs.

We will proceed with the payoff function where $u(A, A) = g(A, \theta)$, $u(B, A) = -g(B, \theta)$, and $u(B, A) = u(B, B) = 0$, as illustrated in the payoff matrix at the top of Figure 1. In this payoff function, since (A, A) is a Pareto superior equilibrium, we are interested in how the probability of occurring (A, A) changes with respect to the accuracy of the players' noise ϵ. Note that the same function of deviation losses also holds for the payoff function where $u(B, A) = -g(A, \theta)$, $u(B, B) = g(B, \theta)$, as illustrated in the payoff matrix at the bottom of Figure 1. In this payoff function, (A, A) becomes a Pareto inferior equilibrium.

1 \ 2	A	B
A	$(g(A, \theta), g(A, \theta))$	$(-g(B, \theta), -g(B, \theta))$
B	(0 , 0)	(0 , 0)

1 \ 2	A	B
A	(0 , 0)	(0 , 0)
B	$(-g(A, \theta), -g(A, \theta))$	$(g(B, \theta), g(B, \theta))$

Figure 1: two games with the same deviation losses

Note that the Bayesian equilibrium is solved by $\pi(k, k) = 0$ and from (1) and (2) $\pi(k,k)$ is given by

$$\begin{aligned}\pi(k,k) &= \frac{1}{2\epsilon}\int_{k-\epsilon}^{k+\epsilon}\{g(A,\theta)(1-\Phi(k-\theta)) - g(B,\theta)\Phi(k-\theta)\}d\theta \\ &= \frac{1}{4\epsilon^2}\int_{k-\epsilon}^{k+\epsilon}\{2\epsilon g(A,\theta) - (g(A,\theta)+g(B,\theta))(k-\theta+\epsilon)\}d\theta.\end{aligned}$$

5.1 Simple Linear Case

Suppose that $g(A,\theta) = m_A\theta$ and $g(B,1) = m_B(1-\theta)$ where $m_A > 0$ and $m_B > 0$.[2] $\pi(k,k)$ is

$$\pi(k,k) = \frac{1}{6}\left\{3k(m_A+m_B) - 3m_B + (m_A - m_B)\epsilon\right\}.$$

Thus, switching point k of an equilibrium is

$$k = \frac{m_B}{m_A+m_B} - \frac{m_A - m_B}{3(m_A+m_B)}\epsilon.$$

Since $dk/d\epsilon$ is always greater than $-1/3$, so that P_A must be decreasing with respect to ϵ for any m_A and m_B.

5.2 An example of increasing the probability of achieving an equilibrium

We give an example of an increase of the probability of achieving (A,A) when ϵ increases. Suppose that $g(A,\theta)+g(B,\theta) = 1$. Then $\pi(k,k)$ is

$$\begin{aligned}\pi(k,k) &= \frac{1}{4\epsilon^2}\int_{k-\epsilon}^{k+\epsilon}\{2\epsilon g(A,\theta) - (g(A,\theta)+g(B,\theta))(k-\theta+\epsilon)\}d\theta \\ &= \frac{1}{2\epsilon}\int_{k-\epsilon}^{k+\epsilon} g(A,\theta)d\theta - \frac{1}{2}.\end{aligned}$$

Let $g(A,\theta)$ be

$$g(A,\theta) = \begin{cases} \theta & \theta < \frac{1}{2} \\ a\left(\theta - \frac{1}{2}\right) + \frac{1}{2} & \frac{1}{2} \le \theta \le \frac{2a+1}{4a} \\ \frac{a}{2a-1}\left(\theta - \frac{2a+1}{4a}\right) + \frac{3}{4} & \theta > \frac{2a+1}{4a} \end{cases}$$

[2]In this example, we immediately find that assumptions A1 and A2 hold. Assumption A3 holds when $m_A = m_B$, or $m_A > m_B$ and $\epsilon < \frac{m_B}{2(m_A - m_B)}$ (or $m_A < m_B$ and $\epsilon < \frac{m_A}{2(m_B - m_A)}$).

for some constant $a > 1$.

When $k < \frac{1}{2} - \epsilon$, $\pi(k,k) = k - 1/2 < 0$. Suppose that $k > \frac{1}{2} + \epsilon$. For any $\theta \in [k-\epsilon, k+\epsilon]$, $g(A,\theta) > \theta$. holds by $a > 1$. This implies that

$$\pi(k,k) > \frac{1}{2\epsilon}\int_{k-\epsilon}^{k+\epsilon} \theta d\theta - \frac{1}{2} = k - \frac{1}{2} > \epsilon > 0.$$

When $\frac{1}{2} - \epsilon < k < \frac{1}{2} + \epsilon$,$\pi(k,k)$ is given by

$$\begin{aligned}\pi(k,k) &= \frac{1}{2\epsilon}\left(\int_{k-\epsilon}^{1/2} \theta d\theta + \int_{1/2}^{k+\epsilon} \left\{a(\theta - \frac{1}{2}) + \frac{1}{2}\right\} d\theta\right) - \frac{1}{2} \\ &= \frac{-(1-2k+2\epsilon)^2 + a(-1+2k+2\epsilon)^2}{16\epsilon}.\end{aligned}$$

Solving $\pi(k^*, k^*) = 0$, we have

$$k^* = \frac{1}{2} - \frac{\epsilon(1 + a \pm 2\sqrt{a})}{a-1}.$$

Since

$$\frac{1}{2} - \frac{\epsilon(1 + a + 2\sqrt{a})}{a-1} < \frac{1}{2} - \epsilon$$

we find that

$$k^* = \frac{1}{2} - \frac{\epsilon(1 + a - 2\sqrt{a})}{a-1}$$

is a solution and the unique equilibrium is a profile of a switching strategy with switching point k^*.

Proposition 4.1 implies that if

$$\frac{dk}{d\epsilon} = -\frac{1 + a - 2\sqrt{a}}{a-1} < -\frac{1}{3}$$

then, P_A is increasing in ϵ. Then, $a > 4$ is a sufficient condition for an increase of P_A.

For example, when $a = 9$, the switching point of the equilibrium strategy is given by

$$k^* = \frac{1}{2} - \frac{\epsilon}{2}$$

and we have

$$P_A = P(I_{AA}) + P(I_A) = \frac{1}{2} + \frac{\epsilon}{6(1+2d)}.$$

We confirm that P_A is increasing in ϵ.

Appendix

In the Appendix, we prove that if k^* uniquely satisfies $\pi(k^*, k^*) = 0$ and $\pi(x, k)$ is continuous with respect to (x, k), then both players selecting a switching strategy with the switching point k^* is the unique Bayesian Nash equilibrium, not only among switching strategies but among all mixed strategy profiles under our assumptions.

Following the literature on global games, such as Carlsson and van Damme (1993) and Morris and Shin (2003), we use the argument of iterated elimination of interim-dominated strategies. While the discussion of equilibrium uniqueness in global games usually applies as ϵ approaches zero, our findings hold for sufficiently small but non-zero values of ϵ.

Since $\pi(x, k)$ is continuous on x, Proposition 3.1 implies that there exists $x \in (-\epsilon, 1+\epsilon)$ satisfying $\pi(x, k) = 0$ for any k by the intermediate value theorem. According to Proposition 3.2, this x that satisfies $\pi(x, k) = 0$ is unique, and we denote it by $b(k)$. Note that $b(k) \in (-\epsilon, 1+\epsilon)$. A switching strategy with switching point $b(k)$ is a best response strategy when the opponent use a switching strategy with switching point k.

The following proposition shows monotonicity of $b(k)$.

Proposition 5.1. *$b(k)$ is strictly increasing, i.e., if $k' > k$, then $b(k') > b(k)$.*

Proof. Suppose that $k' > k$. Since $b(k) \in (-\epsilon, 1+\epsilon)$, Proposition 3.2 implies $\pi(b(k), k') < \pi(b(k), k)$, so that $\pi(b(k), k') < 0$ from $\pi(b(k), k) = 0$. Since $\pi(b(k'), k') = 0$, we have $\pi(b(k), k') < \pi(b(k'), k')$. $\pi(x, k')$ is strictly increasing in x from Proposition 3.2, we conclude that $b(k) < b(k')$. □

To examine the iterated elimination of interim-dominated strategies, we defined $\overline{b}^n$,$\underline{b}^n$ as follows:

$$\overline{b}^n = \begin{cases} 1+\epsilon & n = 1 \\ b(\overline{b}^{n-1}) & n \geq 2, \end{cases} \quad \underline{b}^n = \begin{cases} -\epsilon & n = 1 \\ b(\underline{b}^{n-1}) & n \geq 2. \end{cases}$$

We find that sequence $\{\overline{b}^n\}$ is non-increasing and $\{\underline{b}^n\}$ is a non-decreasing from Proposition 5.1. Let σ_i be a strategy of player i allowing probabilistic choice. $\sigma_i(x_i)$ denotes the probability of choosing A for player i when player i receive signal x_i. Let $V(x_i, \sigma_j)$ be the gain of the expected payoff of player i choosing A rather than B when player i receive signal x_i and opponent player j plays strategy σ_j. We denote a switching strategy with switching point k of player i by $s[k]_i$. $s[k]_i$ is identical to strategy σ_i which satisfies

$$\sigma_i(x_i) = \begin{cases} 1 & x_i > k \\ 0 & x_i < k. \end{cases}$$

Proposition 5.2. *If (σ_1, σ_2) is a Bayesian Nash equilibrium, then for any $n \geq 1$ and $i = 1, 2$, the following conditions hold:*

$$\sigma_i(x_i) = \begin{cases} 1 & x_i > \overline{b}^n \\ 0 & x_i < \underline{b}^n. \end{cases}$$

Proof. The proof is shown by induction on n. Suppose $n = 1$. Then $\overline{b}^n = 1 + \epsilon$ and $\underline{b}^n = -\epsilon$. For $x_i > 1 + \epsilon$, since the probability that θ is less than or equal to 1 is zero, A is a strictly dominant action by assumptions A1 and A2. Hence player i chooses A in an equilibrium, so $\sigma_i(x_i) = 1$. Similarly, $\sigma_i(x_i) = 0$ for $x_i < -\epsilon$ by the same assumptions. We have shown that the statement holds for $n = 1$.

Next, we assume that the statement is true for n and prove it for $n+1$. We will show that $\sigma_i(x_i) = 1$ when $x_i > \overline{b}^{n+1}$. Using a similar argument, we can also show that $\sigma_i(x_i) = 0$ when $x_i < \underline{b}^{n+1}$.

From the inductive hypothesis, we know that $\sigma_i(x_i) = 1$ if $x_i > \overline{b}^n$ and $\sigma_i(x_i) = 0$ if $x_i < \underline{b}^n$, so we only need to consider the case where $\underline{b}^n \leq x_i \leq \overline{b}^n$. Furthermore, since $-\epsilon \leq \underline{b}^n$ and $1 + \epsilon \geq \overline{b}^n$, we only need to consider the case where $-\epsilon \leq x_i \leq 1 + \epsilon$.

The inductive hypothesis also holds for j, so if $x_j > \overline{b}^n$, then $\sigma_j(x_j) = 1$, and if $x_j < \underline{b}^n$, then $\sigma_j(x_j) = 0$. Therefore, for $x_j > \overline{b}^n$ and $x_j < \underline{b}^n$, we have $\sigma_j(x_j) = s[\overline{b}^n]_j(x_j)$, and for $\underline{b}^n \leq x_j < \overline{b}^n$ we have $\sigma_j(x_j) \geq s[\overline{b}^n]_j(x_j)$ since $s[\overline{b}^n]_j(x_j) = 0$. Thus, for any x_j with $x_j \neq \overline{b}^n$, $\sigma_j(x_j) \geq s[\overline{b}^n]_j(x_j)$.

Calculating $V(x_i, \sigma_j) - \pi(x_i, \overline{b}_j^n)$ gives:

$$
\begin{aligned}
&V(x_i, \sigma_j) - \pi(x_i, \overline{b}^n) \\
&= \int_{x_i-\epsilon}^{x_i+\epsilon} \int_{\underline{b}^n}^{\overline{b}^n} \Big\{ (g(A,\theta) + g(B,\theta))(\sigma_j(x_j) - s[\overline{b}^n]_j(x_j)) \Big\} f(x_j, \theta | x_i)\, dx_j\, d\theta \\
&= \int_{x_i-\epsilon}^{x_i+\epsilon} (g(A,\theta) + g(B,\theta)) \int_{\underline{b}^n}^{\overline{b}^n} (\sigma_j(x_j) - s[\overline{b}^n]_j(x_j)) f(x_j, \theta | x_i)\, dx_j\, d\theta
\end{aligned}
$$

where $f(x_j, \theta | x_i)$ is a conditional joint distribution of (x_j, θ) given x_i. Since $x_i - \epsilon \geq -2\epsilon$ and $x_i + \epsilon \leq 1 + 2\epsilon$, Assumption A3 implies $g(A,\theta) + g(B,\theta) > 0$, and thus $V(x_i, \sigma_j) \geq \pi(x_i, \overline{b}^n)$.

Since $\pi(\overline{b}^{n+1}, \overline{b}^n) = 0$ and $\pi(x_i, k)$ is strictly increasing in x_i, we have $\pi(x_i, \overline{b}^n) > 0$ for any $x_i > \overline{b}^{n+1}$.

Therefore, for $x_i > \overline{b}^{n+1}$, we have $V(x_i, \sigma_j) > 0$, which means that A is the unique optimal action, and $\sigma_i(x_i) = 1$ must hold. □

Since both $\{\overline{b}^n\}$ and $\{\underline{b}^n\}$ are monotonic and bounded sequence, they converge to some limit point $\overline{b}^*$ and $\underline{b}^*$, respectively. Continuity of π on (x, k) implies that $\pi(\overline{b}^*, \overline{b}^*) = \pi(\underline{b}^*, \underline{b}^*) = 0$. This ensures the existence of an equilibrium under the assumptions: a strategy profile both players choosing a switching strategy with switching $\overline{b}^*$ or $\underline{b}^*$ is an equilibrium.

Moreover, if k^* satisfying $\pi(k^*, k^*) = 0$ is unique, $\overline{b}^* = \underline{b}^* = k^*$ holds. Hence, we obtain the following proposition from Proposition 5.2.

Proposition 5.3. *If k^* satisfying $\pi(k^*, k^*) = 0$ is unique, then an equilibrium is essentially unique: player i chooses A and B for $x_i > k^*$ and $x_i < k^*$, respectively.*

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