Research Paper Series

No. 52

On the team-maxmin equilibria

Takuya Iimura*

June, 2024

*Faculty of Economics and Business Administration, Tokyo Metropolitan University

On the team-maxmin equilibria

Takuya Iimura*

June 6, 2024

Abstract

We show that team-maxmin equilibrium due to von Stengel and Koller [*Games and Economic Behavior* 21(1997):309–321] exists in a more general setting. We show that it exists, as a pure strategy equilibrium of an infinite game of a team and an adversary, if (i) the strategy sets are compact convex subsets of topological vector spaces, and (ii) the payoff is bounded, upper semicontinuous on the team's strategy profile set, concave on each member's strategy set, and lower semicontinuous and convex on the adversary's strategy set. We also show a corollary on the existence of a mixed strategy team-maxmin equilibrium. Sion's minimax theorem is used for the proof.

Keywords: team-maxmin equilibrium, Sion's minimax theorem, zero-sum games, existence of equilibrium

JEL Classification: C72 (Noncooperative game)

1 Introduction

A *team* is a set of players who have identical payoffs and make a collective strategy choice. Teams appear in games under various guises; e.g., zero-sum games of two teams (Ho and Sun, 1974), non-zero-sum games of two teams (Palfrey and Rosenthal, 1983), and, more recently, non-zero-sum games of any number of teams (Kim et al., 2022). We consider, following von Stengel and Koller (1997), a multi-player, zero-sum, normal form game of one team and one player called the *adversary*. It is assumed that the members of the team can communicate before but not during the play. Thus, they cannot choose correlated strategies. Also they cannot make a joint deviation from the chosen strategy profile; only a unilateral deviation is possible. It is also assumed that, since zero-sum, the team seeks to maximize the worst case payoff, and the members of the team collectively choose a *team-maxmin strategy*

*School of Business Administration, Tokyo Metropolitan University, Tokyo 192-0397, Japan, E-mail: `t.iimura@tmu.ac.jp` (T. Iimura).

profile, a profile of uncorrelated strategies that serves for that purpose. In von Stengel and Koller (1997), the game is a finite game, and the team-maxmin strategy profile is a profile of generally mixed, uncorrelated strategies. They showed, in this setting, that a team-maxmin strategy profile exists, and any team-maxmin strategy profile is part of an equilibrium, called a *team-maxmin equilibrium*, which gives the highest payoff to the team members in the set of equilibria of this multi-player zero-sum game.

If the game is finite, as in von Stengel and Koller (1997), a team-maxmin strategy profile is guaranteed to exist in its mixed extension, in which there also exists an equilibrium by the theorem of Nash (1951). An interesting point of their results is that the existence of an equilibrium of an interesting kind is shown independently of Nash's theorem, using the duality of a linear programming problem. The purpose of the present article is to show that team-maxmin equilibrium exists in a more general setting. To be specific, we show that it exists, as a pure strategy equilibrium of infinite game, if (i) the strategy sets are compact convex subsets of topological vector spaces, and (ii) the payoff is bounded, upper semicontinuous on the team's strategy profile set, concave on each member's strategy set, and lower semicontinuous and convex on the adversary's strategy set. We also show a corollary on the existence of a mixed strategy team-maxmin equilibrium. Our setting covers the original one of von Stengel and Koller (1997),[1] and extends the theory of team-maxmin equilibria from mixed strategy equilibria of finite games to pure and mixed strategy equilibria of infinite games, which may be useful for, say, applications to industrial organization.

Like von Stengel and Koller (1997), we do not use any fixed point theorem. Our proof of main Theorem 3.1 goes in the same way as that of their Theorem, but, instead of using the duality of linear programming problem, we use Sion's minimax theorem (Sion, 1958). Although von Stengel and Koller (1997) rather emphasized the distinction of the adversary's equilibrium strategy from a minmax strategy (in that it is vulnerable to multi-lateral deviations), their result (and ours) seems to imply a special, restricted form of a minimax theorem, which we will present as our Theorem 4.1. We will also argue that, at a team-maxmin equilibrum, we have a collection of zero-sum games of a member and the adversary, and the equilibrium strategy of adversary can be seen as a common minmax strategy of the adversary.

In Section 2, we provide preliminaries. In Section 3, we formally state and prove our main results. In Section 4, we discuss the implications of our results. In Section 5, we conclude

[1]In fact, in the mixed extension of finite game, the strategy sets are identified with unit simplices of Euclidean spaces and the payoff is continuous and multi-linear, satisfying all of our assumpions. For another viewpoint, see a remark after Corollary 3.1 of current article.

with some comments.

2 Preliminaries

Let S and T be sets and $f\colon S\times T \to \mathbb{R}$. This defines a two-player zero-sum game $\Gamma = (S, T, f)$, where player one chooses a strategy s from strategy set S, two a strategy t from strategy set T, and one receives the payoff $f(s,t)$ and two $-f(s,t)$. We only consider bounded payoff functions, namely, f such that there exists $M \in \mathbb{R}$ and $|f(s,t)| < M$ for any $(s,t) \in S \times T$.

A *maxmin strategy* of player one (if any) is a strategy $\underline{s} \in S$ such that

$$\min_{t\in T} f(\underline{s}, t) = \max_{s\in S} \min_{t\in T} f(s,t), \tag{1}$$

i.e., a strategy of player one that sets the highest floor to the payoff of player one. A *minmax strategy* of player two (if any) is a strategy $\bar{t} \in T$ such that

$$\max_{s\in S} f(s, \bar{t}) = \min_{t\in T} \max_{s\in S} f(s,t), \tag{2}$$

i.e., a strategy of player two that sets the lowest ceiling to the payoff of player one.

Suppose that S and T are in some topological spaces. If S and T are compact, and if f is upper semicontinuous (*usc*) on S, and lower semicontinuous (*lsc*) on T, then there exist a maxmin strategy $\underline{s} \in S$ and a minmax strategy $\bar{t} \in T$. Here, f is usc on S iff the upper contour set $U_t(\alpha) = \{s \in S \mid f(s,t) \geq \alpha\}$ is closed for any $\alpha \in \mathbb{R}$ and $t \in T$; lsc on T iff the lower contour set $L_s(\alpha) = \{t \in T \mid f(s,t) \leq \alpha\}$ is closed for any $\alpha \in \mathbb{R}$ and $s \in S$. Clearly,

$$\min_{t\in T} f(\underline{s}, t) \leq f(\underline{s}, \bar{t}) \leq \max_{s\in S} f(s, \bar{t}), \tag{3}$$

so

$$\max_{s\in S} \min_{t\in T} f(s,t) \leq \min_{t\in T} \max_{s\in S} f(s,t). \tag{4}$$

If the equality (called the *maxmin equality*) holds in Eq. (4), and hence in Eq. (3), we have

$$f(s, \bar{t}) \leq f(\underline{s}, \bar{t}) \leq f(\underline{s}, t) \quad \forall s \in S, \forall t \in T, \tag{5}$$

i.e., $(\underline{s}, \bar{t}) \in S \times T$ becomes a Nash *equilibrium* of Γ. Conversely, if Γ has an equilibrium $(s^*, t^*) \in S \times T$, then s^* is a maxmin strategy of player one, and t^* is a minmax strategy of player two (Osborne and Rubinstein, 1994, Proposition 22.2). Thus, for a two-player zero-sum game possessing an equilibrium, a pair of maxmin and minmax strategies is equivalent to an equilibrium. As such, the equilibria are *interchangeable* in that if (s,t) and (s', t') are equilibria of Γ, so are (s', t) and (s, t'); also they are *payoff equivalent* in that $f(s,t)$ is constant for any equilibrium (s,t) of Γ.

In order for the inequality Eq. (4) to hold with equality, however, we need further restrictions for S, T, and f. We will make use of the following theorem.

Theorem 2.1 (Sion (1958))**.** *Let S and T be compact and convex subsets of two topological vector spaces and $f\colon S \times T \to \mathbb{R}$. If f is usc and quasiconcave on S, and if f is lsc and quasiconvex on T, then*

$$\max_{s\in S}\min_{t\in T} f(s,t) = \min_{t\in T}\max_{s\in S} f(s,t). \tag{6}$$

Here, f is *quasiconcave* on S iff the upper contour set $U_t(\alpha)$ is convex for any $\alpha \in \mathbb{R}$ and $t \in T$; *quasiconvex* on T iff the lower contour set $L_s(\alpha)$ is convex for any $\alpha \in \mathbb{R}$ and $s \in S$. If f is *concave* on S, then it is quasiconcave on S; if *convex* on T, then quasiconvex on T. Here, f is concave on S iff $f((1-\theta)s+\theta s', t) \geq (1-\theta)f(s,t)+\theta f(s',t)$ for any $s, s' \in S$ and $\theta \in [0,1]$ for any $t \in T$; convex on T iff $f(s,(1-\theta)t+\theta t') \leq (1-\theta)f(s,t)+\theta f(s,t')$ for any $t, t' \in T$ and $\theta \in [0,1]$ for any $s \in S$.

Now, we introduce an $(n+1)$-player zero-sum game $G = (X_1, \ldots, X_n, Y, u)$ of n members of a team and an adversary of the team, where $X_1, \ldots, X_n, Y$ are non-empty strategy sets of players, X_i for a team member i, $i = 1, \ldots, n$, Y for the adversary, and $u\colon X \times Y \to \mathbb{R}$ is the payoff function of a team member, where $X = X_1 \times \cdots \times X_n$, the set of all strategy profiles of the team. Every member of the team has this identical payoff $u\colon X \times Y \to \mathbb{R}$, and the adversary's payoff is given by the negative of the n times of u, i.e., by $-nu\colon X \times Y \to \mathbb{R}$.

The following definitions are given by von Stengel and Koller (1997). First, a *team-maxmin strategy profile* is a strategy profile $\underline{x} \in X$ such that

$$\min_{y\in Y} u(\underline{x}, y) = \max_{x\in X}\min_{y\in Y} u(x,y), \tag{7}$$

namely, a strategy profile of the team that sets the highest floor to the member's common payoff. Second, as a notion of equilibria of this $(n+1)$-player zero-sum game, a *team-maxmin equilibrium* is an equilibrium $(\underline{x}, \overline{y}) \in X \times Y$ of G, where $\underline{x} \in X$ is a team-maxmin strategy profile and $\overline{y}$ is a strategy of the adversary, namely, $(\underline{x}, \overline{y})$ such that

$$u(\underline{x}_i, \underline{x}_{-i}, \overline{y}) \geq u(x_i, \underline{x}_{-i}, \overline{y})\ \forall x_i \in X_i,\ \forall i, \text{ and } u(\underline{x}, \overline{y}) \leq u(\underline{x}, y)\ \forall y \in Y, \tag{8}$$

where $\underline{x}_{-i} = (\underline{x}_1, \ldots, \underline{x}_{i-1}, \underline{x}_{i+1}, \ldots, \underline{x}_n) \in \prod_{j\neq i} X_j$, as usual.

In von Stengel and Koller (1997), it is shown that a team-maxmin strategy profile exists in the mixed extension of finite games, and any team-maxmin strategy profile is part of a team-maxmin equilibrium (von Stengel and Koller (1997, Theorem)). We will extend the domain of this result in the next section.

3 Main results

Let $G = (X_1, \ldots, X_n, Y, u)$ be a game of a team and an adversary, and let $X = X_1 \times \cdots \times X_n$. We consider the following set of conditions.

Assumption 3.1. 1. $X_1, \dots, X_n, Y$ are compact convex subsets of topological vector spaces.

2. u is bounded, usc on X, concave on each X_i, $i = 1, \dots, n$, and lsc and convex on Y.

Note that X is also compact endowing the product topology, and convex.

Theorem 3.1. *Under Assumption 3.1, a team-maxmin strategy profile exists, and any team-maxmin strategy profile is part of a team-maxmin equilibrium.*

Proof. Since $u(x, \cdot)$ are lsc on compact Y, $\min_{y\in Y} u(x, y)$ exists for every $x \in X$. Since $u(\cdot, y)$ are usc on compact X, and since pointwise minimum of usc function is usc, $\min_{y\in Y} u(\cdot, y)$ is usc, and has a maximizer, which is a team-maxmin strategy profile. Let $\underline{x} \in X$ be a team-maxmin strategy profile, and define $v_i \colon X_i \times Y \to \mathbb{R}$ by $v_i(x_i, y) = u(x_i, \underline{x}_{-i}, y)$ for each $i = 1, \dots, n$. Then their sum $\sum_i v_i = \sum_{i=1}^n v_i$ is usc and concave on X and lsc and convex on Y. By Sion's theorem, it holds that

$$\max_{x\in X} \min_{y\in Y} \sum_i v_i(x_i, y) = \min_{y\in Y} \max_{x\in X} \sum_i v_i(x_i, y). \tag{9}$$

Consider a two-player zero-sum game $\Gamma_1 = (X, Y, \sum_i v_i)$, in which the maxmin equality Eq. (9) holds true. We claim that

$$\max_{x\in X} \min_{y\in Y} \sum_i v_i(x_i, y) = \min_{y\in Y} \sum_i v_i(\underline{x}_i, y), \tag{10}$$

i.e., the team-maxmin strategy profile $\underline{x}$ is a maxmin strategy of x-player in Γ_1. To see this, notice that $\max_x \min_y \sum_i v_i(x_i, y) \geq \min_y \sum_i v_i(\underline{x}_i, y)$, and suppose to the contrary that

$$\max_{x\in X} \min_{y\in Y} \sum_i v_i(x_i, y) > \min_{y\in Y} \sum_i v_i(\underline{x}_i, y). \tag{11}$$

If $\hat{x} \in X$ is a maximizer of the left-hand side, then $\min_y \sum_i v_i(\hat{x}_i, y) > \min_y \sum_i v_i(\underline{x}_i, y)$. If $\hat{y} \in Y$ and $y' \in Y$ are minimizers of $\sum_i v_i(\hat{x}_i, \cdot) = \sum_i u(\hat{x}_i, \underline{x}_{-i}, \cdot)$ and $\sum_i v_i(\underline{x}_i, \cdot) = nu(\underline{x}, \cdot)$, respectively, then

$$\sum_i u(\hat{x}_i, \underline{x}_{-i}, \hat{y}) > nu(\underline{x}, y'). \tag{12}$$

Of course y' also minimizes $v_1(\underline{x}_1, \cdot) = \cdots = v_n(\underline{x}_n, \cdot) = u(\underline{x}, \cdot)$. Now, since u is concave on each X_i, $i = 1, \dots, n$, we have, for any $\epsilon \in]0, 1[$ and $y \in Y$,

$$\begin{aligned}
u((1-\epsilon)\underline{x} + \epsilon\hat{x}, y) &= u((1-\epsilon)\underline{x}_1 + \epsilon\hat{x}_1, \dots, (1-\epsilon)\underline{x}_n + \epsilon\hat{x}_n, y) \\
&\geq (1-\epsilon)u(\underline{x}_1, (1-\epsilon)\underline{x}_2 + \epsilon\hat{x}_2, \dots, (1-\epsilon)\underline{x}_n + \epsilon\hat{x}_n, y) \\
&\quad + \epsilon u(\hat{x}_1, (1-\epsilon)\underline{x}_2 + \epsilon\hat{x}_2, \dots, (1-\epsilon)\underline{x}_n + \epsilon\hat{x}_n, y) \\
&\geq \cdots \\
&\geq (1-\epsilon)^n u(\underline{x}, y) + \epsilon(1-\epsilon)^{n-1}\sum_i u(\hat{x}_i, \underline{x}_{-i}, y) + \epsilon^2 A(\epsilon, \hat{x}, \underline{x}, y) \\
&= u(\underline{x}, y) - n\epsilon u(\underline{x}, y) + \epsilon\sum_i u(\hat{x}_i, \underline{x}_{-i}, y) + \epsilon^2 B(\epsilon, \hat{x}, \underline{x}, y),
\end{aligned}$$

where $A(\epsilon, \hat{x}, \underline{x}, y)$ and $B(\epsilon, \hat{x}, \underline{x}, y)$ are expressions for $\epsilon \in]0, 1[$ that are bounded (due to the boundedness of u).[2] Moreover, for $\epsilon \in]0, \frac{1}{n}[$, we have $(1 - n\epsilon)u(\underline{x}, y) \geq (1 - n\epsilon)u(\underline{x}, y')$ and $\sum_i u(\hat{x}_i, \underline{x}_{-i}, y) \geq \sum_i u(\hat{x}_i, \underline{x}_{-i}, \hat{y})$ for any $y \in Y$ by the definition of y' and $\hat{y}$, so

$$u((1-\epsilon)\underline{x} + \epsilon\hat{x}, y) \geq u(\underline{x}, y') - n\epsilon u(\underline{x}, y') + \epsilon\sum_i u(\hat{x}_i, \underline{x}_{-i}, \hat{y}) + \epsilon^2 B(\epsilon, \hat{x}, \underline{x}, y), \tag{13}$$

for any $y \in Y$. Thus, for $\epsilon \in]0, \frac{1}{n}[$, we have

$$\min_{y\in Y} u((1-\epsilon)\underline{x} + \epsilon\hat{x}, y) \geq u(\underline{x}, y') + \epsilon\left(\sum_i u(\hat{x}_i, \underline{x}_{-i}, \hat{y}) - nu(\underline{x}, y')\right) + \epsilon^2 B(\epsilon, \hat{x}, \underline{x}, y^\epsilon), \tag{14}$$

where y^ϵ is a minimizer of the left-hand side. Since the expression in the parenthesis is positive by assumption and $B(\epsilon, \hat{x}, \underline{x}, y^\epsilon)$ is bounded, we have for sufficiently small ϵ,

$$\min_{y\in Y} u((1-\epsilon)\underline{x} + \epsilon\hat{x}, y) > u(\underline{x}, y') = \min_{y\in Y} u(\underline{x}, y), \tag{15}$$

---

[2] Precise forms of A and B are as follows. Let $N = \{1, \dots, n\}$ and $x_I := (x_i \mid i \in I)$ for $I \subseteq N$. Since

$$\begin{aligned}
u((1-\epsilon)\underline{x} + e\hat{x}, y) &\geq \sum_{k=0}^{n} \epsilon^k (1-\epsilon)^{n-k} \sum_{I\subseteq N, |I|=k} u(\hat{x}_I, \underline{x}_{-I}, y) \\
&= (1-\epsilon)^n u(\underline{x}, y) + \epsilon(1-\epsilon)^{n-1}\sum_i u(\hat{x}_i, \underline{x}_{-i}, y) + \sum_{k=2}^{n} \epsilon^k (1-\epsilon)^{n-k} \sum_{I\subseteq N, |I|=k} u(\hat{x}_I, \underline{x}_{-I}, y),
\end{aligned}$$

we set

$$A(\epsilon, \hat{x}, \underline{x}, y) = \sum_{k=2}^{n} \epsilon^{k-2}(1-\epsilon)^{n-k} \sum_{I\subseteq N, |I|=k} u(\hat{x}_I, \underline{x}_{-I}, y).$$

Since

$$(1-\epsilon)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}(-\epsilon)^k = 1 - n\epsilon + \epsilon^2 \sum_{k=2}^{n} \binom{n}{k} (-\epsilon)^{k-2}$$

and

$$\epsilon(1-\epsilon)^{n-1} = \epsilon\left(1 - (n-1)\epsilon + \epsilon^2 \sum_{k=2}^{n-1} \binom{n-1}{k} (-\epsilon)^{k-2}\right) = \epsilon + \epsilon^2\left(1 - n + \epsilon\sum_{k=2}^{n-1} \binom{n-1}{k} (-\epsilon)^{k-2}\right),$$

we set

$$B(\epsilon, \hat{x}, \underline{x}, y) = A(\epsilon, \hat{x}, \underline{x}, y) + \sum_{k=2}^{n} \binom{n}{k} (-\epsilon)^{k-2} u(\underline{x}, y) + \left(1 - n + \epsilon\sum_{k=2}^{n-1} \binom{n-1}{k} (-\epsilon)^{k-2}\right)\sum_i u(\hat{x}_i, \underline{x}_{-i}, y).$$

i.e., $x' := (1-\epsilon)\underline{x} + \epsilon\hat{x}$ sets a higher floor to the payoff of team members than $\underline{x}$ does, contradicting that $\underline{x}$ is a team-maxmin stratgegy profile. Hence Eq. (10). Having established that $\underline{x}$ is a maxmin strategy of x-player in Γ_1 satisfying the maxmin equality Eq. (9), let $\overline{y} \in Y$ be a minmax strategy of y-player in Γ_1. Then the pair $(\underline{x}, \overline{y})$ constitutes an equilibrium of Γ_1. That $\sum_i v_i(\underline{x}_i, \overline{y}) \geq \sum_i v_i(x_i, \overline{y})$ for any $x \in X$ means that $u(\underline{x}_i, \underline{x}_{-i}, \overline{y}) \geq u(x_i, \underline{x}_{-i}, \overline{y})$ for any $x_i \in X_i$, for every $i = 1, \ldots, n$. That $\sum_i v_i(\underline{x}_i, \overline{y}) \leq \sum_i v_i(\underline{x}, y)$ for any $y \in Y$ means that $u(\underline{x}, \overline{y}) \leq u(\underline{x}, y)$ for any $y \in Y$. Hence, $(\underline{x}, \overline{y})$ is also a team-maxmin equilibrium of G. □

The *mixed extension* of a game $G = (X_1, \ldots, X_n, Y, u)$ is a game $\overline{G} = (M_1, \ldots, M_n, N, \tilde{u})$, where $M_1, \ldots, M_n, N$ are the sets of *mixed* strategies of players, M_i for a team member i and N for the adversary, which are the sets of all probability measures on $X_1, \ldots, X_n, Y$, respectively; $\tilde{u} \colon M_1 \times \cdots \times M_n \times N \to \mathbb{R}$ is the expected payoff of a team member given by

$$\tilde{u}(\mu_1, \ldots, \mu_n, \nu) = \int_{X_1 \times \cdots \times X_n \times Y} u(x_1, \ldots, x_n, y) d(\mu_1 \times \cdots \times \mu_n \times \nu). \tag{16}$$

When we consider $\overline{G}$, we do not need the sets of *pure* strategies $X_1, \ldots, X_n, Y$ of G be convex, nor the payoff u of G be concave on each X_i and convex on Y. The sets $X_1, \ldots, X_n, Y$ even need not be subsets of vector spaces, but, in order to use probability measures on them, we assume that they are subsets of metric spaces. Thus, we assume the following conditions for G:

Assumption 3.2. 1. $X_1, \ldots, X_n, Y$ are compact subsets of metric spaces.

2. u is bounded, usc on X, and lsc on Y.

Inherently, the sets $M_1, \ldots, M_n, N$ are convex subsets of vector spaces and $\tilde{u}$ is linear on each of $M_1, \ldots, M_n, N$. If $X_1, \ldots, X_n, Y$ are compact subsets of metric spaces, then $M_1, \ldots, M_n, N$ are compact under the weak* topologies (Aliprantis and Border, 2006, Theorem 15.11). Let $M = M_1 \times \cdots \times M_n$, endowing the product topology. The boundedness of u ensures the finiteness of the value of integral, and hence, the boundedness of function $\tilde{u}$ on $M \times N$. If $u \colon X \times Y \to \mathbb{R}$ is usc on metrizable X and lsc on metrizable Y, then $\tilde{u} \colon M \times N \to \mathbb{R}$ is usc on M and lsc on N (Aliprantis and Border, 2006, Theorem 15.5). We thus have:

1. $M_1, \ldots, M_n, N$ are compact convex subsets of topological vector spaces.

2. $\tilde{u}$ is bounded, usc on M, linear on each M_i, $i = 1, \ldots, n$, and lsc and linear on N.

Since linear function is both concave and convex, we have the following corollary.

Corollary 3.1. *Under Assumption 3.2, a mixed strategy team-maxmin strategy profile exists, and any team-maxmin strategy profile is part of a team-maxmin equilibrium.*

The mixed-strategy team-maxmin equilibra of finite games can also be viewed as an example of this corollary: If G is a finite game, the finite sets $X_1, \ldots, X_n, Y$ are compact subsets of discrete metric spaces, and the bounded payoff u on finite sets $X \times Y$ is usc on X and lsc on Y under the discrete topology. Thus, it satisfies Assumption 3.2.

4 Discussions

In the proof of Theorem 3.1, we used the functions $v_i \colon X_i \times Y \to \mathbb{R}$ defined by

$$v_i(x_i, y) = u(\underline{x}_1, \ldots, \underline{x}_{i-1}, x_i, \underline{x}_{i+1}, \ldots, \underline{x}_n, y), \ i = 1, \ldots, n, \tag{17}$$

picking an arbitrary team-maxmin strategy profile $\underline{x} \in X$ (v_i depends on $\underline{x}$, but we omitted it for simplicity). We then observed that the team-maxmin strategy profile $\underline{x}$ is a maxmin strategy of x-player of game $\Gamma_1 = (X, Y, \sum_i v_i)$ which satisfies the maxmin equality Eq. (9), and that $\sum_i v_i(x_i, \overline{y}) \leq \sum_i v_i(\underline{x}_i, \overline{y})$ for any $x \in X$ means that $u(x_i, \underline{x}_{-i}, \overline{y}) \leq u(\underline{x}_i, \underline{x}_{-i}, \overline{y})$ for any $x_i \in X_i$ for every $i = 1, \cdots, n$, and that $\sum_i v_i(\underline{x}_i, \overline{y}) \leq \sum_i v_i(\underline{x}_i, y)$ for any $y \in Y$ means that $u(\underline{x}, \overline{y}) \leq u(\underline{x}, y)$ for any $y \in Y$. Thus, Theorem 3.1 also suggests a special type of minimax theorem as follows.

Theorem 4.1. *Suppose $X_1, \ldots, X_n, Y$ and $u \colon X_1 \times \cdots \times X_n \times Y \to \mathbb{R}$ satisfy Assumption 3.1. Let $\underline{x} \in X = X_1 \times \cdots \times X_n$ be such that*

$$\min_{y \in Y} u(\underline{x}, y) = \max_{x \in X} \min_{y \in Y} u(x, y), \tag{18}$$

and let

$$\hat{X} := \bigcup_{i=1}^{n} \left(X_i \times \{\underline{x}_{-i}\} \right). \tag{19}$$

Then

$$\max_{x \in \hat{X}} \min_{y \in Y} u(x, y) = \min_{y \in Y} \max_{x \in \hat{X}} u(x, y). \tag{20}$$

Proof. Since the equilibrium $(\underline{x}, \overline{y})$ of $\Gamma_1 = (X, Y, \sum_i v_i)$ is also an equilibrium of a two-player zero-sum game $\Gamma_2 = (\hat{X}, Y, u|_{\hat{X}})$, $(\underline{x}, \overline{y})$ is a pair of a maxmin and a minmax strategies of Γ_2 satisfying Eq. (20). □

Also using the functions $v_i(x_i, y)$, $i = 1, \ldots, n$, Theorem 3.1 can also be viewed as follows. We have a collection of n zero-sum two-player games $G_i = (X_i, Y, v_i)$, $i = 1, \ldots, n$, in each of which a team member i plays a zero-sum game against a common adversary. Under

Assumption 3.1, v_i are usc and concave on X_i and lsc and convex on Y. Thus, by Sion's theorem, we have for each $i = 1, \dots, n$

$$\max_{x_i \in X_i} \min_{y \in Y} v_i(x_i, y) = \min_{y \in Y} \max_{x_i \in X_i} v_i(x_i, y), \tag{21}$$

and there exists a pair $(\underline{x}_i', \overline{y}^i)$ of a maxmin and a minmax strategies constituting an equilibrium of G_i, i.e., $(\underline{x}_i', \overline{y}^i)$ such that

$$v_i(x_i, \overline{y}^i) \leq v_i(\underline{x}_i', \overline{y}^i) \leq v_i(\underline{x}_i', y) \quad \forall x_i \in X_i, \forall y \in Y. \tag{22}$$

It can be shown (by a contrapositive argument) that the ith element $\underline{x}_i$ of team-maxmin strategy profile $\underline{x}$ is also a maxmin strategy of member i in G_i. Thus, by replacing $\underline{x}_i'$ with $\underline{x}_i$, we have for each $i = 1, \dots, n$

$$v_i(x_i, \overline{y}^i) \leq v_i(\underline{x}_i, \overline{y}^i) \leq v_i(\underline{x}_i, y) \quad \forall x_i \in X_i, \forall y \in Y. \tag{23}$$

The second inequalities are equal to $u(\underline{x}, \overline{y}^i) \leq u(\underline{x}, y)\ \forall y \in Y$, for every i. Hence, $v_i(\underline{x}_i, \overline{y}^i) = \min_{y \in Y} u(\underline{x}, y)$ for every i, and we have

$$v_1(\underline{x}_1, \overline{y}^1) = \cdots = v_n(\underline{x}_n, \overline{y}^n) = \max_{x \in X} \min_{y \in Y} u(x, y). \tag{24}$$

Thus far, we have found n minmax strategies $\overline{y}^i$ of the adversary, each one for each game G_i, only by using Sion's theorem. (This remains valid even if we assume that v_i are *quasiconcave* on X_i, $i = 1, \dots, n$, and *quasiconvex* on Y, namely, even if u is quasiconcave on each X_i and quasiconvex on Y.) Can we choose a *common* minmax strategy that works for all G_i? Theorem 3.1 answers yes, if v_i are concave on X_i, convex on Y, and bounded.[3] In this case the adversary can choose a common minmax strategy $\overline{y}$ by choosing a minimizer $\overline{y}$ of $\max_{x \in X} \sum_i v_i(x_i, y)$.

5 Concluding comments

1. In von Stengel and Koller (1997), it is shown as a Corollary: *Team-maxmin equilibria are precisely the equilibria of the game with highest payoff to the team.* This observation also applies to the game of our setting. Let $(\underline{x}, \overline{y})$ be a team-maxmin equilibrium of $G = (X_1, \dots, X_n, Y, u)$ and (x^*, y^*) any equilibrium of G. Then $u(x^*, \cdot)$ is being minimized by y^*, i.e., $u(x^*, y^*) = \min_{y \in Y} u(x^*, y)$; but, since $u(\underline{x}, \overline{y}) = \max_{x \in X} \min_{y \in Y} u(x, y) \geq \min_{y \in Y} u(x^*, y)$, we have $u(\underline{x}, \overline{y}) \geq u(x^*, y^*)$ for any equilibrium (x^*, y^*) of G.

2. The boundedness of u in Assumptions 3.1 and 3.2 is playing different roles. In Corollary 3.1, it is used for $\tilde{u}$ to be bounded. In Theorem 3.1, it is used for $B = B(\epsilon, \hat{x}, \underline{x}, y)$

[3] See the comment #2 in the next section.

to be bounded, in particular, bounded *below*. Since u is assumed lsc on compact Y, $A = A(\epsilon, \hat{x}, \underline{x}, y)$ is automatically bounded below (see footnote 2 for the precise forms of A and B). Thus, it is acting on the rest part of B, involving the expressions $u(x_i, \underline{x}_{-i}, y) = v_i(x_i, y)$. Therefore, what we really need is that $u(x, \cdot)$ be bounded *above* on Y given any $x \in \hat{X}$ (of Eq. (19)). It seems, however, that u be bounded is not so stringent assumption for the payoff functions of games.

3. It is not known whether the common minmax strategy in the last section survives when we relax the concave–convex assumption to quasiconcave–quasiconvex one. We note that a common minmax strategy, if it exists, sits in the set of minimizers of $u(\underline{x}, \cdot)$. Not every minimizer of $u(\underline{x}, \cdot)$ can be a common minmax strategy (nor even a minmax strategy of a subgame G_i), however. For example, in the first example of von Stengel and Koller (1997), $u(\underline{x}, \cdot)$ is constant and any (mixed) strategy of the adversary is a minimizer, but the common minmax strategy (and every minmax strategy in a subgame) is unique. If u is usc on X and quasiconcave on each X_i, $i = 1, \ldots, n$, and lsc and "strictly" quasiconvex on Y, then we will certainly have a common minmax strategy of the adversary, because then $u(\underline{x}, \cdot)$ is strictly quasiconvex and the set $\arg\min_{y \in Y} u(\underline{x}, y)$ is a singleton, hosting all the $\overline{y}^1, \ldots, \overline{y}^n$ above, implying $\overline{y}^1 = \cdots = \overline{y}^n$. However, this strictness condition precludes the original case of von Stengel and Koller (1997), where u (viewed as a payoff function of mixed extension) is linear on Y.

Acknowledgement This work is supported by JSPS KAKENHI Grant Number 20K01549.

References

Aliprantis, C. D. and Border, K. (2006), *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3rd ed., Springer–Verlag.

Ho, Y. C. and Sun, F. K. (1974), Value of information in two-team zero-sum problems, *Journal of Optimization Theory and Applications*, 14:557–571.

Kim, J., Palfrey, T. R., Zeidel, J. R. (2022), Games played by teams of players, *American Economic Journal: Microeconomics*, 14:122–157.

Nash, J. (1951), Non-cooperative games, *Annals of Mathematics*, 54:286–295.

Osborne, M. J. and Rubinstein, A. (1994), *A Course in Game Theory*, The MIT Press.

Palfrey, T. R. and Rosenthal, H. (1983), A strategic calculus of voting, *Public Choice*, 41:7–53.

Sion, M. (1958), On general minimax theorems, *Pacific Journal of Mathematics*, 8:171–176.

von Stengel, B. and Koller, D. (1997), Team-maxmin equilibria, *Games and Economic Behavior*, 21:309–321.