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A Term Structure Model of Default-free and Defaultable Interest Rates with Regime-Switching Properties: Useful Tool for Risk Evaluation

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Long-term historical data on interest rates and credit spreads can be used to identify different regimes, specifically, a calm regime with lower default risk and volatility and a stressed regime with higher default risk and volatility. We propose a pricing and risk evaluation model of interest rate risk and credit risk with the Markovian regime-switching property. We discuss the dynamics of a regime, the interest rate, and default intensity under a physical measure and a pricing measure, and propose a simple tractable model. In our model, the default-free interest rate and the default intensity are dependent on the regime, and this dependence affects the defaultable zero curve. We propose an appropriate calibration method and demonstrate some numerical examples of the zero curves with different credit ratings, which will show us various types of yield curves in the future. We hope that such yield curve models will reveal some new methods and perspectives for financial risk management, especially for asset liability management.

Keywords: Risk management; Regime-switching model; Term structure of default-free/defaultable interest rates.

1. Introduction

The 21st century has witnessed a few financial distresses, with the 2008 worldwide financial crisis being a predominant one. Financial institutions, financial regulation agencies, and central banks have discussed ways to control financial risks and have been constructing concepts, tools, and systems for risk management, financial regulations, and financial policies. For many financial institutions, especially banks, management of the interest rate and credit risks has been one of the most important challenges.

From the long-term data on interest rates and credit spreads, a few different regimes have been identified: typically, a calm regime with lower default risk and volatility, and a stressed regime with higher credit risk and volatility. Since Hamilton (1989) proposed a regime-switching model and analyzed the economic cycles, many researchers have adopted such models for analyzing and modeling economic and financial data[a]. Restricted to the studies on interest rates, Inoue & Okimoto (2008) used regime-switching models for analyzing monetary policy and private sector behavior in Japan. Dai et al. (2007) used a discrete-time regime-switching Gaussian term structure model, while Wu & Zeng (2005) applied a continuous-time affine term structure model with regime-dependent parameters. Elliott & Siu (2009) considered a bond valuation as a derivative written on a Markovian regime-switching instantaneous spot rate, and derived a Markov-modulated exponential-affine bond price formula. Their results have been applied to the term structures of interest rates, see Elliott & Nishide (2014).

In credit risk analysis, Gourieroux et al. (2013) derived a defaultable bond price in a discrete-time setting and analyzed sovereign yield curves, and Monfort & Renne (2013) analyzed the credit spreads of U.S. bonds. Fisher & Stolper (2019) analyzed the behavior of the credit spreads and their key determinants, and Chun et al. (2014) proposed a regression model of credit spreads with endogenous regimes, and demonstrated that the model enhanced the explanatory power of the determinants. In a continuous-time setting, Hainaut & Le Courtois (2014) proposed a default intensity model described by regime-switching Lévy process to evaluate the survival probability, and analyzed the Credit Default Swap (CDS) market. Li & Ma (2013) discussed pricing options analytically, but the results are limited to the conditional price considering a sample regime path.

In finance, most of the available research on regime-switching models involve analyzing financial data and pricing securities. However, there have been limited research on quantitatively evaluating financial risk using these models. If a high-risk regime is included in the financial risk evaluation, it might give us a new assessment of the future, and one such method is the stress test. In the stress tests, the potential losses are calculated quantitatively under some given high-risk scenarios. Moreover, if such a high-risk regime or scenario can be estimated from the observed data, the forward-looking stress tests can be done with some statistical viewpoints.

In this study, we propose a consistent pricing and risk evaluation model of interest rate risk and credit risk under a regime-switching environment in a continuous-time setting. For the risk evaluation of an asset portfolio, we set a risk horizon, which is a certain future time, and discuss the prices of the surviving securities at the horizon and the accumulated loss by defaults up to the horizon. Such kinds of risk evaluations for a bond portfolio are already discussed in Muromachi (2022). Contrariwise, in this study, we focus on the construction of the default-free and defaultable yield curves and analyses of the movement of the curves in the future. We think that these analyses of the regime-switching property are suitable for the evaluation of interest rate risk and credit risk, especially in asset liability management (ALM) because the banks hold various kinds of liabilities sensitive to interest rate risk. In ALM, the synthetic evaluation of the interest rate risk through a common stochastic model is theoretically desirable, and we suggest that one of the promising

[a]As an example of review papers, see Ang & Timmermann (2012).

tools is the stochastic yield curve model with the regime-switching properties. Our proposed model can combine such evaluation of liabilities with the risk evaluation of the asset portfolio.

This article is organized as follows: Section 2 describes the construction of a simple pricing and risk evaluation model. Based on a Markovian regime-switching model, the default-free spot rate and the default intensity processes are described under the physical measure, and the change of measure is discussed so that the processes under the pricing measure are derived. In Section 3, we propose estimation methods for model parameters and some estimated results. Although our methods are suboptimal, the basic principles are quite straightforward. In Section 4, we show some numerical examples of the term structure of default-free and defaultable zero rates, and Section 5 concludes this article.

2. The Model

In this section, we discuss the construction of a simple pricing and risk evaluation model in a regime-switching environment. Our model follows the framework proposed by Kijima & Muromachi (2000). They discussed why two probability measures, the physical and the pricing probability measures, are necessary for risk evaluation based on market prices. When we calculate risk based on market prices, we set a risk horizon, which is a certain future time, and discuss the future prices of the securities at the horizon, the coupons/dividends, the accumulated loss caused by the defaults, and so on up to the horizon. Given that the accumulated loss and coupons are evaluated as realized values, they must be measured under the physical probability measure. By contrast, when we discuss security prices, we use the no-arbitrage prices not only at present but also at the risk horizon. The no-arbitrage prices are given by the expectation of the discounted future cash flows under the pricing measure equivalent to the physical measure. Therefore, we need two probability measures. In order to maintain consistency between pricing and risk evaluation, stochastic modeling is necessary not only in physical measure but also in pricing measures. The physical measure part of the model is used for generating scenarios up to the risk horizon, while the pricing measure part is used for pricing at the present and the risk horizon. Since the stochastic model in the pricing measure is obtained from the stochastic model in the physical measure and the scheme of the change of measure, it is also enough information for describing the system to give the scheme of the change of measure and the stochastic model either in the physical or the pricing measure.

2.1. *Stochastic processes under the physical probability measure*

Consider a switching regime in a continuous-time and finite-state Markov chain model. Let t, $t \geq 0$, be time, and $t = 0$ is present. We consider a financial market with a finite horizon T, $0 < T < \infty$, and define a filtered probability space

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ under which all stochastic processes can be described. All probability spaces in this study satisfy the usual conditions.

We consider different states $K \in \mathcal{N}$, which are called regimes. Let $\boldsymbol{X}(t) \in \{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_K\}$ be a K-dimensional vector and $\boldsymbol{e}_j$, $j = 1, \cdots, K$, are K-dimensional unit vectors where the i-th component of $\boldsymbol{e}_j$ is the Kronecker's delta δ_{ij}. $\boldsymbol{X}(t)$ implies an economic state at time t, and is a Markov chain on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$. Let $\mathcal{Q} \in \mathcal{R}^K \times \mathcal{R}^K$ denote the infinitesimal generator matrix of $\boldsymbol{X}(t)$, and we assume that $\mathcal{Q}$ is time-independent, so that we can express

$$\mathcal{Q} = (q_{ij})_{i,j=1,\cdots,K}, \;\; q_{ij} = \begin{cases} \lim\limits_{\Delta t \to 0} \dfrac{P\{\boldsymbol{X}(t+\Delta t) = \boldsymbol{e}_j | \boldsymbol{X}(t) = \boldsymbol{e}_i\}}{\Delta t}, & i \neq j \\ -\sum\limits_{k\neq j} q_{kj}, & i = j \end{cases}$$

and we have

$$\boldsymbol{X}(t) = \boldsymbol{X}(0) + \int_0^t \mathcal{Q}^\top \boldsymbol{X}(s)\mathrm{d}s + \boldsymbol{M}(t), \qquad t \ge 0 \tag{2.1}$$

where the K-dimensional vector $\boldsymbol{M}(t)$ is a P–martingale. The transition probability matrix is defined by

$$\mathcal{P}(t) = (p_{ij}(t))_{i,j=1,\cdots,K}, \;\; p_{ij}(t) = P\left\{\boldsymbol{X}(s+t) = \boldsymbol{e}_j | \boldsymbol{X}(s) = \boldsymbol{e}_i\right\}, \;\; s, t \ge 0,$$

and is given by

$$\mathcal{P}(t) = \exp\{t\mathcal{Q}\} = \sum_{k=0}^{\infty} \frac{t^k \mathcal{Q}^k}{k!}. \tag{2.2}$$

The risk-free instantaneous spot rate at time t, denoted by $r(t)$, follows the stochastic differential equation (hereinafter, abbreviated as SDE)

$$\mathrm{d}r(t) = \mu_r(t, r(t), \boldsymbol{X}(t))\mathrm{d}t + \sigma_r(t, r(t), \boldsymbol{X}(t))\mathrm{d}z_r(t), \qquad t \ge 0,$$

where $z_r(t)$ is a standard Brownian motion under P. Consider a risky security, subject to its credit risk. Let $\tau > 0$ be the default time of the security. Define $H(t) = 1_{\{\tau \le t\}}$ as its default process where 1_A is the indicator function[b], and the default intensity at time t, denoted by $h(t)$, follows the SDE

$$\mathrm{d}h(t) = \mu_h(t, h(t), \boldsymbol{X}(t))\mathrm{d}t + \sigma_h(t, h(t), \boldsymbol{X}(t))\mathrm{d}z_h(t), \qquad t \ge 0,$$

where $z_h(t)$ is a standard Brownian motion under P and $\mathrm{d}z_r(t)\mathrm{d}z_h(t) = \rho(t)\mathrm{d}t$. For any arbitrary time t, $0 \le t \le T$, the filtration generated by the default process is defined by $\mathcal{H}_t = \sigma(H(s) : 0 \le s \le t)$ and $\mathcal{H} = (\mathcal{H}_t)_{0\le t\le T}$, while the filtration generated by $X(t)$ is defined by $\mathcal{F}_t^X = \sigma(\boldsymbol{X}(s) : 0 \le s \le t)$ and $\mathcal{F}^X = (\mathcal{F}_t^X)_{0\le t\le T}$. The filtrations generated by $r(t)$ and $h(t)$ are $\mathcal{F}_t^r = \sigma(r(s) : 0 \le s \le t)$, $\mathcal{F}^r = (\mathcal{F}_t^r)_{0\le t\le T}$ and $\mathcal{F}_t^h = \sigma(h(s) : 0 \le s \le t)$, $\mathcal{F}^h = (\mathcal{F}_t^h)_{0\le t\le T}$, respectively. The complete filtration of this system is defined by $\mathcal{F} = \mathcal{F}^X \vee \mathcal{F}^r \vee \mathcal{F}^h \vee \mathcal{H}$, that is, $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^r \vee \mathcal{F}_t^h \vee \mathcal{H}_t$ is satisfied for any arbitrary t, $0 \le t \le T$. Additionally, we use a useful filtration $\mathcal{G}_t = \mathcal{F}_T^X \vee \mathcal{F}_t^r \vee \mathcal{F}_t^h \vee \mathcal{H}_t$, $0 \le t \le T$.

[b] $1_A = 1$ when the event A is true, otherwise $1_A = 0$.

2.2. *Stochastic processes under a pricing probability measure*

Consider a probability measure P^X equivalent to P. Assume that the infinitesimal generator matrix of $\boldsymbol{X}(t)$ under P^X, denoted by $\mathcal{Q}^X = (q_{ij}^X)_{i,j=1,\cdots,K}$, is given by

$$q_{ij}^X = (1+\kappa_{ij})q_{ij}, \qquad i \neq j, \quad i,j = 1,\cdots,K$$

where $\kappa_{ij} > -1$ is constant [c]. Then, the change of probability measure from P to P^X is written as

$$\left.\frac{\mathrm{d}P^X}{\mathrm{d}P}\right|_{\mathcal{F}_T^X} = \eta^X(T),$$

where the Radon-Nikodym derivative $\eta^X(T)$ is defined by

$$\begin{aligned}
\eta^X(t) &= \exp\left\{-\int_0^t \sum_{k,\ell=1}^K \kappa_{k\ell} q_{k\ell} H^\ell(u)\mathrm{d}u\right\} \prod_{0<u\leq t}\left(1+\sum_{k,\ell=1}^K \kappa_{k\ell}\Delta H^{k\ell}(u)\right) \\
H^i(t) &= 1_{\{\boldsymbol{X}(t)=\boldsymbol{e}_i\}}, \qquad i=1,\cdots,K, \\
H^{ij}(t) &= \sum_{0<u\leq t} H^i(u-)H^j(u), \qquad i\neq j, \quad i,j=1,\cdots,K,
\end{aligned}$$

and $\Delta H^{ij}(t) = H^{ij}(t) - H^{ij}(t-)$. Let $\mathcal{P}^X(t)$ be the transition probability matrix under P^X. Then, the similar equations such as (2.1) and (2.2) are obtained only if $(\mathcal{Q}, \mathcal{P}(t), \boldsymbol{M}(t))$ are replaced with $(\mathcal{Q}^X, \mathcal{P}^X(t), \boldsymbol{M}^X(t))$ where $\boldsymbol{M}^X(t)$ is a P^X-martingale.

We follow a standard discussion for modeling credit risk[d]. Let $\beta(t)$ be a $\mathcal{F}_t$-adapted process and $\kappa_h(t) > -1$ be a $\mathcal{F}_t$-predictable process, and define a new probability measure P^c equivalent to P^X by

$$\left.\frac{\mathrm{d}P^c}{\mathrm{d}P^X}\right|_{\mathcal{G}_T} = \rho_h(T),$$

where

$$\rho_h(t) = \int_0^t (\beta_h(s)\mathrm{d}z_h(s) + \kappa_h(s)\mathrm{d}M_h(s))$$

and

$$M_h(t) = H(t) - \int_0^{t\wedge\tau} h(s)\mathrm{d}s.$$

Then, the process

$$z_h^c(t) = z_h(t) - \int_0^t \beta_h(s)\mathrm{d}s$$

[c] Generally, κ_{ij} is a $\mathcal{F}_t$–predictable process.
[d] See, for example, Kusuoka (1999) and Bielecki & Rutkowski (2002).

becomes a standard Brownian motion under P^c, and the process

$$h^c(t) = (1 + \kappa_h(t))h(t)$$

is regarded as a default intensity under P^c, because the process

$$M_h^c(t) = H(t) - \int_0^{t\wedge\tau} h^c(s)\mathrm{d}s$$

becomes a $(\mathcal{F}, P^c)$-martingale, which corresponds to the fact that $M_h(t)$ is a $(\mathcal{F}, P^X)$-martingale.

The following discussion is based on Elliott et al. (2007). Assuming that the price of a risky asset at time t, denoted by $S(t)$, follows

$$\mathrm{d}S(t) = \mu(t, S(t), \boldsymbol{X}(t))S(t)\mathrm{d}t + \sigma(t, S(t), \boldsymbol{X}(t))S(t)\mathrm{d}z_S^c(t), \qquad t \geq 0$$

where $z_S^c(t)$ is a standard Brownian motion under P^c. Define the Radon-Nikodym derivative from P^c to its equivalent probability measure P^η by [e]

$$\left.\frac{\mathrm{d}P^\eta}{\mathrm{d}P^c}\right|_{\mathcal{G}_T} = \exp\left\{\int_0^T \eta(s)\mathrm{d}z_S^c(s) - \frac{1}{2}\int_0^T (\eta(s))^2\,\mathrm{d}s\right\}$$

where

$$\eta(t) = \frac{r(t) - \mu(t, S(t), \boldsymbol{X}(t))}{\sigma(t, S(t), \boldsymbol{X}(t))},$$

then, thanks to the Girsanov's theorem, the process

$$z^\eta(t) = z_S^c(t) - \int_0^t \eta(u)\mathrm{d}u$$

becomes a standard Brownian motion under P^η, and the relative price of $S(t)$ with respect to the bank account $B(t) = \exp\left\{\int_0^t r(u)\mathrm{d}u\right\}$ becomes a $(\mathcal{G}, P^\eta)$–local martingale. Assuming that the relative price is a $(\mathcal{G}, P^\eta)$–martingale, the price of a European derivative given $\mathcal{G}_T$ is given by

$$V(t|\mathcal{G}_t) = E^{P^\eta}\left[\exp\left\{-\int_t^M r(u)\mathrm{d}u\right\}G(S(M))\middle|\,\mathcal{G}_t\right],$$

where $G(\cdot)$ is the payoff function of the derivative at its maturity M, $t \leq M \leq T$, and $E^{P^\eta}[\cdot|\cdot]$ is the conditional expectation operator under P^η. From the chain rule of the conditional expectation, given $\{\tau > t\}$, the price $V(t)$ is given by

$$V(t) = E^{P^\eta}\left[V(t|\mathcal{G}_t)|\,\mathcal{F}_t\right] = E^{P^\eta}\left[\exp\left\{-\int_t^M r(u)\mathrm{d}u\right\}G(S(M))\middle|\,\mathcal{F}_t\right]. \qquad (2.3)$$

Hereafter, we call P^η the risk-neutral probability measure and denote it as $\widetilde{P}$, and denote a standard Brownian motion under $\widetilde{P}$ as $\widetilde{z}$. Similarly, $h^c(t)$, $\mathcal{Q}^X$, $\mathcal{P}^X(t)$ and

[e] This change of measure is called the risk-neutral regime-switching Esscher transform.

$\boldsymbol{M}^X(t)$ are denoted by $\widetilde{h}(t)$, $\widetilde{\mathcal{Q}}$, $\widetilde{\mathcal{P}}(t)$ and $\widetilde{\boldsymbol{M}}(t)$, respectively. Generally, a regime-switching model derives an incomplete market so that many risk-neutral measures might exist, and P^η is one of them. However, according to Elliott et al. (2005), the measure P^η is the minimum entropy martingale measure of P, so that the price given by (2.3) is a reasonable one.

Let $\mathcal{F}_t$-predictable process $\lambda_r(t)$ be the market price of risk against $z_r(t)$, then the stochastic processes describing the market under $\widetilde{P}$ are as follows:

$$\mathrm{d}r(t) = \widetilde{\mu}_r(t, r(t), \boldsymbol{X}(t))\mathrm{d}t + \sigma_r(t, r(t), \boldsymbol{X}(t))\mathrm{d}\widetilde{z}_r(t), \tag{2.4}$$

$$\mathrm{d}h(t) = \widetilde{\mu}_h(t, h(t), \boldsymbol{X}(t))\mathrm{d}t + \sigma_h(t, h(t), \boldsymbol{X}(t))\mathrm{d}\widetilde{z}_h(t) \tag{2.5}$$

$$\widetilde{h}(t) = (1 + \kappa_h(t, \boldsymbol{X}(t)))h(t) \tag{2.6}$$

$$\widetilde{\mu}_r(t, r(t), \boldsymbol{X}(t)) = \mu_r(t, r(t), \boldsymbol{X}(t)) - \lambda_r(t)\sigma_r(t, r(t), \boldsymbol{X}(t)) \tag{2.7}$$

$$\widetilde{\mu}_h(t, h(t), \boldsymbol{X}(t)) = \mu_h(t, h(t), \boldsymbol{X}(t)) + \beta(t)\sigma_h(t, h(t), \boldsymbol{X}(t)). \tag{2.8}$$

Consider a default-free discount bond with maturity M, $t \le M \le T$. From (2.3), its price at time t is given by

$$v(t, M, r, \boldsymbol{X}) = \widetilde{E}\left[\exp\left\{-\int_t^M r(u)\mathrm{d}u\right\}\middle| r(t) = r,\ \boldsymbol{X}(t) = \boldsymbol{X}\right].$$

Next, consider a defaultable discount bond with maturity M, and let $\tau > 0$ be its default time. Suppose the holder of the bond receives \$1 at M if the bond survives at M, while the holder receives \$$\delta$, $0 \le \delta < 1$ at M if default occurs up to M. From (2.3), given $\{\tau > t\}$, the price of the bond is given as

$$\begin{aligned} D(t, M, r, \widetilde{h}, \boldsymbol{X}) &= \widetilde{E}\left[\exp\left\{-\int_t^M r(u)\mathrm{d}u\right\}\left\{1_{\{\tau > M\}} + \delta 1_{\{\tau \le M\}}\right\}\middle| r(t) = r,\ \widetilde{h}(t) = \widetilde{h},\ \boldsymbol{X}(t) = \boldsymbol{X}\right] \\ &= \delta v(t, M, r, \boldsymbol{X}) + (1 - \delta)p(t, M, r, \widetilde{h}, \boldsymbol{X}), \end{aligned} \tag{2.9}$$

$$p(t, M, r, \widetilde{h}, \boldsymbol{X}) = \widetilde{E}\left[\exp\left\{-\int_t^M (r(u) + \widetilde{h}(u))\mathrm{d}u\right\}\middle| r(t) = r,\ \widetilde{h}(t) = \widetilde{h},\ \boldsymbol{X}(t) = \boldsymbol{X}\right]. \tag{2.10}$$

Hereafter, we call $p(t, M, r, \widetilde{h}, \boldsymbol{X})$ the price of the "survival discount bond", whose payoff is \$1 if and only if the bond survives at M.

2.3. *A simple model*

Here, we propose a simple model. Under the physical measure P, $r(t)$ follows

$$\mathrm{d}r(t) = a\left(\langle \boldsymbol{m}, \boldsymbol{X}(t)\rangle - r(t)\right)\mathrm{d}t + \langle \boldsymbol{\sigma}, \boldsymbol{X}(t)\rangle \mathrm{d}z_r(t), \qquad t \ge 0, \tag{2.11}$$

where $\boldsymbol{m} = (m_1, \cdots, m_K)^\top$ and $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_K)^\top$ are K-dimensional constant vectors, a and σ_k, $k = 1, \cdots, K$ are positive constants, and $\langle \boldsymbol{A}, \boldsymbol{B}\rangle$ is the inner product of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Similarly, $h(t)$ follows

$$\mathrm{d}h(t) = a_h\left(\langle \boldsymbol{m}_h, \boldsymbol{X}(t)\rangle - h(t)\right)\mathrm{d}t + \langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle \mathrm{d}z_h(t), \qquad t \ge 0, \tag{2.12}$$

where $\boldsymbol{m}_h = (m_{h,1}, \cdots, m_{h,K})^\top$ and $\boldsymbol{\sigma}_h = (\sigma_{h,1}, \cdots, \sigma_{h,K})^\top$ are K-dimensional constant vectors, a_h and $\sigma_{h,k}$, $k = 1, \cdots, K$ are positive constants, and $\mathrm{d}z_r(t)\mathrm{d}z_h(t) = \rho(t)\mathrm{d}t$ [f]. For simplicity, we assume that $\boldsymbol{X}(t)$ is independent of $(z_r(t), z_h(t))$, and select the Vasicek model because model parameters can be calibrated easily by using the existing statistical tools such as Matlab[g].

To make our model more tractable, we apply the following assumption.

Assumption 2.1. *The market price of risk $\lambda_r(t)$ is written by*

$$\lambda_r(t) = \langle \boldsymbol{L}_r, \boldsymbol{X}(t) \rangle + \psi_r(t) \tag{2.13}$$

where $\boldsymbol{L}_r = (\lambda_{r,1}, \cdots, \lambda_{r,K})^\top$ is a K-dimensional constant vector and $\psi_r(t)$ is a deterministic function of time t.

From (2.4), (2.7), (2.11) and (2.13), we have

$$\mathrm{d}r(t) = (\langle \boldsymbol{\phi}(t), \boldsymbol{X}(t) \rangle - ar(t))\,\mathrm{d}t + \langle \boldsymbol{\sigma}, \boldsymbol{X}(t) \rangle \mathrm{d}\widetilde{z}_r(t), \tag{2.14}$$

where the deterministic function vector $\boldsymbol{\phi}(t) = (\phi_1(t), \cdots \phi_K(t))^\top$ is given as

$$\boldsymbol{\phi}(t) = a\boldsymbol{m} - \boldsymbol{\sigma} \otimes \boldsymbol{L}_r - \boldsymbol{\sigma}\psi_r(t),$$

and $\boldsymbol{A} \otimes \boldsymbol{B}$ is the Hadamard product of K-dimensional vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Then, the discount bond price $v(t, M, r, \boldsymbol{X})$ is given by

$$v(t, M, r, \boldsymbol{X}) = \exp\left\{\langle \boldsymbol{A}(t, M), \boldsymbol{X} \rangle - B(t, M)r\right\}, \tag{2.15}$$

where

$$B(t, M) = \frac{1 - \mathrm{e}^{-a(M-t)}}{a}$$

and $\boldsymbol{A}(t, M) = (A(t, M, \boldsymbol{e}_1), \cdots, A(t, M, \boldsymbol{e}_K))^\top$. Here, define $\bar{A}_i(t, M) = \exp\{A(t, M, \boldsymbol{e}_i)\}$ and $\bar{\boldsymbol{A}}(t, M) = (\bar{A}_1(t, M), \cdots, \bar{A}_K(t, M))^\top$, and they are the solution of the following ordinary differential equations (hereinafter, abbreviated by ODEs),

$$\frac{\mathrm{d}\bar{A}_i(t, M)}{\mathrm{d}t} + \left\{\frac{1}{2}\sigma_i^2 B^2(t, M) - \phi_i(t)B(t, M)\right\}\bar{A}_i(t, M) + \langle \bar{\boldsymbol{A}}(t, M), Q^\top \boldsymbol{e}_i \rangle = 0, \tag{2.16}$$

with the terminal conditions $\bar{A}_i(M, M) = 1$, $i = 1, \cdots, K$. A numerical solution of (2.16) is obtained easily, for example, by using the Runge-Kutta method. The zero rate (yield to maturity of the discount bond) with maturity M at time t is given by

$$R(t, M, r(t), \boldsymbol{X}(t)) = -\frac{\log v(t, M, r(t), \boldsymbol{X}(t))}{M - t}, \qquad 0 \le t < M.$$

Additionally, to manage the default intensity process similarly and to make the calibration tractable, we introduce the following assumption:

[f] An extension of (a, a_h) to the deterministic functions of time t is obvious.
[g] In Hatakeyama (2022), a slightly modified Matlab program was used to estimate parameters.

Assumption 2.2. *The stochastic process $\beta(t)$ is written as*

$$\beta(t) = \langle \boldsymbol{L}_h, \boldsymbol{X}(t)\rangle + \psi_h(t) \tag{2.17}$$

where $\boldsymbol{L}_h = (\lambda_{h,1}, \cdots, \lambda_{h,K})^\top$ is a K-dimensional constant vector and $\psi_h(t)$ is a deterministic function of time t. Further, define

$$\ell(t) = \langle \boldsymbol{\ell}(t), \boldsymbol{X}(t)\rangle, \qquad \boldsymbol{\ell}(t) = (\ell_1(t), \cdots, \ell_K(t))^\top \tag{2.18}$$

where $\ell_j(t) = \kappa_{h,j}(t)h(t)$, $j = 1, \cdots, K$ are differentiable deterministic functions of time t[h].

From (2.5), (2.6), (2.8), (2.12), (2.17) and (2.18), we have [i]

$$\mathrm{d}h(t) = (\langle \boldsymbol{\phi}_h(t), \boldsymbol{X}(t)\rangle - a_h h(t))\,\mathrm{d}t + \langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle \mathrm{d}\widetilde{z}_h(t) \tag{2.19}$$

$$\boldsymbol{\phi}_h(t) = a_h \boldsymbol{m}_h + \boldsymbol{\sigma}_h \otimes \boldsymbol{L}_h + \boldsymbol{\sigma}_h \psi_h(t), \tag{2.20}$$

$$\widetilde{h}(t) = h(t) + \langle \boldsymbol{\ell}(t), \boldsymbol{X}(t)\rangle. \tag{2.21}$$

Now, consider a survival discount bond with maturity M. From (2.10), the time t price of the survival discount bond is written as

$$p(t, M, r, \widetilde{h}, \boldsymbol{X}) = \widetilde{E}\left[\exp\left\{-\int_t^M (r(u) + \widetilde{h}(u))\mathrm{d}u\right\}\middle| r(t) = r, \widetilde{h}(t) = \widetilde{h}, \boldsymbol{X}(t) = \boldsymbol{X}\right].$$

From Appendix A, we obtain

$$p(t, M, r, \widetilde{h}, \boldsymbol{X}) = \exp\left\{\langle \boldsymbol{C}(t, M), \boldsymbol{X}\rangle - B(t, M)r - B_h(t, M)\widetilde{h}\right\}, \tag{2.22}$$

where

$$B_h(t, M) = \frac{1 - \mathrm{e}^{-a_h(M-t)}}{a_h}$$

and $\boldsymbol{C}(t, M) = (C(t, M, \boldsymbol{e}_1), \cdots, C(t, M, \boldsymbol{e}_K))^\top$. Here, define $\bar{C}_i(t, M) = \exp\{C(t, M, \boldsymbol{e}_i)\}$ and $\bar{\boldsymbol{C}}(t, M) = (\bar{C}_1(t, M), \cdots, \bar{C}_K(t, M))^\top$, and they are the solution of the ODE system:

$$\frac{\mathrm{d}\bar{C}_i(t, M)}{\mathrm{d}t} + \left\{\frac{\sigma_i^2}{2}B^2(t, M) + \frac{\sigma_{h,i}^2}{2}B_h^2(t, M) - \phi_i(t)B(t, M) - a_h m'_{h,i}(t)B_h(t, M) + \rho(t)\sigma_i\sigma_{h,i}B(t, M)B_h(t, M)\right\}\bar{C}_i(t, M) + \langle \bar{\boldsymbol{C}}(t, M), Q^\top \boldsymbol{e}_i\rangle = 0 \tag{2.23}$$

with the terminal conditions $\bar{C}_i(M, M) = 1$, $i = 1, \cdots, K$, where

$$m'_{h,i}(t) = m_{h,i} + \ell_i(t) + \frac{1}{a_h}\left(\frac{\mathrm{d}\ell_i(t)}{\mathrm{d}t} + \sigma_{h,i}\lambda_{h,i} + \sigma_{h,i}\psi_h(t) + \langle \boldsymbol{\ell}(t), Q^\top \boldsymbol{e}_i\rangle\right).$$

[h]This differentiability seems implausible because $h(t)$ is not differentiable; however, it makes the calibration easier.

[i]If we set $\boldsymbol{\ell}(t)$ as a constant vector $\boldsymbol{\ell}$, $\boldsymbol{L}_h$ has the same effect as $\boldsymbol{\ell}$, so that $\boldsymbol{L}_h$ would be dropped.

Our valuation is different from Li & Ma (2013)'s results because their results are the conditional price given the sample path of $\boldsymbol{X}(\cdot)$ up to M, while our results are the unconditional price.

When $\widetilde{h}(t) = 0$, $0 \leq t \leq T$, this implies that the bond is default-free, (2.23) reduces to (2.16). By contrast, consider a discount bond with the payoff $1_{\{\tau > M\}} \exp\left\{\int_t^M r(u)\mathrm{d}u\right\}$ at M, then, the price at time t is written as

$$\bar{v}(t, M, \widetilde{h}, \boldsymbol{X}) = \widetilde{E}\left[\exp\left\{-\int_t^M \widetilde{h}(u)\mathrm{d}u\right\}\middle| \widetilde{h}(t) = \widetilde{h}, \boldsymbol{X}(t) = \boldsymbol{X}\right], \tag{2.24}$$

and is given by

$$\bar{v}(t, M, \widetilde{h}, \boldsymbol{X}) = \exp\left\{\langle \boldsymbol{D}(t, M), \boldsymbol{X}\rangle - B_h(t, M)\widetilde{h}\right\}$$

where $\boldsymbol{D}(t, M) = (D(t, M, \boldsymbol{e}_1), \cdots, D(t, M, \boldsymbol{e}_K))^\top$. Similarly, define $\bar{\boldsymbol{D}}(t, M) = (\bar{D}_1(t, M), \cdots, \bar{D}_K(t, M))^\top$ where $\bar{D}_i(t, M) = \exp\{D(t, M, \boldsymbol{e}_i)\}$, and they are the solution of the ODE system:

$$\frac{\mathrm{d}\bar{D}_i(t, M)}{\mathrm{d}t} + \left\{\frac{\sigma_{h,i}^2}{2} B_h^2(t, M) - a_h m'_{h,i}(t) B_h(t, M)\right\} \bar{D}_i(t, M) + \langle \bar{\boldsymbol{D}}(t, M), Q^\top \boldsymbol{e}_i\rangle = 0$$

with the terminal conditions $\bar{D}_i(M, M) = 1$, $i = 1, \cdots, K$. Note that from (2.24), $\bar{v}(t, M, \widetilde{h}, \boldsymbol{X})$ is the conditional survival probability at M under $\widetilde{P}$ on $\{\tau > t\}$, that is, $\bar{v}(t, M, \widetilde{h}, \boldsymbol{X}) = \widetilde{P}\{\tau > M | \tau > t, \widetilde{h}(t) = \widetilde{h}, \boldsymbol{X}(t) = \boldsymbol{X}\}$. From the above discussion, generally, and even if $\rho(t) = 0$, it follows that

$$p(t, M, r, \widetilde{h}, \boldsymbol{X}) \neq v(t, M, r, \boldsymbol{X})\widetilde{P}\{\tau > M | \tau > t, \widetilde{h}(t) = \widetilde{h}, \boldsymbol{X}(t) = \boldsymbol{X}\}.$$

This is because $r(t)$ and $\widetilde{h}(t)$ are dependent through $X(t)$ even if $\rho(t) = 0$.

A definite disadvantage of this model is a negative default intensity in the future with a positive probability. To avoid the disadvantage, for example, the Cox-Ingersoll-Ross (CIR) process or the squared Gaussian process can be proposed.

3. Estimation of model parameters

We discuss on estimating the model parameters used in the simple model in Section 2.3. They are (1) $\mathcal{Q}$, $(a, \boldsymbol{m}, \boldsymbol{\sigma})$ and $(a_h, \boldsymbol{m}_h, \boldsymbol{\sigma}_h)$ on stochastic processes under P, and (2) $\widetilde{\mathcal{Q}}$ (or $\boldsymbol{\kappa} = (\kappa_{ij})_{i,j=1,\cdots,K}$), $(\boldsymbol{L}_r, \psi_r(t))$, $(\boldsymbol{L}_h, \psi_h(t))$ and $\boldsymbol{\ell}(t)$ on stochastic processes under $\widetilde{P}$. If the time-series data of $r(t)$ and $h(t)$ can be observed, it is natural that the parameters in (1) are estimated from the time-series data, while those in (2) are estimated from the term structures of default-free and defaultable interest rates at $t = 0$. However, given that we can only observe the time-series data of the credit spreads instead of $h(t)$, we have to devise another method.

3.1. Estimation procedure

From (2.19) and (2.21), the default intensity $\widetilde{h}(t)$ under P is followed by

$$\mathrm{d}\widetilde{h}(t) = a_h \left\{ \langle \widetilde{\boldsymbol{m}}_h(t), \boldsymbol{X}(t) \rangle - \widetilde{h}(t) \right\} \mathrm{d}t + \langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t) \rangle \mathrm{d}z_h(t) + \langle \boldsymbol{\ell}(t), \mathrm{d}\boldsymbol{M}(t) \rangle \tag{3.1}$$

where the i-th component of $\widetilde{\boldsymbol{m}}_h(t)$ is given by

$$\widetilde{m}_{h,i}(t) = m_{h,i} + \ell_i(t) + \frac{1}{a_h} \left(\frac{\mathrm{d}\ell_i(t)}{\mathrm{d}t} + \langle \boldsymbol{\ell}(t), Q^\top \boldsymbol{e}_i \rangle \right), \qquad i = 1, \cdots, K.$$

We have little information on $\boldsymbol{M}(t)$. So, as the approximated $\widetilde{h}(t)$ process, we use

$$\mathrm{d}\widetilde{h}(t) \simeq a_h \left\{ \langle \widetilde{\boldsymbol{m}}_h(t), \boldsymbol{X}(t) \rangle - \widetilde{h}(t) \right\} \mathrm{d}t + \langle \widetilde{\boldsymbol{\sigma}}_h(t), \boldsymbol{X}(t) \rangle \mathrm{d}z_h(t), \tag{3.2}$$

where

$$(\widetilde{\sigma}_{h,i}(t))^2 = (\sigma_{h,i})^2 + \sum_{k=1}^{K} q_{ki} (\ell_i(t) - \ell_k(t))^2. \tag{3.3}$$

The second term of the above equation is the influence of $\langle \boldsymbol{\ell}(t), \mathrm{d}\boldsymbol{M}(t) \rangle$ on the instantaneous variance of $\widetilde{h}(t)$ under P.

Then, we propose the following parameter estimation procedure:

(1) Estimate $(\mathcal{Q}, a, \boldsymbol{m}, \boldsymbol{\sigma}, a_h, \widetilde{\boldsymbol{m}}_h, \widetilde{\boldsymbol{\sigma}}_h(t))$ from the time-series data of $r(t)$ and $\widetilde{h}(t)$. Here, we assume $\widetilde{h}(t)$ follows the process (3.2) under P.
(2) Estimate $\boldsymbol{\ell}(t)$ from the observed survival function of τ under P.
(3) Estimate $(\widetilde{\mathcal{Q}}, \boldsymbol{L}_r, \psi_r(t))$ in order to fit the present term structure of the default-free interest rates.
(4) Estimate $(\boldsymbol{L}_h, \psi_h(t))$ in order to fit the present term structure of the credit spreads (or defaultable interest rates).

In Step (1), we can use some existing tools; for example, VAR(1) (1-st order vector autoregression) model with regimes offered by Matlab [j]. Moreover, instead of $\widetilde{h}(t)$ (which is related to the instantaneous credit spread and usually cannot be observed in the market), we can use the credit spreads of a defaultable discount bond, $CS(t)$, with a certain maturity with some adjustments; for example, $\widetilde{h}(t) \simeq CS(t)/(1-\delta)$ where δ is the recovery rate of the bond [k]. The main purpose of Step (2) is fitting the term structure of the default probabilities (the distribution function of τ) in P, and as the same time, the effect of $\langle \boldsymbol{\ell}(t), \mathrm{d}\boldsymbol{M}(t) \rangle$ on $\widetilde{\sigma}_{h,i}(t)$ is evaluated through (3.3). The derived $\boldsymbol{\sigma}_h$ must be positive. Detailed procedures and results are omitted here. [l]. In Step (3), $(\widetilde{\mathcal{Q}}, \boldsymbol{L}_r, \psi_r(t))$ are calibrated in order to reproduce the present term structure of the default-free interest rates. First, $\widetilde{\mathcal{Q}}$ and $\boldsymbol{L}_r$ are calibrated so that the theoretical term structure of the default-free interest

[j] For details, see Hatakeyama (2022). We are currently preparing to publish his results in English.
[k] This approximation implies the credit spreads are flat. If several bonds are issued by the same company, we might estimate more useful information about $\widetilde{h}(t)$.
[l] We are preparing another paper on the estimation methods and their results.

rates without $\psi_r(t)$ fits the observed one as much as possible. Here, we use the ridge regression; the objective function with the ridge parameter λ_1 is

$$F_1 \equiv \sum_{j=1}^{10}(R(0,j,r(0),\boldsymbol{X}(0)) - R_{obs}(0,j))^2 + \lambda_1 \left(\sum_{i\neq j}(\widetilde{q}_{ij} - q_{ij})^2 + \sum_{i=1}^{K} L_{r,i}^2 \right) \quad (3.4)$$

where $R_{obs}(0,j)$ is the observed zero rate with maturity j years. The residuals between the observed and theoretical zero rates are compensated by $\psi_r(t)$, therefore, this simple model can reproduce the present term structures of interest rates perfectly. Step (4) is similar to Step (3) without $\widetilde{Q}$.

Although these obtained estimates are not optimal, the relations between the data and the parameters are clear, and each step is simple.

3.2. *Setting and estimated parameters*

In Step (1) in our estimation procedure, we follow Hatakeyama(2021). He used the monthly data of the US Treasury Bond Yields and ICE BofA US Corporate Index OAS (Option Adjusted Spread) from January 2006 to May 2021. Since the instantaneous default intensity $\widetilde{h}(t)$ is difficult to observe, we assume the recovery rate $\delta = 0.4$, which is typical for pricing corporate bonds and regard the credit spread (OAS) divided by $(1-\delta)$ as the default intensity $\widetilde{h}(t)$, then we transform the parameters obtained by Hatakeyama (2021) into those of $\widetilde{h}(t)$'s process.

In Step (2), as the observed survival probabilities, we use S&P's corporate average cumulative default rates, 1981–2019 by Kraemer et al. (2020). Consider default-free discount bonds and three kinds of defaultable discount bonds with different levels of credit risk: AAA, BBB, and CCC/C-rated bonds, respectively.

We choose $K = 3$. In our estimations, statistical information criteria such as AIC (Akaike's Information Criterion) and BIC (Bayesian Information Criterion) decrease as K increases. On the other hand, the estimation of model parameters becomes more difficult with K because the number of parameters increases by the order of K^2. In time-series data analysis, for simplicity, we assume that $(\widetilde{\boldsymbol{m}}_h(t), \widetilde{\boldsymbol{\sigma}}(t))$ are constant, and assume $a = a_h^{AAA} = a_h^{BBB} = a_h^{CCC} = 1$ because the simultaneous estimation of the mean-reverting power and the volatility is difficult. We assume the present default-free and defaultable zero rate curves as

$$R(0,M) = a + bM + cM^2 + dM^3 + eM^4, \quad (\%)$$

given in Table 1 where $R(0,M)$ is the zero rate with maturity M (years) [m]. Additionally, for simplicity, we assume $\phi_r(t)$ and $\phi_h(t)$ are piecewise constant in $t \in [j-1, j]$, $j = 1, 2, \cdots, 10$ years, and $\boldsymbol{\ell}(t)$ is constant in each term. [n], $t \in [0,1]$

[m] We use these values as examples during these several years.
[n] It was difficult to reproduce the observed term structures of the default probability under P on constant $\boldsymbol{\ell}$ setting, and several trials taught us that the difficulty was caused mainly by the differences in the very short term. We ignore the differentiability of $\boldsymbol{\ell}(t)$ at $t = 1$ year.

and $t \in [1, 10]$ years. In the above setting, $(\boldsymbol{m}, \boldsymbol{\sigma})$ become constant in each term $[0, 1]$ and $[1, 10]$ years, respectively.

Table 1. Parameters (%) of the present zero rate curves.

	a	b	c	d	e
default-free	0.00	0.546	−0.0606	0.00233	0.00000
AAA	0.66	0.380	0.0036	−0.00650	0.00040
BBB	0.80	0.740	−0.0820	0.00200	0.00009
CCC/C	4.20	1.000	−0.1000	0.00300	0.00003

The estimated parameters on the processes are shown in Table 2, and the estimated generators are given by

$$\mathcal{Q} = \begin{pmatrix} -1.120 & 0.967 & 0.153 \\ 1.659 & -2.027 & 0.368 \\ 1.222 & 1.575 & -2.797 \end{pmatrix}, \quad \widetilde{\mathcal{Q}} = \begin{pmatrix} -1.292 & 1.291 & 0.001 \\ 1.393 & -1.394 & 0.001 \\ 1.222 & 1.576 & -2.798 \end{pmatrix}. \tag{3.5}$$

The estimated market prices of risks are shown in Table 3.

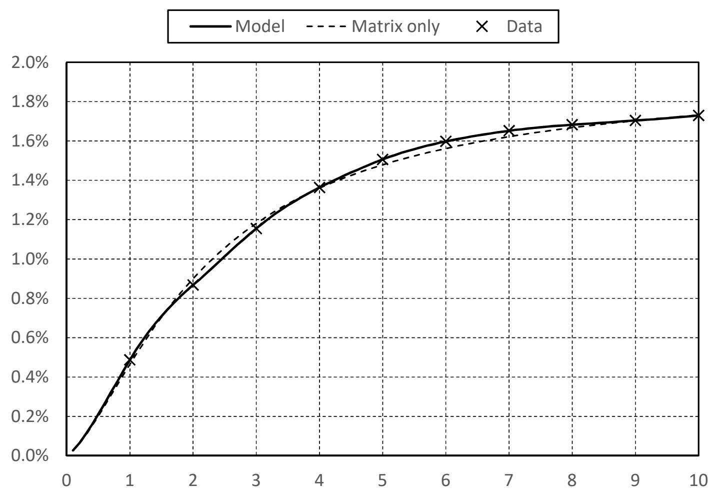

Fig. 1. Term structures of default-free zero rates. This figure compares the term structures of default-free zero rates. The horizontal axis represents maturity (years), and the vertical represents the zero rate. "Model" is the term structure where all calibrated parameters are used, "Matrix Only" is that where the calibrated $\widetilde{\mathcal{Q}}$ is used, while "Data" are given data.

Table 2. Parameters (%) of default-free interest rate and default intensity processes under P.

default-free

regime k	m_k	σ_k
0	0.330	0.460
1	1.230	3.380
2	−2.630	4.310

AAA

regime k	$\widetilde{m}_{h,k}$	$\widetilde{\sigma}_{h,k}$	$m_{h,k}$ $[0,1]$	$m_{h,k}$ $[1,10]$	$\sigma_{h,k}$ $[0,1]$	$\sigma_{h,k}$ $[1,10]$	ℓ_k $[0,1]$	ℓ_k $[1,10]$
0	0.933	0.300	−0.209	−0.407	0.150	0.299	0.981	1.325
1	1.833	0.550	0.935	0.517	0.531	0.550	1.121	1.338
2	3.183	3.817	2.284	1.868	3.816	3.817	1.149	1.341

BBB

regime k	$\widetilde{m}_{h,k}$	$\widetilde{\sigma}_{h,k}$	$m_{h,k}$ $[0,1]$	$m_{h,k}$ $[1,10]$	$\sigma_{h,k}$ $[0,1]$	$\sigma_{h,k}$ $[1,10]$	ℓ_k $[0,1]$	ℓ_k $[1,10]$
0	2.267	0.667	−0.175	−0.886	0.637	0.667	2.318	3.148
1	4.383	1.317	2.130	1.237	1.312	1.317	2.426	3.152
2	10.533	6.317	8.273	7.392	6.316	6.317	2.445	3.155

CCC/C

regime k	$\widetilde{m}_{h,k}$	$\widetilde{\sigma}_{h,k}$	$m_{h,k}$ $[0,1]$	$m_{h,k}$ $[1,10]$	$\sigma_{h,k}$ $[0,1]$	$\sigma_{h,k}$ $[1,10]$	ℓ_k $[0,1]$	ℓ_k $[1,10]$
0	11.300	4.050	29.150	−3.060	2.163	4.049	−19.994	14.390
1	21.583	6.500	42.686	7.211	6.221	6.499	−18.126	14.374
2	46.550	26.833	67.560	32.015	26.819	26.833	−17.791	14.295

Figure 1 shows the observed and the theoretical term structures of the default-free zero rates, while Figure 2 shows those of the CCC/C. In these figures, the horizontal axis represents the maturity, and the vertical represents the zero rates. If all calibrated parameters are used, all term structures can be reproduced perfectly due to the time-dependent parameters $\psi_r(t)$ and $\psi_h(t)$.

Figure 3 shows the observed and the theoretical survival probabilities of CCC/C firms. Notice that in our model the survival probability cannot be reproduced perfectly. This figure shows that the risk premia adjustments (RPA), $\boldsymbol{\ell}(t)$, play an

Table 3. Estimated market prices of risk (%).

	default-free		AAA	BBB	CCC/C
$L_{r,1}$	-0.1259	$L_{h,1}$	-2.98786	-0.05383	-2.16136
$L_{r,2}$	-0.7595	$L_{h,2}$	0.42513	-1.65209	0.50460
$L_{r,3}$	-0.0004	$L_{h,3}$	0.00143	-0.00530	0.00010
term (years)	$\psi_r(t)$		$\psi_h^{AAA}(t)$	$\psi_h^{BBB}(t)$	$\psi_h^{CCC}(t)$
$[0,1]$	-0.0505		0.09766	-0.12207	0.21362
$[1,2]$	0.1761		-0.58594	0.29297	-0.67749
$[2,3]$	-0.1546		0.48828	0.19531	0.07935
$[3,4]$	-0.0296		0.29297	0.19531	-0.12207
$[4,5]$	-0.1105		0.48828	-0.17090	0.34180
$[5,6]$	0.0268		-0.39063	-0.34180	0.29297
$[6,7]$	0.0339		-0.68359	-0.58594	0.64697
$[7,8]$	0.1020		-0.78125	-0.58594	0.73242
$[8,9]$	-0.0457		0.48828	-0.24414	1.13525
$[9,10]$	-0.0575		3.71094	0.68359	1.51367

important role on the fitting of the survival probabilities, especially in the lower credit ratings. Since the fitting between the observed curve and the theoretical curve is difficult on the constant $\boldsymbol{\ell}(t)$ setting, mainly due to the difference in the very short term, we use a 2-term model, $[0,1]$ years and $[1,10]$ years, and assume $\boldsymbol{\ell}(t)$ is constant in each term. Here, we show the results for CCC/C rating only, but we obtain better results in AAA and BBB zero curves.

To our regret, our proposed estimation procedure is not appropriate enough, that is, some estimated values are unreasonable: for example, some $m_{h,0}$ in Table 1 are negative. More future research will be necessary for robust and sophisticated estimation methods.

4. Numerical examples

In this section, we show some numerical examples of zero curves at some future times by using a Monte Carlo simulation. We set $\delta = 0.4$, and $\rho(t) = 0$.

4.1. *Procedure of the Monte Carlo simulation*

The Monte Carlo simulation consists of the following steps:

1. Set the initial values $(\boldsymbol{X}(0), r(0), h(0))$.

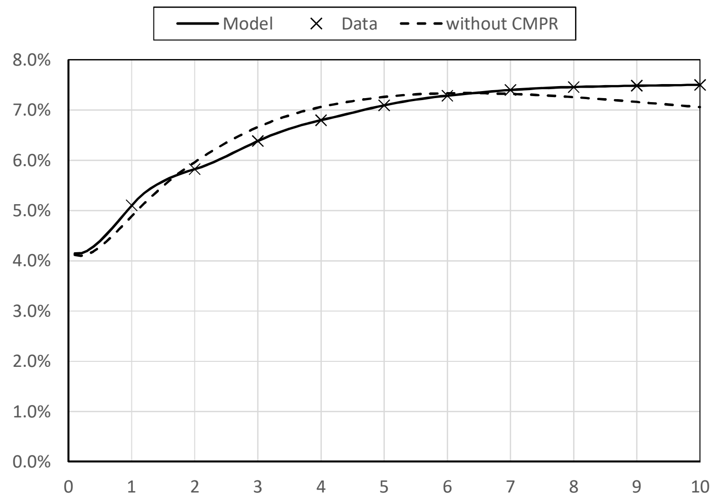

Fig. 2. Term structures of CCC/C zero rates. This figure compares the term structures of CCC/C zero rates. The horizontal axis represents maturity (years), and the vertical represents the zero rate. "Model" is the term structure where all calibrated parameters are used, "Data" are given data, and "without CMPR" is that where the calibrated $\boldsymbol{L}_h$ and $\psi_h(t)$ are not used.

2. Generate a sample path $(\boldsymbol{X}(t_i), r(t_i), h(t_i))$, $0 = t_0 < t_1 < \cdots < t_F = 10$ (years) under P. First, generate a path of $\boldsymbol{X}(\cdot)$. Then, given a path $\boldsymbol{X}(\cdot)$, generate a path of $(r(\cdot), h(\cdot))$.
3. Calculate the prices of the bonds $D(t_i, t_i + M, r(t_i), \widetilde{h}(t_i), \boldsymbol{X}(t_i))$ for various maturities M at time t_i.
4. Calculate the zero rates from the bond prices.

In Step 2, each sample path is generated from $t = 0$ to $t = T_F$ under P, while in Step 3 the no-arbitrage price is calculated based on the stochastic processes under $\widetilde{P}$. The input values $(r(t_i), h(t_i), \boldsymbol{X}(t_i))$ in Step 3 are given as the values at $t = t_i$ on each path generated in Step 2. Notice that $r(t_i)$ and $\boldsymbol{X}(t_i)$ are the same values under P and $\widetilde{P}$, while $\widetilde{h}(t_i) = h(t_i) + \langle \boldsymbol{\ell}^{cr(i)}(t_i), \boldsymbol{X}(t_i)\rangle$. We simulate 100,000 runs.

4.2. *Distribution of discount bond price in future*

First, we show the future price distributions of CCC/C discount bond with maturity $M = 10$ years. Figure 4 is its histogram at 1-year-after ($t = 1$); the horizontal axis represents price $D^{CCC}(t = 1, M, r(1), \widetilde{h}(1), \boldsymbol{X}(1))$ (bond's face amount is one), and the vertical represents frequency. Here, $(r(1), \widetilde{h}(1), \boldsymbol{X}(1))$ are simulated values. The peak around 0.34 is the distribution of the recovered price when the default occurs before 1-year-after, while the distribution above 0.4 is one when the bond survives at 1-year-after. Two peaks can be seen around 0.54 and 0.50, however, these do not correspond with the regime just at 1-year-after directly.

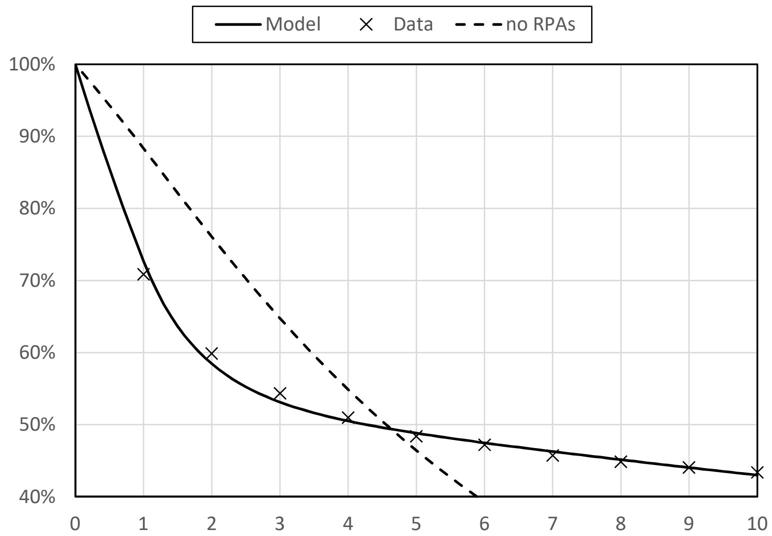

Fig. 3. Survival probabilities of CCC/C firms under P. This figure compares the survival probabilities of CCC/C firms in the physical measure. The horizontal axis represents maturity (years), and the vertical represents the survival probability. “Model” is the survival probability where all calibrated parameters are used, “Data” are the given data, and “no RPAs” is where the calibrated $\boldsymbol{\ell}(t)$ are not used.

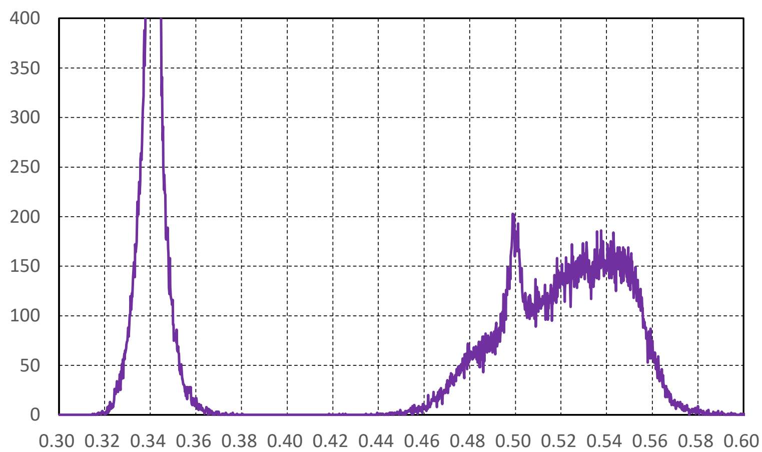

Fig. 4. Distribution of CCC/C discount bond price with 10-year maturity. The horizontal axis represents price, and the vertical axis represents frequency.

Figure 5 is the detailed version of Figure 4; each distribution corresponds to each regime at 1-year-after, “Regime 0”, “Regime 1” and “Regime 2”, respectively,

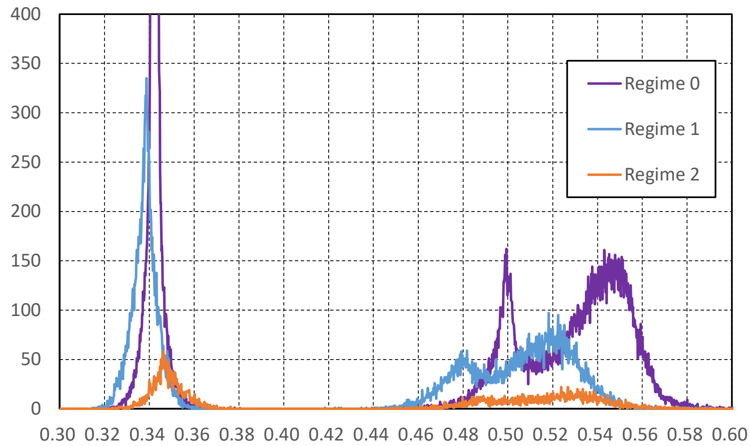

Fig. 5. Distribution of CCC/C discount bond price with 10-year maturity. The horizontal axis represents price, and the vertical axis represents frequency. Each distribution corresponds to the regime at 1-year-after.

and when summing up the three distributions, we obtain Figure 4. All distributions for surviving bond in Figure 5 have two definite peaks, and the left narrow peak is mainly from parameters in Regime 2, while the right wide peak is mainly from parameters in Regime 0 and Regime 1. Since the parameters in Regime 1 are not very different from those in Regime 0, they make a distribution with only one peak. The distributions in AAA and BBB in surviving bond have only one peak because the differences among parameters in different regimes are not as evident as in CCC/C.

4.3. *Term structures of zero rates*

Figure 6 depicts the intial theoretical (calculated by our model) term structures of zero rates with different credit ratings. The horizontal axis represents maturity (years), and the vertical represents the zero rate (%). In Figure 6, there are four curves: from below, default-free, AAA, BBB, and CCC/C zero curves, respectively. Notice that the theoretical curve does not match the given curve at each credit rating perfectly here because we construct the objective function for calibration using discrete grid points (0, 1, 2, ...) and are limited to 10 years. There is a tendency for the curve in the short term to sometimes bend unnaturally, but it does not look so remarkable. Hereafter, we show some examples of various shapes of theoretical future zero curves obtained by the Monte Carlo simulations. We assume the initial regime is "Regime 0" [o].

The default-free zero rates at 0-, 2-, 4-, 6-, 8-years-after on a certain sample path are shown in Figure 7. The zero curve at 2-years-after begins from 2 years to

[o]This assumption is consistent with the calibration described in Section 3.2

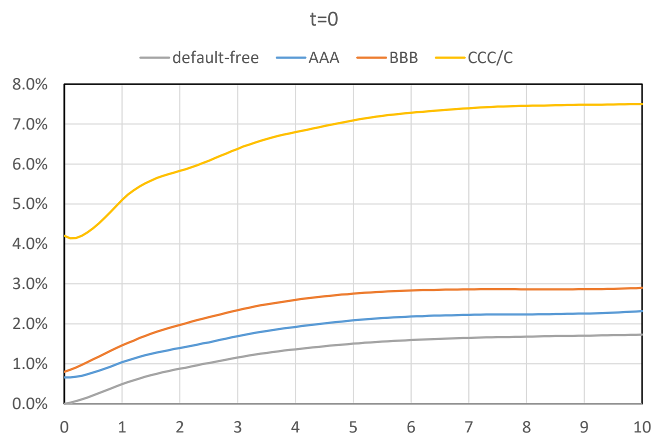

Fig. 6. Initial term structures of zero rates (initial regime 0). The horizontal axis represents maturity, and the vertical represents the zero rate.

10 years on the horizontal axis; on this curve, the rate on the horizontal 2 years is the instantaneous spot rate (0-year zero rate) at 2-years-after, and the rate on the horizontal 10 years is the 8-year zero rate at 2-years-after, and so on. The regime has much influence on the shape of the zero curve, but the shape does not correspond directly to the regime at the same time because the shape is the result of the cumulative effect from $t = 0$.

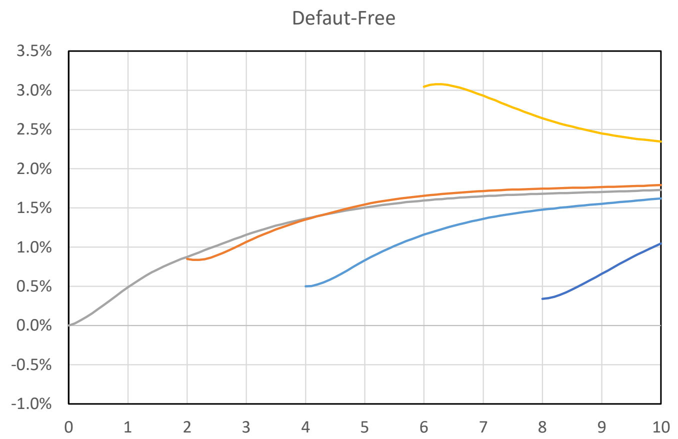

Fig. 7. Future term structures of default-free zero rates on a certain sample path. The horizontal axis represents maturity, and the vertical represents the zero rate. The curve that begins at 2 years means the term structures after 2 years, and so on.

Figure 8, 9 and 10 depict the future zero curves with AAA, BBB and CCC/C,

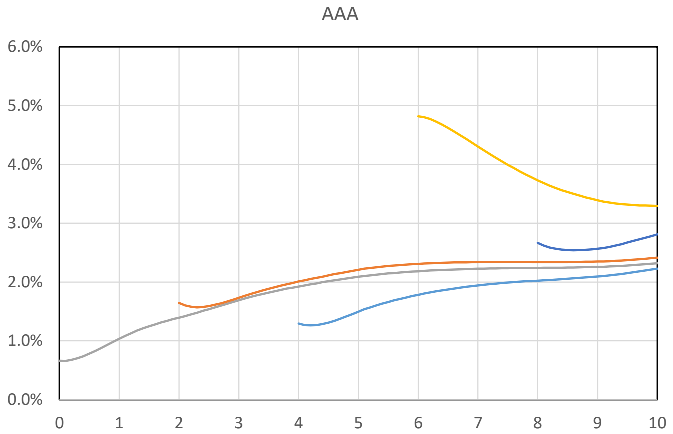

Fig. 8. Future term structures of AAA zero rates on a certain sample path. The horizontal axis represents maturity, and the vertical represents zero rate. The curve that begins at 2 years means the term structures after 2 years, and so on.

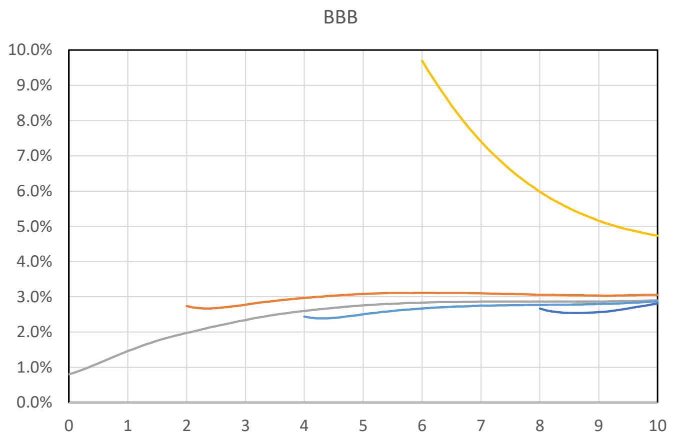

Fig. 9. Future term structures of BBB zero rates on a certain sample path. The horizontal axis represents maturity, and the vertical represents the zero rate. The curve that begins at 2 years means the term structures after 2 years, and so on.

respectively. These figures are described in the same way as Figure 7. These figures have certain common features.

(1) The shapes of these curves are similar at the same future time, especially betweeen the higher credit qualities, default-free and AAA. This is because, as indicated by (2.9), (2.15) and (2.22), the zero curves at time t are monotonically increasing functions of $r = r(t)$, which is common for all credit ratings. The intensity $\widetilde{h} = \widetilde{h}(t)$ is different from each other, but its volatility is low in

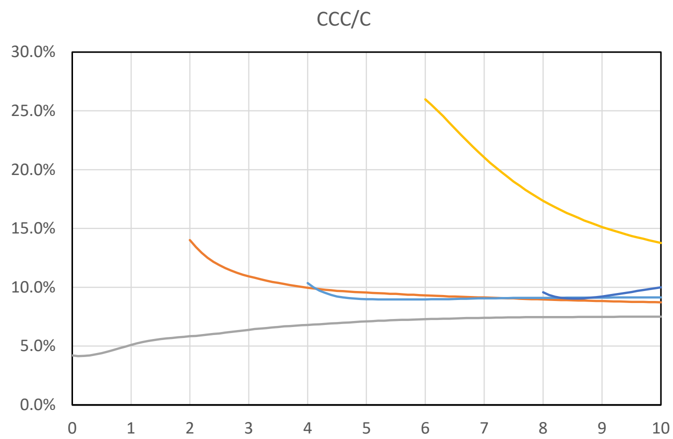

Fig. 10. Future term structures of CCC/C zero rates on a certain sample path. The horizontal axis represents maturity, and the vertical represents the zero rate. The curve that begins at 2 years means the term structures after 2 years, and so on.

higher credit ratings, so that the fluctuation by $\widetilde{h}(t)$ is small. In CCC/C, σ_h is so high that the shape of the zero curve becomes different from those in higher credit ratings.

(2) The zero rates in the short term fluctuate widely with time, while the zero rates in the long term do not fluctuate so much. This is because the instantaneous zero rate, which is the simulated $r(t) + (1 - \delta)\widetilde{h}(t)$, moves stochastically with time and it has much influence on shorter zero rates, while the long-term zero rate would approach gradually the limiting distribution so that the fluctuation would decrease.

(3) The zero curve moves more widely as the credit rating decreases, which corresponds to the increase of σ_h.

(4) In each moment, the zero rates are usually arranged from lower to upper against the credit ratings, which can be seen in Figure 11 (zero curves with different credit ratings at 2-year-after [p]).

(5) Compared with other credit ratings, future CCC/C zero curves become higher than the initial curve because $\widetilde{h}(t) = h(t) + \langle \ell(t), X(t) \rangle$ in future becomes much higher than $\widetilde{h}(0)$ due to high $m_{h,k}^{CCC/C}(t)$ and $\ell_k^{CCC/C}(t)$.

In general terms, the curves with different credit ratings have similar shapes at the same time, and the main difference is the level and the fluctuation with time. Moreover, we can see various shapes of zero curves: for example, normal-yield (default-free at 2-, 4- and 8-years-after), inverted-yield (CCC/C at 2- and 6-years-after), weakly humped shape (BBB at 2-years-after), and inverted-humped shape (AAA,

[p] We do not impose any restrictions for the order of the level at all. This is the result of our calibration.

BBB and CCC/C at 8-years-after).

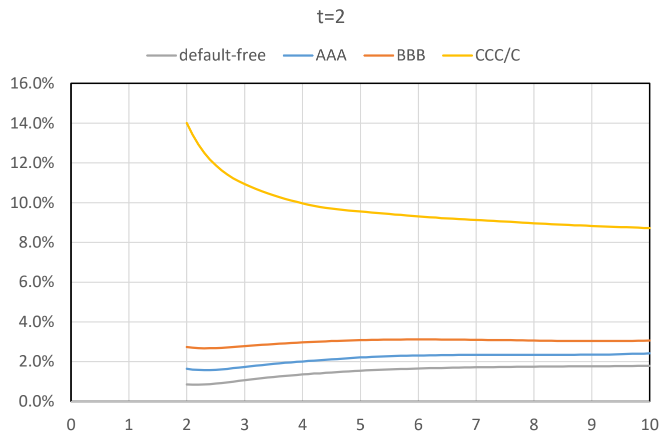

Fig. 11. Future term structures of zero rates with different credit ratings on a certain sample path. The horizontal axis represents maturity, and the vertical represents the zero rate.

We can show various other shapes. Among them, notably, the normal-yields and the default-free rate are negative but other rates are positive in the short term, and so on.

4.4. *Future distribution of zero rates*

Figure 12 shows the distributions of the default-free zero rates with different maturities at 1-year-after. The "1-yr zeros" is the distribution of 1-year zero rates ($M = 1$) at 1-year-after ($t = 1$), and "3-yr zeros" is that of 3-year zero rates ($M = 3$) at the same time, and so on. The distribution moves upward with maturity, and its width becomes narrow. This feature corresponds to the second feature (denoted by "(2)") of the zero curves described in Section 4.3, that is, the zero rates in the short term fluctuate widely with time, while the zero rates in the long term do not fluctuate to that degree. The above feature can be also seen in other credit ratings, Figure 13 and Figure 14.

In general terms, the definite upward movement of the distribution such as Figure 12 and 13 implies that the normal-yield curve would be usually seen, but the inverted yield curve would appear rarely. The inverted-yield curve corresponds to the curve whose shorter zero rate is higher than the longer zero rate. In Figure 12 and 13, since the distribution of 1-yr zeros is wide but not significantly higher than the distributions of longer zeros, the inverted-yield curve is less predominant. However, in Figure 14, the distribution of 1-yr zeros is so wide that there exists some non-negligible probability mass where 1-yr zeros are higher than the longer zeros probability mass, therefore, the inverted-yield curve often appears in CCC/C curves.

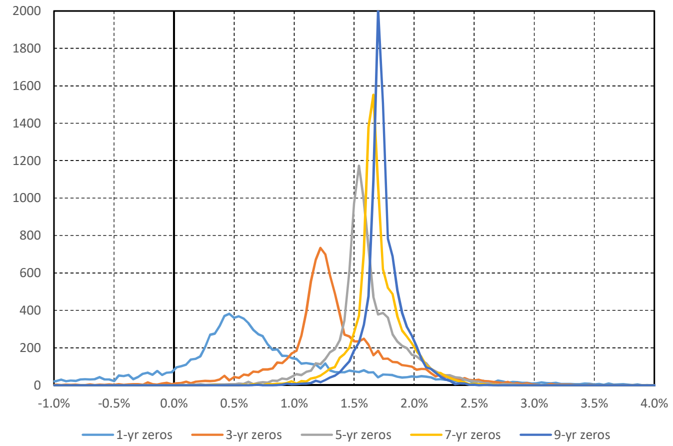

Fig. 12. Distributions of the default-free zero rates with different maturities at 1-year after.

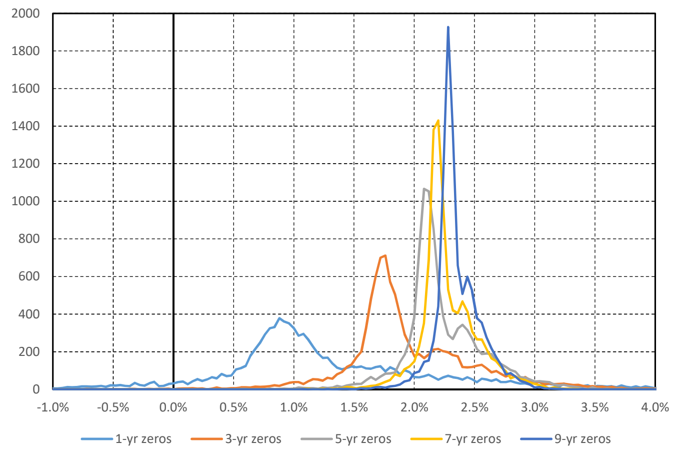

Fig. 13. Distributions of the AAA zero rates with different maturities at 1-year after.

5. Concluding remarks

In this study, we propose a term structure model for risk evaluation, sensitive to the interest rate and the credit risks with Markovian regime-switching property. First, we discuss the dynamics of a regime, interest rate, and default intensity in the physical probability measure and the change of measure to an equivalent martingale measure (pricing measure), and derive the dynamics in the pricing measure and a pricing formula. Second, we propose and demonstrate a simple model to calculate the discount bond price, which is a minor extension of model proposed in Elliott & Siu (2009). Third, we propose a calibration method for the simple model. Fourth, based on the simple model, we present some calibration results and some numerical

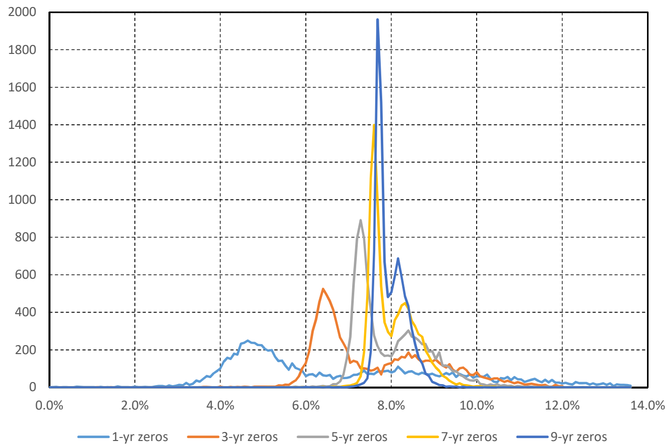

Fig. 14. Distributions of the CCC/C zero rates with different maturities at 1-year after.

examples of term structures of zero rates with different credit ratings at some future times. As the default-free interest rate and the default intensity are dependent on $\boldsymbol{X}(t)$, the term structure of the zero rates depends on the regime dynamics. As a result, we can obtain various shapes of yield curves with different credit ratings in future.

However, considering the existing limitations, additional research is required. First, more efficient parameter estimation methods should be urgently developed. In this study, although we propose a simple calibration method, it is apparently suboptimal because the parameter estimation is performed sequentially. Second, as we do not have enough knowledge about the regime-switching properties of financial data, more studies on analyzing various kinds of data are necessary. For example, because of the dependence among the financial variables might be dissimilar in various regimes, the estimation of appropriate copula functions in each regime would be another plausible research theme. Additionally, some minor changes to the model are required; we use the Vasicek model as an intensity process, but it is desirable to replace it with, for example, the CIR model and so on, in order to avoid the negative intensity.

Finally, there is considerable interest in data analyses of liabilities combined with information about regimes. Some kinds of regime-switching properties could be seen not only in the interest rate risk and credit risk but also in the equity risk, the currency risk, and so on. Additionally, the dependence of each asset and liability class on financial variables might change in each regime and class. For example, we consider the growth rates of the liquid deposits' volumes to be dependent on the interest rates, however, it is widely known that the dependence disappears under a very low-interest rate environment. We think that, for financial institutions such as banks, the regime-switching models would be promising and useful tools to grasp the risk precisely in a wide range, especially in the field of ALM. By analyzing the properties of assets and liabilities with the information about regimes obtained through a reasonable regime-switching model, and evaluating the risks of assets and liabilities synthetically through the model, new features and prospects of the risks might be clarified, and the results could be reflected in the practical risk management. To the best of our knowledge, there are not any studies on assets and liabilities from such a standpoint. We hope that such studies will be encouraged and this study can be used as a part of the theoretical foundation.

Appendix A. Derivation of (2.23)

Consider the model in Section 2.3. Under Assumption 2.2, we have

$$\mathrm{d}\ell(t) = \mathrm{d}\langle \boldsymbol{\ell}(t), \boldsymbol{X}(t)\rangle = \left(\left\langle \frac{\mathrm{d}\boldsymbol{\ell}(t)}{\mathrm{d}t}, \boldsymbol{X}(t)\right\rangle + \langle \boldsymbol{\ell}(t), Q^\top \boldsymbol{X}(t)\rangle\right)\mathrm{d}t + \langle \boldsymbol{\ell}(t), \mathrm{d}\widetilde{\boldsymbol{M}}(t)\rangle \tag{A.1}$$

under $\widetilde{P}$. The term $\langle \boldsymbol{\ell}(t), Q\boldsymbol{X}(t)\rangle$ is the contribution of the regime-switching. It follows from Assumption 2.2, (2.19)–(2.21) and (A.1) that

$$\begin{aligned} \mathrm{d}\widetilde{h}(t) &= \mathrm{d}h(t) + \mathrm{d}\ell(t) \\ &= a_h\left(\langle \boldsymbol{m}'_h(t), \boldsymbol{X}(t)\rangle - \widetilde{h}(t)\right)\mathrm{d}t + \langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle \mathrm{d}\widetilde{z}_h(t) + \langle \boldsymbol{\ell}, \mathrm{d}\widetilde{\boldsymbol{M}}(t)\rangle, \end{aligned}$$

where $\boldsymbol{A}\otimes\boldsymbol{B}$ is the Hadamard product of K-dimensional vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, and

$$\begin{aligned} \boldsymbol{m}'_h(t) = \boldsymbol{m}_h + \boldsymbol{\ell}(t) + \frac{1}{a_h}\Bigg(&\frac{\mathrm{d}\boldsymbol{\ell}(t)}{\mathrm{d}t} + \boldsymbol{\sigma}_h \otimes \boldsymbol{L}_h + \psi_h(t)\boldsymbol{\sigma}_h \\ &+ (\langle \boldsymbol{\ell}(t), Q^\top \boldsymbol{e}_1\rangle, \cdots, \langle \boldsymbol{\ell}(t), Q^\top \boldsymbol{e}_K\rangle)^\top\Bigg). \end{aligned} \tag{A.2}$$

Define $r_{t,u}(r) = r(u|r(t) = r)$, $\widetilde{h}_{t,u}(\widetilde{h}) = \widetilde{h}(u|\widetilde{h}(t) = \widetilde{h})$, $0 \le t \le u \le T$, and

$$\begin{aligned} p(t, T, r, \widetilde{h}, \boldsymbol{X}) &= \widetilde{E}_t\left[\exp\left\{-\int_t^T r_{t,u}(r)\mathrm{d}u\right\} 1_{\{\tau > T\}}\right] \\ &= \widetilde{E}_t\left[\exp\left\{-\int_t^T (r_{t,u}(r) + \widetilde{h}_{t,u}(\widetilde{h}))\mathrm{d}u\right\}\right]. \end{aligned} \tag{A.3}$$

It follows from (2.14) that

$$r_{t,s}(r) = r + \int_t^s (\langle \boldsymbol{\phi}(u), \boldsymbol{X}(u)\rangle - a r_{t,u}(r))\,\mathrm{d}u + \int_t^s \langle \boldsymbol{\sigma}, \boldsymbol{X}(u)\rangle \mathrm{d}\widetilde{z}_r(u),$$

so that $D_{t,s} = \partial r_{t,s}(r)/\partial r$ satisfies

$$D_{t,s} = 1 - a\int_t^s D_{t,u}\mathrm{d}u. \tag{A.4}$$

Differentiating (A.6) with respect to s, we have

$$\frac{\mathrm{d}D_{t,s}}{\mathrm{d}s} = -aD_{t,s}, \qquad D_{t,t} = 1.$$

Therefore, we obtain

$$D_{t,s} = \mathrm{e}^{-a(s-t)}. \tag{A.5}$$

From (A.3) and (A.5), we have

$$\frac{\partial}{\partial r} p(t,T,r,\widetilde{h},\boldsymbol{X}) = -\left(\int_t^T D_{t,u}\mathrm{d}u\right) p(t,T,r,\widetilde{h},\boldsymbol{X}) = -B(t,T)p(t,T,r,\widetilde{h},\boldsymbol{X}),$$

and similarly, we have

$$\frac{\partial}{\partial \widetilde{h}} p(t,T,r,\widetilde{h},\boldsymbol{X}) = -B_h(t,T)p(t,T,r,\widetilde{h},\boldsymbol{X}).$$

Define

$$\begin{aligned}\widetilde{p}(t,T,r,\widetilde{h},\boldsymbol{X}) &= \exp\left\{-\int_0^t (r_{t,u}(r) + \widetilde{h}_{t,u}(\widetilde{h}))\mathrm{d}u\right\} p(t,T,r,\widetilde{h},\boldsymbol{X}) \\ &= \widetilde{E}_t\left[\exp\left\{-\int_0^T (r_{t,u}(r) + \widetilde{h}_{t,u}(\widetilde{h}))\mathrm{d}u\right\}\right],\end{aligned}$$

which is a Doob's martingale under $\widetilde{P}$. Applying Ito's lemma to $\widetilde{p}(t,T,r,\widetilde{h},\boldsymbol{X})$ and putting $\widetilde{\boldsymbol{p}} = (\widetilde{p}_1,\cdots,\widetilde{p}_K)^\top$ with $\widetilde{p}_i = \widetilde{p}(t,T,r,\widetilde{h},\boldsymbol{e}_i)$, $i=1,\cdots,K$, we have

$$\begin{aligned}&\widetilde{p}(t,T,r,\widetilde{h},\boldsymbol{X}) = \widetilde{p}(0,T,r(0),\widetilde{h}(0),\boldsymbol{X}(0)) \\ &\quad + \int_0^t \left[\frac{\partial \widetilde{p}}{\partial u} + \frac{\partial \widetilde{p}}{\partial r}(\langle \boldsymbol{\phi}(u),\boldsymbol{X}(u)\rangle - ar(u)) + \frac{\partial \widetilde{p}}{\partial \widetilde{h}}\left(\langle a_h \boldsymbol{m}'_h(u),\boldsymbol{X}(u)\rangle - a_h\widetilde{h}(u)\right)\right. \\ &\qquad \left. + \frac{1}{2}\langle \boldsymbol{\sigma},\boldsymbol{X}(u)\rangle^2 \frac{\partial^2 \widetilde{p}}{\partial r^2} + \frac{1}{2}\langle \boldsymbol{\sigma}_h,\boldsymbol{X}(u)\rangle^2 \frac{\partial^2 \widetilde{p}}{\partial \widetilde{h}^2} + \rho(t)\langle \boldsymbol{\sigma}\otimes\boldsymbol{\sigma}_h,\boldsymbol{X}(u)\rangle \frac{\partial^2 \widetilde{p}}{\partial r \partial \widetilde{h}}\right]\mathrm{d}u \\ &\quad + \int_0^t \frac{\partial \widetilde{p}}{\partial r}\langle \boldsymbol{\sigma},\boldsymbol{X}(u)\rangle \mathrm{d}\widetilde{z}_r(u) + \int_0^t \frac{\partial \widetilde{p}}{\partial \widetilde{h}}\langle \boldsymbol{\sigma}_h,\boldsymbol{X}(u)\rangle \mathrm{d}\widetilde{z}_h(u) + \int_0^t \langle \widetilde{\boldsymbol{p}},\widetilde{Q}^\top \boldsymbol{X}(u)\rangle \mathrm{d}u \\ &\quad + \int_0^t \left\langle \widetilde{\boldsymbol{p}} + \frac{\partial \widetilde{p}}{\partial \widetilde{h}}\otimes \boldsymbol{\ell}(u), \mathrm{d}\widetilde{\boldsymbol{M}}(u)\right\rangle.\end{aligned}$$

Because of the martingale property of $\widetilde{p}$, the integrand of du term must be zero, so that we have the following partial differential equation (abbreviated as PDE)

$$\begin{aligned}0 &= \frac{\partial \widetilde{p}}{\partial t} + \frac{\partial \widetilde{p}}{\partial r}(\langle \boldsymbol{\phi}(t),\boldsymbol{X}(t)\rangle - ar(t)) + \frac{\partial \widetilde{p}}{\partial \widetilde{h}}\left(\langle a_h \boldsymbol{m}'_h(t),\boldsymbol{X}(t)\rangle - a_h\widetilde{h}(t)\right) \\ &\quad + \frac{1}{2}\langle \boldsymbol{\sigma},\boldsymbol{X}(t)\rangle^2 \frac{\partial^2 \widetilde{p}}{\partial r^2} + \frac{1}{2}\langle \boldsymbol{\sigma}_h,\boldsymbol{X}(t)\rangle^2 \frac{\partial^2 \widetilde{p}}{\partial \widetilde{h}^2} + \rho(t)\langle \boldsymbol{\sigma}\otimes\boldsymbol{\sigma}_h,\boldsymbol{X}(t)\rangle \frac{\partial^2 \widetilde{p}}{\partial r \partial \widetilde{h}} + \langle \widetilde{\boldsymbol{p}},\widetilde{Q}^\top \boldsymbol{X}(t)\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}0 &= \frac{\partial p}{\partial t} - rp - \widetilde{h}p + \frac{\partial p}{\partial r}(\langle \boldsymbol{\phi}(t),\boldsymbol{X}(t)\rangle - ar(t)) + \frac{\partial p}{\partial \widetilde{h}}\left(\langle a_h \boldsymbol{m}'_h(t),\boldsymbol{X}(t)\rangle - a_h\widetilde{h}(t)\right) \\ &\quad + \frac{1}{2}\langle \boldsymbol{\sigma},\boldsymbol{X}(t)\rangle^2 \frac{\partial^2 p}{\partial r^2} + \frac{1}{2}\langle \boldsymbol{\sigma}_h,\boldsymbol{X}(t)\rangle^2 \frac{\partial^2 p}{\partial \widetilde{h}^2} + \rho(t)\langle \boldsymbol{\sigma}\otimes\boldsymbol{\sigma}_h,\boldsymbol{X}(t)\rangle \frac{\partial^2 p}{\partial r \partial \widetilde{h}} + \langle \boldsymbol{p},\widetilde{Q}^\top \boldsymbol{X}(t)\rangle,\end{aligned} \tag{A.6}$$

where $\boldsymbol{p} = (p_1, \cdots, p_K)^\top$ with $p_i = p(t, T, r, \widetilde{h}, \boldsymbol{e}_i)$, $i = 1, \cdots, K$, with the terminal condition $p(T, T, \cdot, \cdot, \cdot) = 1$. (A.6) means a system of K coupled PDEs

$$0 = \frac{\partial p_i}{\partial t} - rp_i - \widetilde{h}p_i + \frac{\partial p_i}{\partial r}(\phi_i(t) - ar(t)) + \frac{\partial p_i}{\partial \widetilde{h}}\left(a_h m'_{h,i}(t) - a_h\widetilde{h}(t)\right)$$
$$+ \frac{1}{2}\sigma_i^2\frac{\partial^2 p_i}{\partial r^2} + \frac{1}{2}\sigma_{h,i}^2\frac{\partial^2 p_i}{\partial \widetilde{h}^2} + \rho(t)\sigma_i\sigma_{h,i}\frac{\partial^2 p_i}{\partial r\partial\widetilde{h}} + \langle \boldsymbol{p}, \widetilde{Q}^\top \boldsymbol{e}_i(t)\rangle, \qquad 0 \le t \le T,$$

with $p_i(T, T, \cdot, \cdot) = 1$, $i = 1, \cdots, K$. Here, assume that

$$p_i(t, T, r, \widetilde{h}) = \exp\left\{A_i(t, T) - B(t, T)r - B_h(t, T)\widetilde{h}\right\}, \quad i = 1, \cdots, K,$$

where $A_i(t, T) = A(t, T, \boldsymbol{e}_i)$, then, the following ODEs

$$0 = \mathrm{e}^{A_i}\left[\frac{\mathrm{d}A_i}{\mathrm{d}t} - \phi_i(t)B - a_h m'_{h,i}(t)B_h + \frac{\sigma_i^2}{2}B^2 + \frac{\sigma_{h,i}^2}{2}B_h^2 + \rho(t)\sigma_i\sigma_{h,i}BB_h\right]$$
$$+ \langle \bar{\boldsymbol{A}}, Q^\top \boldsymbol{e}_i\rangle, \tag{A.7}$$
$$\frac{\mathrm{d}B}{\mathrm{d}t} = aB - 1, \tag{A.8}$$
$$\frac{\mathrm{d}B_h}{\mathrm{d}t} = a_h B_h - 1 \tag{A.9}$$

must be satisfied where $\bar{\boldsymbol{A}} = (\bar{A}_1(t, T), \cdots, \bar{A}_K(t, T))^\top$ with $\bar{A}_i(t, T) = \mathrm{e}^{A_i(t,T)}$, $i = 1, \cdots, K$. From (A.8) and (A.9), we have

$$B(t, T) = \frac{1 - \mathrm{e}^{-a(T-t)}}{a} \quad \text{and} \quad B_h(t, T) = \frac{1 - \mathrm{e}^{-a_h(T-t)}}{a_h},$$

and (A.7) is rewritten as

$$0 = \frac{\mathrm{d}\bar{A}_i(t, T)}{\mathrm{d}t} + \left\{\frac{\sigma_i^2}{2}B^2(t, T) + \frac{\sigma_{h,i}^2}{2}B_h^2(t, T) - \phi_i(t)B(t, T) - a_h m'_{h,i}(t)B_h(t, T)\right.$$
$$\left. + \rho(t)\sigma_i\sigma_{h,i}B(t, T)B_h(t, T)\right\}\bar{A}_i(t, T) + \langle \bar{\boldsymbol{A}}, Q^\top \boldsymbol{e}_i\rangle \tag{A.10}$$

where

$$m'_{h,i}(t) = m_{h,i} + \ell_i(t) + \frac{1}{a_h}\left(\frac{\mathrm{d}\ell_i(t)}{\mathrm{d}t} + \sigma_{h,i}\lambda_{h,i} + \sigma_{h,i}\psi_h(t) + \langle \boldsymbol{\ell}, Q^\top \boldsymbol{e}_i\rangle\right)$$

with the terminal conditions $\bar{A}_i(T, T) = 1$, $i = 1, \cdots, K$.

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References

A. Ang & A. Timmermann (2012) Regime changes and financial markets, *Annual Review of Financial Economics* **4**, 313–337.

T. R. Bielecki & M. Rutkowski (2002) *Credit Risk: Modelling, Valuation and Hedging*, Springer-Ferlag.

O. M. Chun, G. Dionne & P. Francois (2014) Credit spread changes within switching regimes, *Journal of Banking & Finance* **49**, 41–55.

Q. Dai, K. J. Singleton & W. Yang (2007) Regime shifts in a dynamic term structure model of US treasury bond yields, *Review of Financial Studies* **20**(5), 1669–1706.

R. J. Elliott, L. Chan & T. K. Siu (2005) Option pricing and Esscher transform under regime switching, *Annals of Finance* **1**, 423–432.

R. J. Elliott & K. Nishide (2014) Pricing of discount bonds with a Markov switching regime, *Annals of Finance* **10**(3), 509–522.

R. J. Elliott & T. K. Siu (2009) On Markov-modulated exponential-affine bond price formulae, *Applied Mathematical Finance* **16**(1), 1–15.

R. J. Elliott, T. K. Siu & H. Yang (2007) Martingale representation for contingent claims with regime switching, *Communications on Stochastic Analysis* **1**(2), 279–292.

H. Fischer & O. Stolper (2019) *The nonlinear dynamics of corporate bond spreads: Regime-dependent effects of their determinants.* Discussion Paper.

C. Gourieroux, A. Monfort, F. Pegoraro & J.-P. Renne (2014) Regime switching and bond pricing, *Journal of Financial Econometrics* **12**(2), 237–277.

D. Hainaut, & O. Le Courtois (2014) An intensity model for credit risk with switching Lévy processes, *Quantitative Finance* **14**(8), 1453–1465.

J. D. Hamilton (1989) A new approach to the economic analysis of nonstationary time series and the business cycle, *Econometrica* **57**(2), 357–384.

N. Hatakeyama (2022) *Time series analysis of interest rate risk and credit risk by using regime-switching model.* Master Thesis, Tokyo Metropolitan University *(in Japanese).*

T. Inoue & T. Okimoto (2008) Were there structural breaks in the effect of Japanese monetary policy? Re-evaluation the policy effects of the lost decade, *Journal of Japanese and International Economics* **22**, 320–342.

R.A. Jarrow & S.M. Turnbull (1995) Pricing derivatives on financial securities subject to credit risk, *Journal of Finance* **50**, 53–86.

M. Kijima & Y. Muromachi (2000) Evaluation of credit risk of a portfolio with stochastic interest rate and default processes, *Journal of Risk* **3** (1), 5–36.

N.W. Kraemer, E.M. Gunter, J. Palmer, N.M. Richhariya & L. Fernandes (2020) Defaults, transition, and recovery: 2019 annual U.S. corporate default and rating transition study: https://www.spglobal.com/ratings/en/research/articles/200617-default-transition-and-recovery-2019-annual-u-s-corporate-default-and-rating-transition-study-11498915

S. Kusuoka (1999) A remark on default risk models, *Advances in Mathematical Economics* **1**, 69–82.

J. Li & S. Ma (2013) Pricing options with credit risk in Markovian regime-switching markets, *Journal of Applied Mathematics* **2013**(2), 1–9.

A. Monfort & J. -P. Renne (2013) Default, liquidity, and crises: an econometric framework, *Journal of Financial Econometrics* **11**(2), 221–262.

Y. Muromachi (2022) *Pricing and risk evaluation of interest rate risk and credit risk under regime-switching environment.* Research Paper Series, 34, Graduate School of Management, Tokyo Metropolitan University.

S. Wu & Y. Zeng (2005) A general equilibrium model of the term structure of interest rates under regime-switching risk, *International Journal of Theoretical and Applied Finance* **8** (7), 839–869.