Research Paper Series

No. 34

Pricing and Risk Evaluation of Interest Rate Risk and Credit Risk under Regime-Switching Environment

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March, 2022

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March 16, 2022

Abstract

Long term historical data of interest rates and credit spreads imply that there exist some different regimes; a calm regime with lower default risk and volatility, and a stressed regime with higher default risk and volatility. In this article, we propose a pricing and risk evaluation model of interest rate risk and credit risk with Markovian regime-switching property. We discuss the dynamics of regime, interest rate and default intensity under the physical probability and the change of measure to an equivalent martingale one, and we propose a simple tractable model. In our model, the default-free interest rate and the default intensity are dependent through the regime, and the dependence affects the price of the defaultable bond. We calculate distributions of a bond portfolio's price at a risk horizon, which reflects the actual default loss up to the horizon and the decrease of market prices due to the transition to a stressed regime. Numerical examples show that the price distribution has a short right tail and a long left tail, and that the distribution depends strongly on the present regime. Such results would be applicable to the financial risk management, especially on the stress tests.

Keywords: financial risk management, regime-switching model, interest rate risk, credit risk, term structure of interest rates, change of measure

JEL Classification Numbers: G12, G13, G17.

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1 Introduction

In the 21st century some financial distresses have already happened, and one of them was the worldwide financial crisis. Many financial institutions, financial regulation agencies and central banks have discussed to control financial risks, and have been constructing concepts, tools and systems for risk management, financial regulations, and financial policies. For many financial institutions, especially for banks, management of the interest rate risk and the credit risk has been one of the most important challenges.

From the long term data of interest rates and credit spreads, some feels that there exist some different regimes; typically, a calm regime with lower default risk and volatility, and a stressed regime with higher credit risk and volatility. Since Hamilton [12] proposed a regime-switching model and analyzed the economic cycles, such models have been used by many researchers, mainly for analyzing and modeling economic and financial data[1]. Restricted to the researches on interest rates, Inoue and Okimoto [14] used Markovian regime-switching models for analyzing monetary policy and private sector behavior in Japan. Dai et al. [4] used a discrete-time regime-switching Gaussian term structure model, while Wu and Zeng [20] applied a continuous-time affine term structure model with regime-dependent parameters. Elliott and Siu [7] considered a bond valuation as a derivative written on a Markovian regime-switching instantaneous spot rate, and derived a Markov-modulated exponential-affine bond price formula. Their results have been applied to the term structures of interest rates, see Elliott and Nishide [6] and so on.

On credit risk analyses, Gourieroux et al. [10] derived a defaultable bond price in a discrete-time setting and analyzed sovereign yield curves, and Monfort and Renne [19] analyzed the credit spreads of U.S. bonds. Fischer and Stolper [9] analyzed the behavior of the credit spreads and their key determinants, and Chun et al. [3] proposed a regression model of credit spreads with endogenous regimes, and showed the model enhanced the explanatory power of the determinants. In a continuous-time setting, Hainaut and Le Courtois [11] proposed a default intensity model described by regime-switching Lévy process to evaluate the survival probability, and analyzed the CDS market. Li and Ma [18] discussed pricing options analytically, but their results are limited to the conditional price given a sample path of the regime.

Many research papers have already existed on the regime-switching models. However, we do not know papers in which such models are used for evaluating the financial risk

[1]As an example of review papers, see Ang and Timmermann [1].

quantitatively. If an extremely high-risk regime is included in the risk evaluation model, it might give us fresh views against the future risk. Moreover, if such a high-risk regime can be estimated from the observed data, another kind of forward-looking stress tests could be obtained statistically.

In this article, we propose a pricing and risk evaluation model of interest rate risk and credit risk with regime-switching property, under a continuous-time setting. For evaluating risk, we set a risk horizon which is a certain future time, and discuss the price at the horizon and the accumulated loss up to the horizon totally. For pricing, we use the no-arbitrage price not only at present but also at the risk horizon because the no-arbitrage price at the horizon can reflect the influence of the loss which could occur after the horizon, while the loss up to the horizon can be evaluated as a realized loss on a simulation path. In order to keep the consistency between pricing and risk evaluation, stochastic modelings are necessary not only under the physical measure but also under the pricing measure such as the risk-neutral measure. The physical measure part of the model is used for generating scenarios up to the risk horizon, while the pricing measure part is used for pricing at the risk horizon.

This article is organized as follows. In Section 2, we describe constructions of a pricing and risk evaluation model, and propose a simple model. Based on a Markovian regime-switching model, the default-free spot rate and the default intensity processes are described under the physical measure, and the change of measure to the equivalent one is discussed, so that the processes under the pricing measure are derived explicitly. In Section 3, we show some numerical examples of the term structure of default-free and defaultable zero rates, and explain the effect of the existence of regimes. In Section 4, the price distributions of a bond portfolio are shown by the Monte Carlo simulation, and some implications are discussed. Section 5 concludes this article.

2 The Model

In Section 2.1 and 2.2, constructions of a pricing and risk evaluation model with a regime-switching property is discussed, while in Section 2.3, a simple tractable model is proposed. If pricing securities is the only one object, discussion in Section 2.1 is not necessary, however, since we would like to proceed the risk evaluation which is consistent with pricing, we begin to discuss from Section 2.1. See Kijima and Muromachi [16] for detail, in which constructions of a risk evaluation model consistent with pricing was discussed. One of their main discussions is the reason why two probability measures, the physical measure and the pricing measure, must be needed. Throughout this article, we use a simple homogeneous model as a regime-switching part.

2.1 Stochastic processes under physical probability measure

Consider a switching regime under a continuous-time and finite-state Markov chain model. Let t be time, and $t = 0$ is present. We consider a financial market with a finite horizon T, $0 < T < \infty$, and define a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ under which all stochastic processes can be described. All probability spaces appeared in this article satisfy the usual conditions.

There exist K, $K \in \mathcal{N}$, different states which are called regimes. Let $\boldsymbol{X}(t) \in \{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_K\}$ be a K-dimensional vector and $\boldsymbol{e}_j$, $j = 1, \cdots, K$, are K-dimensional unit vectors where the i-th component of $\boldsymbol{e}_j$ is the Kronecker's delta δ_{ij}. $\boldsymbol{X}(t)$ implies an economic state at time t, and is a Markov chain on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$. Let $\mathcal{Q} \in \mathcal{R}^K \times \mathcal{R}^K$ denote the infinitesimal generator matrix of $\boldsymbol{X}(t)$, and we assume that $\mathcal{Q}$ is steady, so that we can express

$$\mathcal{Q} = (q_{ij})_{i,j=1,\cdots,K}, \quad q_{ij} = \begin{cases} \lim\limits_{\Delta t\to 0} \dfrac{P\{\boldsymbol{X}(t+\Delta t) = \boldsymbol{e}_j | \boldsymbol{X}(t) = \boldsymbol{e}_i\}}{\Delta t}, & j \neq i \\ -\sum\limits_{k\neq i} q_{ik}, & j = i \end{cases} \tag{2.1}$$

and

$$\boldsymbol{X}(t) = \boldsymbol{X}(0) + \int_0^t \mathcal{Q}\boldsymbol{X}(s)\mathrm{d}s + \boldsymbol{M}(t), \qquad t \ge 0 \tag{2.2}$$

where the K-dimensional vector $\boldsymbol{M}(t)$ is a P–martingale. The transition probability matrix defined by

$$\mathcal{P}(t) = (p_{ij}(t))_{i,j=1,\cdots,K}, \quad p_{ij}(t) = P\left\{\boldsymbol{X}(s+t) = \boldsymbol{e}_j | \boldsymbol{X}(s) = \boldsymbol{e}_i\right\}, \; s, t \ge 0, \tag{2.3}$$

is the solution of the following differential equation

$$\frac{\mathrm{d}\mathcal{P}(t)}{\mathrm{d}t} = \mathcal{Q}\mathcal{P}(t), \qquad \mathcal{P}(0) = \mathcal{I}_K, \tag{2.4}$$

where $\mathcal{I}_K$ is the K-dimensional indentical matrix, and is given by

$$\mathcal{P}(t) = \exp\{t\mathcal{Q}\} = \sum_{k=0}^{\infty} \frac{t^k \mathcal{Q}^k}{k!}. \tag{2.5}$$

The risk-free instantaneous spot rate at time t, denoted by $r(t)$, follows the stochastic differential equation (hereafter, abbreviated by SDE)

$$\mathrm{d}r(t) = \mu_r(t, r(t), \boldsymbol{X}(t-))\mathrm{d}t + \sigma_r(t, r(t), \boldsymbol{X}(t-))\mathrm{d}z_r(t), \qquad t \geq 0, \tag{2.6}$$

where $z_r(t)$ is the standard Brownian motion under P. Consider a risky security subject to its credit risk. Let $\tau > 0$ be the default time of the issuer of the security or the counterparty of a contract of the security. Define $H(t) = 1_{\{\tau \leq t\}}$ as its default process where 1_A is the indicator function[2], and the default intensity at time t, denoted by $h(t)$, follows the SDE

$$\mathrm{d}h(t) = \mu_h(t, h(t), \boldsymbol{X}(t-))\mathrm{d}t + \sigma_h(t, h(t), \boldsymbol{X}(t-))\mathrm{d}z_h(t), \qquad t \geq 0, \tag{2.7}$$

where $z_h(t)$ is a standard Brownian motion under P and $\mathrm{d}z_r(t)\mathrm{d}z_h(t) = \rho(t)\mathrm{d}t$.

The filtrations in this article are as follows; for any arbitrary time t, $0 \leq t \leq T$, the filtration generated by the default process is defined by $\mathcal{H}_t = \sigma(H(s) : 0 \leq s \leq t)$ and $\mathcal{H} = (\mathcal{H}_t)_{0 \leq t \leq T}$, while the filtration generated by $X(t)$ is defined by $\mathcal{F}_t^X = \sigma(\boldsymbol{X}(s) : 0 \leq s \leq t)$ and $\mathcal{F}^X = (\mathcal{F}_t^X)_{0 \leq t \leq T}$. Similarly, the filtrations generated by $r(t)$ and $h(t)$ are $\mathcal{F}_t^r = \sigma(r(s) : 0 \leq s \leq t)$, $\mathcal{F}^r = (\mathcal{F}_t^r)_{0 \leq t \leq T}$ and $\mathcal{F}_t^h = \sigma(h(s) : 0 \leq s \leq t)$, $\mathcal{F}^h = (\mathcal{F}_t^h)_{0 \leq t \leq T}$, respectively. The complete filtration of this system is defined by $\mathcal{F} = \mathcal{F}^X \vee \mathcal{F}^r \vee \mathcal{F}^h \vee \mathcal{H}$, that is, $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^r \vee \mathcal{F}_t^h \vee \mathcal{H}_t$ is satisfied for any arbitrary t, $0 \leq t \leq T$. Additionally, we use a useful filtration $\mathcal{G}_t = \mathcal{F}_T^X \vee \mathcal{F}_t^r \vee \mathcal{F}_t^h \vee \mathcal{H}_t$, $0 \leq t \leq T$.

2.2 Stochastic processes under risk-neutral probability measure

Consider a probability measure P^X equivalent to P. Assume that the infinitesimal generator matrix of $\boldsymbol{X}(t)$ under P^X, denoted by $\mathcal{Q}^X = (q_{ij}^X)_{i,j=1,\cdots,K}$, is given by

$$q_{ij}^X = (1 + \kappa_{ij})q_{ij}, \qquad i \neq j, \quad i, j = 1, \cdots, K \tag{2.8}$$

where $\kappa_{ij} > -1$ are constant. Then, the change of probability measure from P to P^X is written by

$$\left.\frac{\mathrm{d}P^X}{\mathrm{d}P}\right|_{\mathcal{F}_T^X} = \eta^X(T), \tag{2.9}$$

[2] $1_A = 1$ when the event A is true, otherwise $1_A = 0$.

where the Radon-Nykodym derivative $\eta^X(T)$ is defined by

$$\eta^X(t) = \exp\left\{-\int_0^t \sum_{k,\ell=1}^K \kappa_{k\ell} q_{k\ell} H^k(u)\mathrm{d}u\right\} \prod_{0<u\le t}\left(1+\sum_{k,\ell=1}^K \kappa_{k\ell}\Delta H^{k\ell}(u)\right) \tag{2.10}$$

$$H^i(t) = 1_{\{\boldsymbol{X}(t)=\boldsymbol{e}_i\}}, \qquad i=1,\cdots,K, \tag{2.11}$$

$$H^{ij}(t) = \sum_{0<u\le t} H^i(u-)H^j(u), \qquad i\ne j, \quad i,j=1,\cdots,K, \tag{2.12}$$

and $\Delta H^{ij}(u) = H^{ij}(u) - H^{ij}(u-)$. Let $\mathcal{P}^X(t)$ be the transition probability matrix under P^X. Then, the similar equations such as (2.2), (2.4) and (2.5) are obtained only if $(\mathcal{Q}, \mathcal{P}(t), \boldsymbol{M}(t))$ are replaced with $(\mathcal{Q}^X, \mathcal{P}^X(t), \boldsymbol{M}^X(t))$ where $\boldsymbol{M}^X(t)$ is a P^X-martingale.

We follow a standard discussion in the credit risk model[3]. Let $\beta(t)$ be a $\mathcal{F}_t$-adapted process and $\kappa_h(t) > -1$ be a $\mathcal{F}_t$-predictable process, and define

$$\rho_h(t) = \int_0^t (\beta_h(s)\mathrm{d}z_h(s) + \kappa_h(s)\mathrm{d}M_h(s)) \tag{2.13}$$

and a new probability measure P^c equivalent to P^X by

$$\left.\frac{\mathrm{d}P^c}{\mathrm{d}P^X}\right|_{\mathcal{G}_T} = \rho_h(T). \tag{2.14}$$

Then, the process

$$z_h^c(t) = z_h(t) - \int_0^t \beta_h(s)\mathrm{d}s \tag{2.15}$$

becomes a standard Brownian motion under P^c, and the process

$$h^c(t) = (1+\kappa_h(t))h(t) \tag{2.16}$$

is regarded as a default intensity under P^c, because the process

$$M_h^c(t) = H(t) - \int_0^{t\wedge\tau} h^c(s)\mathrm{d}s \tag{2.17}$$

becomes a $(\mathcal{F}, P^c)$-martingale, which corresponds to the fact that the process

$$M_h(t) = H(t) - \int_0^{t\wedge\tau} h(s)\mathrm{d}s, \tag{2.18}$$

is a $(\mathcal{F}, P^X)$-martingale.

[3]See Kusuoka [17], Bielecki and Rutkovski [2], and so on.

The following discussion is according to Elliott et al.[8]. Assuming that the time t price of a risky asset, denoted by $S(t)$, follows

$$\mathrm{d}S(t) = \mu(t, S(t), \boldsymbol{X}(t))S(t)\mathrm{d}t + \sigma(t, S(t), \boldsymbol{X}(t))S(t)\mathrm{d}z_S^c(t), \qquad t \geq 0 \tag{2.19}$$

where $z_S^c(t)$ is a standard Brownian motion under P^c. Define the Radon-Nikodym derivative from P^c to its equivalent probability measure P^η by [4]

$$\left.\frac{\mathrm{d}P^\eta}{\mathrm{d}P^c}\right|_{\mathcal{G}_T} = \exp\left\{\int_0^T \eta(s)\mathrm{d}z_S^c(s) - \frac{1}{2}\int_0^T (\eta(s))^2\,\mathrm{d}s\right\} \tag{2.20}$$

where

$$\eta(t) = \frac{r(t) - \mu(t, S(t), \boldsymbol{X}(t))}{\sigma(t, S(t), \boldsymbol{X}(t))}, \tag{2.21}$$

then, thanks to Girsanov's theorem, the process

$$z^\eta(t) = z_S^c(t) - \int_0^t \eta(u)\mathrm{d}u \tag{2.22}$$

becomes a standard Brownian motion under P^η, and the relative price of $S(t)$ with respect to the bank account $B(t) = \exp\left\{\int_0^t r(u)\mathrm{d}u\right\}$ becomes a $(\mathcal{G}, P^\eta)$–local martingale. Assuming that the relative price is a $(\mathcal{G}, P^\eta)$–martingale, the price of a European derivarive given $\mathcal{G}_T$ is given by

$$V(t|\mathcal{G}_t) = E^{P^\eta}\left[\exp\left\{-\int_t^M r(u)\mathrm{d}u\right\}G(S(M))\middle|\,\mathcal{G}_t\right], \tag{2.23}$$

where $G(\cdot)$ is the payoff function of the derivative at its maturity M, $t \leq M \leq T$, and $E^{P^\eta}[\cdot|\cdot]$ is the conditional expectation operator under P^η. From the chain rule of the conditional expectation, given $\{\tau > t\}$, the price $V(t)$ is given by

$$V(t) = E^{P^\eta}\left[V(t|\mathcal{G}_t)|\,\mathcal{F}_t\right] = E^{P^\eta}\left[\exp\left\{-\int_t^M r(u)\mathrm{d}u\right\}G(S(M))\middle|\,\mathcal{F}_t\right]. \tag{2.24}$$

Hereafter, we call P^η the risk-neutral probability measure, and denote it $\widetilde{P}$, and a standard Brownian motion under $\widetilde{P}$ is denoted by $\widetilde{z}$. Similarly, $h^c(t)$, $\mathcal{Q}^X$, $\mathcal{P}^X(t)$ and $\boldsymbol{M}^X(t)$ are denoted by $\widetilde{h}(t)$, $\widetilde{\mathcal{Q}}$, $\widetilde{\mathcal{P}}(t)$ and $\widetilde{\boldsymbol{M}}(t)$, respectively.

Generally speaking, a regime-switching model derives an incomplete market so that many risk-neutral measures might exist, and P^η is one of them. However, according to Elliott et

[4] This change of measure is called the risk-neutral regime-switching Esscher transform.

al.[5], since the measure P^η is the minimum entropy martingale measure of P, the price given by (2.24) might be a reasonable one.

Let $\mathcal{F}_t$-predictable process $\lambda_r(t)$ be the market price of risk for $z_r(t)$, then the stochastic processes describing the market under $\widetilde{P}$ are as follows:

$$\mathrm{d}r(t) = \widetilde{\mu}_r(t, r(t), \boldsymbol{X}(t-))\mathrm{d}t + \sigma_r(t, r(t), \boldsymbol{X}(t-))\mathrm{d}\widetilde{z}_r(t), \tag{2.25}$$

$$\mathrm{d}h(t) = \widetilde{\mu}_h(t, h(t), \boldsymbol{X}(t-))\mathrm{d}t + \sigma_h(t, h(t), \boldsymbol{X}(t-))\mathrm{d}\widetilde{z}_h(t) \tag{2.26}$$

$$\widetilde{h}(t) = (1 + \kappa_h(t, \boldsymbol{X}(t-)))h(t) \tag{2.27}$$

$$\widetilde{\mu}_r(t, r(t), \boldsymbol{X}(t-)) = \mu_r(t, r(t), \boldsymbol{X}(t-)) - \lambda_r(t)\sigma_r(t, r(t), \boldsymbol{X}(t-)) \tag{2.28}$$

$$\widetilde{\mu}_h(t, h(t), \boldsymbol{X}(t-)) = \mu_h(t, h(t), \boldsymbol{X}(t-)) + \beta(t)\sigma_h(t, h(t), \boldsymbol{X}(t-)). \tag{2.29}$$

Consider a default-free discount bond with maturity M, $t \le M \le T$. From (2.24), its price at time t is given by

$$v(t, M, r, \boldsymbol{X}) = \widetilde{E}\left[\exp\left\{-\int_t^M r(u)\mathrm{d}u\right\}\middle| r(t) = r,\ \boldsymbol{X}(t) = \boldsymbol{X}\right]. \tag{2.30}$$

Next, consider a defaultable discount bond with maturity M, and let $\tau > 0$ its default time. Suppose the holder of the bond receives \$1 at M if the bond survives at M, while he/she receives \$$\delta$, $0 \le \delta < 1$, at M if default occurs up to M. From (2.24), given $\{\tau > t\}$, the price of the bond is given by

$$\begin{aligned}
&D(t, M, r, \widetilde{h}, \boldsymbol{X}) \\
&= \widetilde{E}\left[\exp\left\{-\int_t^M r(u)\mathrm{d}u\right\}\left\{1_{\{\tau>M\}} + \delta 1_{\{\tau\le M\}}\right\}\middle| r(t) = r,\ \widetilde{h}(t) = \widetilde{h},\ \boldsymbol{X}(t) = \boldsymbol{X}\right] \\
&= \delta v(t, M, r, \boldsymbol{X}) + (1-\delta)p(t, M, r, \widetilde{h}, \boldsymbol{X}), \qquad (2.31) \\
&p(t, M, r, \widetilde{h}, \boldsymbol{X}) \\
&= \widetilde{E}\left[\exp\left\{-\int_t^M (r(u) + \widetilde{h}(u))\mathrm{d}u\right\}\middle| r(t) = r,\ \widetilde{h}(t) = \widetilde{h},\ \boldsymbol{X}(t) = \boldsymbol{X}\right]. \qquad (2.32)
\end{aligned}$$

We call $p(t, M, r, \widetilde{h}, \boldsymbol{X})$ the price of a "survival discount bond", whose payoff is \$1 if and only if the bond survives at M.

2.3 A simple model

Here, we propose a simple tractable model. Under the physical measure P, $r(t)$ follows

$$\mathrm{d}r(t) = a\left(\langle \boldsymbol{m}, \boldsymbol{X}(t)\rangle - r(t)\right)\mathrm{d}t + \langle \boldsymbol{\sigma}, \boldsymbol{X}(t)\rangle \mathrm{d}z_r(t), \qquad t \ge 0, \tag{2.33}$$

where $\boldsymbol{m} = (m_1, \cdots, m_K)^\top$ and $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_K)^\top$ are K-dimensional constant vectors, a and σ_k, $k = 1, \cdots, K$ are positive constants, $\langle \boldsymbol{A}, \boldsymbol{B} \rangle$ is the inner product of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, and $z_r(t)$ is a standard Brownian motion under P. Similarly, $h(t)$ follows

$$\mathrm{d}h(t) = a_h \left(\langle \boldsymbol{m}_h, \boldsymbol{X}(t) \rangle - h(t) \right) \mathrm{d}t + \langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t) \rangle \mathrm{d}z_h(t), \qquad t \geq 0, \tag{2.34}$$

where $\boldsymbol{m}_h = (m_{h,1}, \cdots, m_{h,K})^\top$ and $\boldsymbol{\sigma}_h = (\sigma_{h,1}, \cdots, \sigma_{h,K})^\top$ are K-dimensional constant vectors, a_h and $\sigma_{h,k}$, $k = 1, \cdots, K$, are positive constants, $z_h(t)$ is a standard Brownian motion under P and $\mathrm{d}z_r(t)\mathrm{d}z_h(t) = \rho(t)\mathrm{d}t$ [5]. For simplicity, we assume that $\boldsymbol{X}(t)$ is independent of $(z_r(t), z_h(t))$. As you see, we select the Vasicek model because model parameters can be calibrated easily by using the existing statistical tools such as Matlab[6].

In order to simplify the mode, we use the following assumption.

Assumption 2.1. The market price of risk $\lambda_r(t)$ against $z_r(t)$ is written by

$$\lambda_r(t) = \langle \boldsymbol{L}_r, \boldsymbol{X}(t) \rangle + \psi_r(t) \tag{2.35}$$

where $\boldsymbol{L}_r = (\lambda_{r,1}, \cdots, \lambda_{r,K})^\top$ is a K-dimensional constant vector and $\psi_r(t)$ is a deterministic function of time t.

From (2.25), (2.28), (2.33) and Assumption 2.1, we have

$$\mathrm{d}r(t) = \left(\langle \boldsymbol{\phi}(t), \boldsymbol{X}(t) \rangle - ar(t) \right) \mathrm{d}t + \langle \boldsymbol{\sigma}, \boldsymbol{X}(t) \rangle \mathrm{d}\widetilde{z}_r(t), \tag{2.36}$$

where the deterministic function vector $\boldsymbol{\phi}(t) = (\phi_1(t), \cdots \phi_K(t))^\top$ is given by

$$\boldsymbol{\phi}(t) = a\boldsymbol{m} - \boldsymbol{\sigma} \otimes \boldsymbol{L}_r - \boldsymbol{\sigma}\psi_r(t), \tag{2.37}$$

and $\boldsymbol{A} \otimes \boldsymbol{B}$ is the Hadamard product of K-dimensional vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Then, the discount bond price $v(t, M, r, \boldsymbol{X})$ is given by

$$v(t, M, r, \boldsymbol{X}) = \exp \left\{ \langle \boldsymbol{A}(t, M), \boldsymbol{X} \rangle - B(t, M)r \right\}, \tag{2.38}$$

where

$$B(t, M) = \frac{1 - \mathrm{e}^{-a(M-t)}}{a}, \tag{2.39}$$

[5] The extension of (a, a_h) to the deterministic functions of time t is obvious, not essential.

[6] In Hatakeyama [13], a slightly modified program based on Matlab was used, which released our labor much.

and $\bar{\boldsymbol{A}}(t, M) = (\bar{A}_1(t, M), \cdots, \bar{A}_K(t, M))^\top$, $\bar{A}_i(t, M) = \exp\{A(t, M, \boldsymbol{e}_i)\}$, $i = 1, \cdots, K$, is the solution of the following ordinary differential equations (hereafter, abbreviated by ODEs, and these are Kolmogorov's backward equations)

$$\frac{\mathrm{d}\bar{A}_i(t, M)}{\mathrm{d}t} + \left\{\frac{1}{2}\sigma_i^2 B^2(t, M) - \phi_i(t)B(t, M)\right\} \bar{A}_i(t, M) + \sum_{j=1}^{K} \widetilde{q}_{ij}\bar{A}_j(t, M) = 0, \tag{2.40}$$

with the terminal conditions $\bar{A}_i(M, M) = 1$, $i = 1, \cdots, K$. A numerical solution of (2.40) is obtained easily, for example, by using the Runge-Kutta method. The zero rate (Yield To Maturity of the discount bond) at time t is given by

$$R(t, M, r(t), \boldsymbol{X}(t)) = -\frac{\log v(t, M, r(t), \boldsymbol{X}(t))}{M - t}, \qquad 0 \leq t < M. \tag{2.41}$$

In order to treat the default intensity process similarly above, and in order to make the calibration tractable, we introduce the following assumption.

Assumption 2.2. The stochastic process $\beta(t)$ is written as

$$\beta(t) = \langle \boldsymbol{L}_h, \boldsymbol{X}(t)\rangle + \psi_h(t) \tag{2.42}$$

where $\boldsymbol{L}_h = (\lambda_{h,1}, \cdots, \lambda_{h,K})^\top$ is a K-dimensional constant vector and $\psi_h(t)$ is a deterministic function of time t. And, define

$$\ell(t) = \langle \boldsymbol{\ell}(t), \boldsymbol{X}(t)\rangle, \qquad \boldsymbol{\ell}(t) = (\ell_1(t), \cdots, \ell_K(t))^\top \tag{2.43}$$

where $\ell_j(t) = \kappa_{h,j}(t)h(t)$, $j = 1, \cdots, K$, are differentiable deterministic functions[7] of time t.

From (2.26), (2.27), (2.29), (2.34) and Assumption 2.2, we have

$$\mathrm{d}h(t) = (\langle \boldsymbol{\phi}_h(t), \boldsymbol{X}(t)\rangle - a_h h(t))\,\mathrm{d}t + \langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle \mathrm{d}\widetilde{z}_h(t) \tag{2.44}$$

$$\boldsymbol{\phi}_h(t) = a_h \boldsymbol{m}_h + \boldsymbol{\sigma}_h \otimes \boldsymbol{L}_h + \boldsymbol{\sigma}_h \psi_h(t), \tag{2.45}$$

$$\widetilde{h}(t) = h(t) + \langle \boldsymbol{\ell}(t), \boldsymbol{X}(t)\rangle. \tag{2.46}$$

Now, consider a survival discount bond with the maturity M. From (2.32), the time t price of the bond is written by

$$\begin{aligned} &p(t, M, r, \widetilde{h}, \boldsymbol{X}) \\ &\quad = \widetilde{E}\left[\exp\left\{-\int_t^M (r(u) + \widetilde{h}(u))\mathrm{d}u\right\}\middle| r(t) = r, \widetilde{h}(t) = \widetilde{h}, \boldsymbol{X}(t) = \boldsymbol{X}\right], \end{aligned} \tag{2.47}$$

[7]This seems impossible because $h(t)$ cannot be differentiable, however, it is very useful for calibration.

and from the results in Appendix A, it is given by

$$p(t,M,r,\widetilde{h},\boldsymbol{X}) = \exp\left\{\langle \boldsymbol{A}(t,M),\boldsymbol{X}\rangle - B(t,M)r - B_h(t,M)\widetilde{h}\right\}, \tag{2.48}$$

where

$$B_h(t,M) = \frac{1-\mathrm{e}^{-a_h(M-t)}}{a_h}, \tag{2.49}$$

and $\bar{\boldsymbol{A}}(t,M) = (A_1(t,M),\cdots,A_K(t,M))^\top$, $\bar{A}_i(t,M) = \exp\{A_i(t,M,\boldsymbol{e}_i)\}$, $i=1,\cdots,K$, is the solution of the following ODE system:

$$\begin{aligned}\frac{\mathrm{d}\bar{A}_i(t,M)}{\mathrm{d}t} + \Bigg\{\frac{\sigma_i^2}{2}B^2(t,M) + \frac{\sigma_{h,i}^2}{2}B_h^2(t,M) - \phi_i(t)B(t,M)\\ -a_h m''_{h,i}(t)B_h(t,M) + \rho(t)\sigma_i\sigma_{h,i}B(t,M)B_h(t,M)\Bigg\}\bar{A}_i(t,M)\\ +\sum_{j=1}^{K}\widetilde{q}_{ij}\bar{A}_j(t,M) = 0.\end{aligned} \tag{2.50}$$

where

$$m''_{h,i}(t) = m_{h,i} + \ell_i(t) + \frac{1}{a_h}\left(\frac{\mathrm{d}\ell_i(t)}{\mathrm{d}t} + \sigma_{h,i}\lambda_{h,i} + \sigma_{h,i}\psi_h(t) + \sum_{j=1}^{K}\widetilde{q}_{ij}\ell_j(t)\right) \tag{2.51}$$

with the terminal condition $\bar{A}_i(M,M) = 1$, $i=1,\cdots,K$. Since our valuation method considers the regime-dependence of $r(t)$ and $h(t)$ directly, it is different from Li and Ma's result [18].

When $\widetilde{h}(t) = 0$ for $0 \le t \le T$, which implies that the bond is default-free, (2.50) reduces to (2.40). In contrast, consider a discount bond with the payoff $1_{\{\tau>M\}}\exp\left\{\int_t^M r(u)\mathrm{d}u\right\}$ at M, then, its time t price is written as [8]

$$\bar{v}(t,M,\widetilde{h},\boldsymbol{X}) = \widetilde{E}\left[\exp\left\{-\int_t^M \widetilde{h}(u)\mathrm{d}u\right\}\middle|\widetilde{h}(t)=\widetilde{h},\boldsymbol{X}(t)=\boldsymbol{X}\right], \tag{2.52}$$

and is given by

$$\bar{v}(t,M,\widetilde{h},\boldsymbol{X}) = \exp\left\{\langle \boldsymbol{A}(t,M),\boldsymbol{X}\rangle - B_h(t,M)\widetilde{h}\right\} \tag{2.53}$$

where $\bar{\boldsymbol{A}}(t,M)$ is the solution of the following ODE system: for $i=1,\cdots,K$,

$$\frac{\mathrm{d}\bar{A}_i(t,M)}{\mathrm{d}t} + \left\{\frac{\sigma_{h,i}^2}{2}B_h^2(t,M) - a_h m'_{h,i}(t)B_h(t,M)\right\}\bar{A}_i(t,M) + \sum_{j=1}^{K}\widetilde{q}_{ij}\bar{A}_j(t,M) = 0. \tag{2.54}$$

[8] The price does not depend on the short rate $r(t)$.

Notice that from (2.52), $\bar{v}(t, M, \widetilde{h}, \boldsymbol{X})$ is the conditional survival probability at M on $\{\tau > t\}$, that is, $\bar{v}(t, M, \widetilde{h}, \boldsymbol{X}) = \widetilde{P}\{\tau > M | \tau > t\}$. From the above discussion, generally, and even if $\rho(t) = 0$, it follow that

$$p(t, M, r, \widetilde{h}, \boldsymbol{X}) \neq v(t, M, r, \boldsymbol{X}) \widetilde{P}\{\tau > M | \tau > t\}. \tag{2.55}$$

This is because even if $\rho(t) = 0$, $r(s)$ and $\widetilde{h}(s)$, $t \leq s \leq T$, are $\mathcal{F}^X$-conditionally independent, which does not mean independent.

The necessary model parameters are (1) $\mathcal{Q}$, $(a, \boldsymbol{m}, \boldsymbol{\sigma})$ and $(a_h, \boldsymbol{m}_h, \boldsymbol{\sigma}_h)$ on stochastic processes under P, and (2) $\widetilde{\mathcal{Q}}$ (or $\boldsymbol{\kappa} = (\kappa_{ij})_{i,j=1,\cdots,K}$), $(\boldsymbol{L}_r, \psi_r(t))$, $(\boldsymbol{L}_h, \psi_h(t))$ and $\boldsymbol{\ell}(t)$ on stochastic processes under $\widetilde{P}$ (or, change of measure). The parameters in (1) are estimated from the time-series data of interest rates and credit spreads, while those in (2) are estimated from the term structures of default-free and defaultable intererest rates at $t = 0$.

A definite disadvantage of this model is that the distribution of the future intensity $h(t)$, $t > 0$, is constructed basically from normal distributions, which means that a negative intensity appears with a positive probability. In order to avoid it, the CIR process or the squared Gaussian process can be proposed. However, since more than 99% of the credit spreads in future becomes positive in our preliminary results, we might not worry so much against negative $h(t)$.

3 Term structures of interest rates

In this section we show some numerical examples of the term structures of default-free and defaultable zero rates (we call "zero curves"). Since we have been trying to estimate all model parameters from observed data but not succeeded yet, some parameters used below are estimates from observed data, but others are not. Many parameters are estimated from the time-series data of credit spreads of bonds in the United States, but the details are omitted here.

3.1 Model parameters

First, we set $K = 3$. In all our preliminary estimations, statistical information criteria, AIC and SBIC, decrease as K increases, however, the cases with $K \geq 4$ are difficult for us because the number of parameters increases by the order of K^2. Consider three defaultable discount bonds with different levels of credit risk; here we call AAA-, BBB-, and CCC- rated bond, respectively. The mean-reverting powers of the default-free interest rate process and the default intensity processes are assumed $a = a_h^{AAA} = a_h^{BBB} = a_h^{CCC} = 1.0$, and other parameters of the processes are listed in Table 1. The infinitesimal generator $\mathcal{Q}$ is shown in (3.56). These parameters are estimated from the time-series data[9]. The present default-free zero rate curve is given by $R(0,M) = a + bM + cM^2 + dM^3(\%)$ where $(a,b,c,d) = (0, 0.546, -0.0606, 0.00233)$, and assume that $\boldsymbol{L}_r = \mathbf{0}$ and the deterministic function $\psi_r(t)$ is piecewise constant. The infinitesimal generator under $\widetilde{P}$, denoted by $\widetilde{\mathcal{Q}}$, and the term structure of $\psi_r(t)$ are calibrated from the above conditions, and are shown in (3.56) and Table 2.

$$\mathcal{Q} = \begin{pmatrix} -1.5 & 1.3 & 0.2 \\ 2.6 & -2.6 & 0.0 \\ 0.0 & 2.5 & -2.5 \end{pmatrix}, \quad \widetilde{\mathcal{Q}} = \begin{pmatrix} -6.6086 & 6.2546 & 0.3540 \\ 2.576 & -2.576 & 0.0 \\ 0.0 & 2.5005 & -2.5005 \end{pmatrix} \tag{3.56}$$

Other parameters are not from observed data. For simplicity, we set $\rho(t) = 0$, and the risk-premia adjustments are assumed to be constant with time, and are as follows[10];

$$\left(\boldsymbol{\ell}^{AAA}\ \boldsymbol{\ell}^{BBB}\ \boldsymbol{\ell}^{CCC}\right) = \begin{pmatrix} 0.0042 & 0.0102 & 0.0508 \\ 0.0082 & 0.0197 & 0.0971 \\ 0.0143 & 0.0474 & 0.2095 \end{pmatrix}.$$

3.2 Term structures of zero rates with different credit ratings

We show some numerical examples of zero curves. The recovery rate is $\delta = 0$, and the initial value of default intensity for i-th bond is $h_i(0) = m_{h,k(0)}^{cr(i)}$ where $k(0)$ is the initial regime-number and $cr(i)$ is the credit rating of i-th bond, that is, each initial value is typical for its initial regime. The initial value of the short rate must be $r(0) = R(0,0) = 0.0\%$.

Figure 1 and 2 show the default-free and defaultable (AAA, BBB and CCC) zero curves with the initial regime 0 and 2, respectively. The horizontal axis is the maturity (years), and the vertical axis is the zero rate (%). In Figure 1, since each initial value of the intensity for i-th bond, $\widetilde{h}_i(0)$, is lower than the average value on the steady state of the credit rating $cr(i)$, each zero curve goes upwards gradually. On the other hand, since BBB and CCC

[9] If you would like to see details and can read Japanese, see Hatakeyama [13].

[10] In our preliminary results, we estimated the mean-reverting levels of the default intensities under $\widetilde{P}$. Roughly, one-fourth of the estimated level is allocated as the mean-reverting level under P (in Table 1), and the residual (three-fourth) is allocated as the risk-premia adjustments.

initial values in Figure 2 are higher than the average values on the steady state, these zero curves go downwards.

Next, compare the same credit zero curves with different initial regimes. Figure 3 and 4 show the CCC and AAA curves with different initial regimes. Figure 3 seems natural because the zero curve becomes higher as the initial regime is higher ("higher" means "more risky"). On the other hand, in Figure 4, except in the short maturities, the zero curve with initial regime 2 is lower than those with initial regime 0 and 1. Such an unnatural feature can be explained by the regime-dependence of $r(t)$ and $\widetilde{h}(t)$. See Figure 5, which is the same figure as Figure 4 in the case where the default intensity $\widetilde{h}(t)$ only has a regime-switching property[11]. Figure 5 seems natural because the zero curve becomes higher as the initial regime is higher, likely to Figure 3. The difference between Figure 4 and Figure 5 is caused by the joint regime movement of $r(t)$ and $\widetilde{h}(t)$ through $\boldsymbol{X}(t)$. In the case of Figure 5, since $r(t)$ and $\widetilde{h}(t)$ are independent, the following equation $p(0, M, r(0), \widetilde{h}(0), \boldsymbol{X}(0)) = v(0, M, r(0), \boldsymbol{X}(0))\widetilde{P}\{\tau > M\}$ is always satisfies, so that the relative difference $v(0, M, r(0), \boldsymbol{X}(0))\widetilde{P}\{\tau > M\}/p(0, M, r(0), \widetilde{h}(0), \boldsymbol{X}(0)) - 1$ is always zero. Figure 6 shows the term structure of the relative difference in the same case as Figure 4 (AAA). The horizontal axis is the maturity (years), and the vertical axis is the relative difference. Figure 6 says that the regime-dependence of $r(t)$ and $\widetilde{h}(t)$ decreases the bond prices (increases the zero rates) for the initial regime 1 case, while increases the bond prices (decreases the zero rates) for the initial regime 2 case at most 1% compared to the independent case. The effect of regime-dependence is consistent with the difference between Figure 3 and Figure 4 quantitatively[12]. Also in Figure 3, the same effect has similar influence on the zero curves, however, an apparent change cannot be seen because of the large difference between $\widetilde{h}(t)$ with different regimes. Changing the correlation $\rho(t)$ has a little influence on the relative difference curve. When $\rho(t) = -0.7$, the curves in Figure 6 move upwards and have positive slopes, but since the difference is at most 0.03%, the zero curves change little.

4 Risk evaluation of bond portfolio

In this section we consider a risk evaluation of a bond portfolio consisting of N discount bonds by the Monte Carlo simulation. The risk horizon is denoted by T_R, $0 < T_R \leq M$. Here, we consider $T_R = 1$ year, $M = 5$ years and $N = 20$ discount bonds; 5 bonds of

[11] $r(t)$ follows the Vasicek model with regime-1 parameters.

[12] Roughly speaking, the duration effect on the zero rate is $\Delta p/p \simeq -t\Delta R(0, t)$ where p is the bond price.

them are AAA, other 5 bonds are BBB, and the residual 10 are CCC. Most of the model parameters are the same in Section 3.2. And, all the correlations between the Brownian motions including $\rho(t)$ are assumed to be zero. As initial values, we use $r(0) = 0$, and the initial value of i-th bond's default intensity, $h_i(0)$, is set to be the mean-reverting level of the regime 0 for each credit rating $cr(i)$[13]. The number of simulation runs is 50,000.

4.1 Monte Carlo simulation for risk evaluation

We run the Monte Carlo simulation by using our simple model, and its procedure consists of the following steps. Here, we denote $\boldsymbol{h}(t) = (h_1(t), \cdots, h_N(t))$.

1. Set the initial values $(\boldsymbol{X}(0), r(0), \boldsymbol{h}(0))$.
2. Generate a sample path $(\boldsymbol{X}(t_i), r(t_i), \boldsymbol{h}(t_i))$, $0 = t_0 < t_1 < \cdots < t_F = T_R$ under P.
3. Under P, judge whether each bond survives or has defaulted at t_i independently. The default probability of j-th bond during $(t_i, t_{i+1}]$ is given by $h_j(t_i)(t_{i+1} - t_i)$ if $t_{i+1} - t_i$ is small enough.
4. Calculate the price of each bond at time T_R; for i-th bond, $D(T_R, M, r(T_R), \widetilde{h}_i(T_R), \boldsymbol{X}(T_R))$ if the bond survives at T_R, or $\delta v(T_R, M, r(T_R), \boldsymbol{X}(T_R))$ if the bond has defaulted up to T_R[14].
5. Calculate the portfolio price at T_R.
6. Repeat the above steps until enough scenarios have been obtained in order to calculate necessary statistics accurately.

In Step 2, each sample path is generated from $t = 0$ to $t = T_R$ under P, while in Step 4 the no-arbitrage price is calculated based on the stochastic processes under $\widetilde{P}$. Notice that the input $(r(T_R), \widetilde{h}_i(T_R), \boldsymbol{X}(T_R))$ in Step 4 is given based on the values at $t = T_R$ on the sample path generated in Step 3. $r(T_R)$ and $\boldsymbol{X}(T_R)$ are the same under P and $\widetilde{P}$, but $\widetilde{h}_i(T_R) = h_i(T_R) + \langle \boldsymbol{\ell}^{cr(i)}(T_R), \boldsymbol{X}(T_R) \rangle$.

[13] It is different from the setting in Section 3.2.

[14] This recovery setting is consistent with that used in Jarrow and Turnbull [15].

4.2 Price distribution at risk horizon

Figure 7 depicts the distributions of the CCC discount bond prices at T_R. The horizontal axis is the price, and the vertical axis is the frequency. As the number of initial regime becomes high, the distribution moves left and becomes wider because the default probability in future tends to increase as the initial regime number is higher. The numbers of defaulted scenarios are 974, 1,504 and 3,268 for initial regime 0, 1, and 2, respectively. These numbers reflect the mean values of the initial regime, $m_{h,k}^{CCC}$, $k = 0, 1, 2$.

Figure 8 depicts the distributions of the portfolio price at T_R with different initial regimes. All distributions have short right tails and long left tails, and each has two or three peaks; the highest peak is the distribution consisting of no-default scenarios, and the second highest peak is the distribution of one-default scenarios. The third peak can be seen slightly only in the distribution with initial regime 2. This figure shows that even if the initial observable values $(r(0), \boldsymbol{h}(0))$ and model parameters are the same, the distribution of the future price depends strongly on the initial regime $\boldsymbol{X}(0)$, which can not be observed.

Table 3 summarizes some basic statistics for the distributions in Figure 8. In this table, $100\alpha\%$-VaR is defined by the difference between the average and the $100(1-\alpha)$-pecentile for $0 < \alpha < 1$. The results clearly show that each distribution has a long left tail, which becomes longer as the initial regime number is higher, and imply that the existence of regimes might be important for risk evaluation. Although we cannot observe the regime directly, we could use the probability distribution of regimes estimated from time-series data. When the initial distribution is assumed, for example, $(p_1, p_2, p_3) = (0.5, 0, 3, 0.2)$ where p_i is the probability of regime i, a corresponding distribution is obtained in Figure 9, and the risk measures such as VaR (Value at Risk) and ES (Expected Shortfall) are calculated based on the distribution. Such information might be useful for risk management.

The results such as Table 3 could be applied to stress tests. For example, the results labelled as "initial regime 2" in Table 3 can be regarded as a case study when the regime changes suddenly from regime 0 to regime 2. Since our model includes the dynamics of variables, we can evaluate the time-evolution of the cumulative loss after the regime change; we can see the enlarging aggregate loss and the recovering process from the stressed regime with different risk horizons. Such kinds of dynamic stress tests will give new information about risk management to financial institutions.

Samples of the time-evolution of the zero curves are shown in Figure 10 (default-free) and Figure 11 (CCC). The vertical axes are the zero rates, while the horizontal axes are the times. In each figure, a zero curve at future time $t_1 \geq 0$ starts from t_1 on the horizontal axis, and the zero rate on $t_1 + 1$ on the curve shows one-year zero rate at future t_1, and generally, the zero rate on $t_1 + n$ on the curve shows n-year zero rate at t_1. So, Figure 10 and Figure 11 show the time-evolution of default-free and CCC zero curves on a certain future scenario, respectively. You can see various kinds of curves (increasing ones, humped ones, and decreasing ones), and various curves appear on a future scenario. If you use our model, you can evaluate the time-evolving properties of your portfolio on time-evolving interest rate scenarios with various shapes.

5 Concluding remarks

In this article we propose a pricing and risk evaluation model of interest rate risk and credit risk with Markovian regime-switching property. First, we discuss the dynamics of regime, interest rate and default intensity under the physical probability measure and the change of measure to an equivalent martingale measure (pricing measure), and derive the dynamics under the pricing measure. Second, we propose a simple model, and derive a method to calculate the discount bond price, which is a slight extension of the existing papers such as Elliott and Siu [7]. Based on the simple model, we show some numerical examples for pricing, and clarify the effects of regime-switching on pricing. Since the default-free interest rate and the default intensity are dependent through regimes, the term structure of the zero rates depends on the regime dynamics. And we calculate the distribution of a bond portfolio's price at the horizon, which reflects the actual default loss up to the horizon and the decrease of market prices due to the transition to a stressed regime. Numerical examples show that the price distribution has a short right tail and a long left tail, and that the distribution depends strongly on the present regime. The former result is derived from a general feature of regimes; the existence of a calm regime with low default risk and volatility and a stressed regime with high default risk and volatility. The latter result seems applicable to the financial risk management, especially on the stress tests. We think that stochastic models with regime-switching property would be promising and useful for pricing and financial risk management in future.

However, there remain necessary future works. First of all, it is desired to develop an efficient and stable parameter estimation method. Since we have not developed yet, we used some existing packages such as Matlab in order to analyse the time-series data, therefore, the stochastic processes are limited to the Vasicek model, and the number of regimes are limited up to 3. Additionally, a stable calibration method to fit theoretical prices to market prices is also necessary. Second, since we do not have enough knowledge about the regime-switching properties on financial and economic data, more researches on various kinds of data are needed. For example, since the dependence between the financial variables might be different in different regimes, such analyses would be fruitful. The dependence described by a copula, which is frequently used in credit risk models, would be different in each regime, More knowledge about regime-switching properties will make models sophisticated to be more useful for the financial risk management.

A Derivation of (2.50)

Consider the model in Section 2.3. Under Assumption 2.2, we have

$$\mathrm{d}\ell(t) = \mathrm{d}\langle \boldsymbol{\ell}(t), \boldsymbol{X}(t)\rangle = \left(\left\langle \frac{\mathrm{d}\boldsymbol{\ell}(t)}{\mathrm{d}t}, \boldsymbol{X}(t)\right\rangle + \langle \boldsymbol{\ell}(t), Q\boldsymbol{X}(t)\rangle\right)\mathrm{d}t + \langle \boldsymbol{\ell}(t), \mathrm{d}\widetilde{\boldsymbol{M}}(t)\rangle \tag{A.1}$$

under $\widetilde{P}$. The term $\langle \boldsymbol{\ell}(t), Q\boldsymbol{X}(t)\rangle$ means the jump due to the regime-switching. It follows from Assumption 2.2, (2.44)–(2.46) and (A.1) that

$$\begin{aligned}
\mathrm{d}\widetilde{h}(t) &= \mathrm{d}h(t) + \mathrm{d}\ell(t) = a_h\left(\langle \boldsymbol{m}_h, \boldsymbol{X}(t)\rangle - h(t)\right)\mathrm{d}t + \langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle \mathrm{d}z_h(t) + \mathrm{d}\langle \boldsymbol{\ell}(t), \boldsymbol{X}(t)\rangle \\
&= \left(\langle a_h\boldsymbol{m}_h, \boldsymbol{X}(t)\rangle - a_h(\widetilde{h}(t) - \langle \boldsymbol{\ell}(t), \boldsymbol{X}(t)\rangle)\right)\mathrm{d}t + \langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle\left(\mathrm{d}\widetilde{z}_h(t) + (\langle \boldsymbol{L}_h, \boldsymbol{X}(t)\rangle + \psi_h(t))\mathrm{d}t\right) \\
&\qquad + \left(\left\langle \frac{\mathrm{d}\boldsymbol{\ell}(t)}{\mathrm{d}t}, \boldsymbol{X}(t)\right\rangle + \langle \boldsymbol{\ell}(t), Q\boldsymbol{X}(t)\rangle\right)\mathrm{d}t + \langle \boldsymbol{\ell}(t), \mathrm{d}\widetilde{\boldsymbol{M}}(t)\rangle \\
&= \left(\left\langle a_h\boldsymbol{m}_h + a_h\boldsymbol{\ell}(t) + \boldsymbol{\sigma}_h \otimes \boldsymbol{L}_h + \psi_h(t)\boldsymbol{\sigma}_h + \frac{\mathrm{d}\boldsymbol{\ell}(t)}{\mathrm{d}t}, \boldsymbol{X}(t)\right\rangle + \langle \boldsymbol{\ell}(t), Q\boldsymbol{X}(t)\rangle - a_h\widetilde{h}(t)\right)\mathrm{d}t \\
&\qquad + \langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle \mathrm{d}\widetilde{z}_h(t) + \langle \boldsymbol{\ell}(t), \mathrm{d}\widetilde{\boldsymbol{M}}(t)\rangle \\
&= a_h\left(\langle \boldsymbol{m}'_h(t), \boldsymbol{X}(t)\rangle - \widetilde{h}(t)\right)\mathrm{d}t + \langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle \mathrm{d}\widetilde{z}_h(t) + \langle \boldsymbol{\ell}, \mathrm{d}\widetilde{\boldsymbol{M}}(t)\rangle,
\end{aligned} \tag{A.2}$$

where $\boldsymbol{A} \otimes \boldsymbol{B}$ is the Hadamard product of K-dimensional vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, and

$$\boldsymbol{m}'_h(t) = \boldsymbol{m}_h + \boldsymbol{\ell}(t) + \frac{1}{a_h}\left(\frac{\mathrm{d}\boldsymbol{\ell}(t)}{\mathrm{d}t} + \boldsymbol{\sigma}_h \otimes \boldsymbol{L}_h + \psi_h(t)\boldsymbol{\sigma}_h + (\langle \boldsymbol{\ell}(t), Q\boldsymbol{e}_1\rangle, \cdots, \langle \boldsymbol{\ell}(t), Q\boldsymbol{e}_K\rangle)^\top\right). \tag{A.3}$$

We define $r_{t,u}(r) \equiv r(u|r(t) = r)$, $\widetilde{h}_{t,u}(\widetilde{h}) \equiv \widetilde{h}(u|\widetilde{h}(t) = \widetilde{h})$ for $0 \le t \le u \le T$, and

$$p(t, T, r, \widetilde{h}, \boldsymbol{X}) = \widetilde{E}_t\left[\exp\left\{-\int_t^T r_{t,u}(r)\mathrm{d}u\right\} 1_{\{\tau > T\}}\right] = \widetilde{E}_t\left[\exp\left\{-\int_t^T (r_{t,u}(r) + \widetilde{h}_{t,u}(\widetilde{h}))\mathrm{d}u\right\}\right]. \tag{A.4}$$

It follows from (2.36) that

$$r_{t,s}(r) = r + \int_t^s (\langle \boldsymbol{\phi}(u), \boldsymbol{X}(u)\rangle - a r_{t,u}(r))\,\mathrm{d}u + \int_t^s \langle \boldsymbol{\sigma}, \boldsymbol{X}(u)\rangle \mathrm{d}\widetilde{z}_r(u), \tag{A.5}$$

so that $D_{t,s} = \partial r_{t,s}(r)/\partial r$ satisfies

$$D_{t,s} = 1 - a\int_t^s D_{t,u}\mathrm{d}u. \tag{A.6}$$

Differentiating (A.6) with respect to s, we have

$$\frac{\mathrm{d}D_{t,s}}{\mathrm{d}s} = -aD_{t,s}, \qquad D_{t,t} = 1. \tag{A.7}$$

Therefore, we obtain

$$D_{t,s} = \mathrm{e}^{-a(s-t)}. \tag{A.8}$$

From (A.4) and (A.8), we have

$$\frac{\partial}{\partial r} p(t,T,r,\widetilde{h},\boldsymbol{X}) = -\left(\int_t^T D_{t,u}\mathrm{d}u\right) p(t,T,r,\widetilde{h},\boldsymbol{X}) = -B(t,T)p(t,T,r,\widetilde{h},\boldsymbol{X}), \tag{A.9}$$

and similarly, we have

$$\frac{\partial}{\partial \widetilde{h}} p(t,T,r,\widetilde{h},\boldsymbol{X}) = -B_h(t,T)p(t,T,r,\widetilde{h},\boldsymbol{X}). \tag{A.10}$$

Define

$$\begin{aligned}\widetilde{p}(t,T,r,\widetilde{h},\boldsymbol{X}) &\equiv \exp\left\{-\int_0^t (r_{t,u}(r) + \widetilde{h}_{t,u}(\widetilde{h}))\mathrm{d}u\right\} p(t,T,r,\widetilde{h},\boldsymbol{X}) \\ &= \widetilde{E}_t\left[\exp\left\{-\int_0^T (r_{t,u}(r) + \widetilde{h}_{t,u}(\widetilde{h}))\mathrm{d}u\right\}\right],\end{aligned} \tag{A.11}$$

which is a Doob's martingale under $\widetilde{P}$. Applying Ito's lemma to $\widetilde{p}(t,T,r,\widetilde{h},\boldsymbol{X})$ and putting $\widetilde{\boldsymbol{p}} = (\widetilde{p}_1,\cdots,\widetilde{p}_K)^\top$ with $\widetilde{p}_i = \widetilde{p}(t,T,r,\widetilde{h},\boldsymbol{e}_i)$, $i = 1,\cdots,K$, we have

$$\begin{aligned}\widetilde{p}(t,T,r,\widetilde{h},\boldsymbol{X}) &= \widetilde{p}(0,T,r(0),\widetilde{h}(0),\boldsymbol{X}(0)) \\ &+ \int_0^t \left[\frac{\partial\widetilde{p}}{\partial u} + \frac{\partial\widetilde{p}}{\partial r}(\langle\boldsymbol{\phi}(u),\boldsymbol{X}(u)\rangle - ar(u)) + \frac{\partial\widetilde{p}}{\partial\widetilde{h}}\left(\langle a_h\boldsymbol{m}_h'(u),\boldsymbol{X}(u)\rangle - a_h\widetilde{h}(u)\right)\right. \\ &\qquad \left. + \frac{1}{2}\langle\boldsymbol{\sigma},\boldsymbol{X}(u)\rangle^2\frac{\partial^2\widetilde{p}}{\partial r^2} + \frac{1}{2}\langle\boldsymbol{\sigma}_h,\boldsymbol{X}(u)\rangle^2\frac{\partial^2\widetilde{p}}{\partial\widetilde{h}^2} + \rho(t)\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}_h,\boldsymbol{X}(u)\rangle\frac{\partial^2\widetilde{p}}{\partial r\partial\widetilde{h}}\right]du \\ &+ \int_0^t \frac{\partial\widetilde{p}}{\partial r}\langle\boldsymbol{\sigma},\boldsymbol{X}(u)\rangle\mathrm{d}\widetilde{z}_r(u) + \int_0^t \frac{\partial\widetilde{p}}{\partial\widetilde{h}}\langle\boldsymbol{\sigma}_h,\boldsymbol{X}(u)\rangle\mathrm{d}\widetilde{z}_h(u) + \int_0^t \langle\widetilde{\boldsymbol{p}},\widetilde{Q}\boldsymbol{X}(u)\rangle\mathrm{d}u \\ &+ \int_0^t \left\langle\widetilde{\boldsymbol{p}} + \frac{\partial\widetilde{p}}{\partial\widetilde{h}}\otimes\boldsymbol{\ell}(u),\mathrm{d}\widetilde{\boldsymbol{M}}(u)\right\rangle.\end{aligned} \tag{A.12}$$

Because of the martingale property of $\widetilde{p}$, the integrand of $\mathrm{d}u$ term must be zero, that is, we have the following partial differential equation (hereafter, abbreviated by PDE)

$$0 = \frac{\partial \widetilde{p}}{\partial t} + \frac{\partial \widetilde{p}}{\partial r}\left(\langle \boldsymbol{\phi}(t), \boldsymbol{X}(t)\rangle - ar(t)\right) + \frac{\partial \widetilde{p}}{\partial \widetilde{h}}\Big(\langle a_h \boldsymbol{m}_h'(t), \boldsymbol{X}(t)\rangle - a_h \widetilde{h}(t)\Big)$$
$$+ \frac{1}{2}\langle \boldsymbol{\sigma}, \boldsymbol{X}(t)\rangle^2 \frac{\partial^2 \widetilde{p}}{\partial r^2} + \frac{1}{2}\langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle^2 \frac{\partial^2 \widetilde{p}}{\partial \widetilde{h}^2} + \rho(t)\langle \boldsymbol{\sigma}\otimes\boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle \frac{\partial^2 \widetilde{p}}{\partial r \partial \widetilde{h}} + \langle \widetilde{\boldsymbol{p}}, \widetilde{Q}\boldsymbol{X}(t)\rangle,$$

which implies that

$$0 = \frac{\partial p}{\partial t} - rp - \widetilde{h}p + \frac{\partial p}{\partial r}\left(\langle \boldsymbol{\phi}(t), \boldsymbol{X}(t)\rangle - ar(t)\right) + \frac{\partial p}{\partial \widetilde{h}}\Big(\langle a_h \boldsymbol{m}_h'(t), \boldsymbol{X}(t)\rangle - a_h \widetilde{h}(t)\Big)$$
$$+ \frac{1}{2}\langle \boldsymbol{\sigma}, \boldsymbol{X}(t)\rangle^2 \frac{\partial^2 p}{\partial r^2} + \frac{1}{2}\langle \boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle^2 \frac{\partial^2 p}{\partial \widetilde{h}^2} + \rho(t)\langle \boldsymbol{\sigma}\otimes\boldsymbol{\sigma}_h, \boldsymbol{X}(t)\rangle \frac{\partial^2 p}{\partial r \partial \widetilde{h}} + \langle \boldsymbol{p}, \widetilde{Q}\boldsymbol{X}(t)\rangle, \tag{A.13}$$

where $\boldsymbol{p} = (p_1, \cdots, p_K)^\top$ with $p_i = p(t, T, r, \widetilde{h}, \boldsymbol{e}_i)$, $i = 1, \cdots, K$, with the terminal condition $p(T, T, r, \widetilde{h}, \boldsymbol{\cdot}) = 1$. (A.13) means a system of K coupled PDEs:

$$0 = \frac{\partial p_i}{\partial t} - rp_i - \widetilde{h}p_i + \frac{\partial p_i}{\partial r}\left(\phi_i(t) - ar(t)\right) + \frac{\partial p_i}{\partial \widetilde{h}}\Big(a_h m_{h,i}'(t) - a_h \widetilde{h}(t)\Big)$$
$$+ \frac{1}{2}\sigma_i^2 \frac{\partial^2 p_i}{\partial r^2} + \frac{1}{2}\sigma_{h,i}^2 \frac{\partial^2 p_i}{\partial \widetilde{h}^2} + \rho(t)\sigma_i\sigma_{h,i}\frac{\partial^2 p_i}{\partial r \partial \widetilde{h}} + \langle \boldsymbol{p}, \widetilde{Q}\boldsymbol{e}_i(t)\rangle, \qquad 0 \le t \le T,$$

with $p_i(T, T, r, \widetilde{h}) = 1$, $i = 1, \cdots, K$. Here, assume that

$$p_i(t, T, r, \widetilde{h}) = \exp\left\{A_i(t, T) - B(t, T)r - B_h(t, T)\widetilde{h}\right\}, \quad i = 1, \cdots, K, \tag{A.14}$$

where $A_i(t, T) = A(t, T, \boldsymbol{e}_i)$, then, $A_i(t, T)$ satisfies the following ODEs:

$$0 = \mathrm{e}^{A_i}\left[\frac{\mathrm{d}A_i}{\mathrm{d}t} - \phi_i(t)B - a_h m_{h,i}'(t)B_h + \frac{\sigma_i^2}{2}B^2 + \frac{\sigma_{h,i}^2}{2}B_h^2 + \rho(t)\sigma_i\sigma_{h,i}BB_h\right]$$
$$+ \langle \boldsymbol{A}, Q\boldsymbol{e}_i(t)\rangle, \tag{A.15}$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = aB - 1, \tag{A.16}$$

$$\frac{\mathrm{d}B_h}{\mathrm{d}t} = a_h B_h - 1. \tag{A.17}$$

where $\boldsymbol{A} = (\mathrm{e}^{A_1(t,T)}, \cdots, \mathrm{e}^{A_K(t,T)})^\top$. From (A.16) and (A.17), we have

$$B(t, T) = \frac{1 - \mathrm{e}^{-a(T-t)}}{a} \quad \text{and} \quad B_h(t, T) = \frac{1 - \mathrm{e}^{-a_h(T-t)}}{a_h}. \tag{A.18}$$

Let $\bar{A}_i(t, T) = \mathrm{e}^{A_i(t,T)}$, $i = 1, \cdots, K$, then (A.15) can be rewritten as

$$0 = \frac{\mathrm{d}\bar{A}_i(t, T)}{\mathrm{d}t} + \left\{\frac{\sigma_i^2}{2}B^2(t, T) + \frac{\sigma_{h,i}^2}{2}B_h^2(t, T) - \phi_i(t)B(t, T) - a_h m_{h,i}'(t)B_h(t, T)\right.$$
$$\left. + \rho(t)\sigma_i\sigma_{h,i}B(t, T)B_h(t, T)\right\}\bar{A}_i(t, T) + \sum_{j=1}^{K}\bar{A}_j(t, T)\widetilde{q}_{ji}. \tag{A.19}$$

In practice, we will solve numerically the following Kolmogorov's backward equations:

$$0 = \frac{\mathrm{d}\bar{A}_i(t,T)}{\mathrm{d}t} + \left\{ \frac{\sigma_i^2}{2} B^2(t,T) + \frac{\sigma_{h,i}^2}{2} B_h^2(t,T) - \phi_i(t)B(t,T) - a_h m''_{h,i}(t) B_h(t,T) \right.$$
$$\left. + \rho(t)\sigma_i\sigma_{h,i} B(t,T) B_h(t,T) \right\} \bar{A}_i(t,T) + \sum_{j=1}^{K} \widetilde{q}_{ij} \bar{A}_j(t,T), \quad i = 1, \cdots, K, \qquad \text{(A.20)}$$

where[15]

$$m''_{h,i}(t) = m_{h,i} + \ell_i(t) + \frac{1}{a_h} \left(\frac{\mathrm{d}\ell_i(t)}{\mathrm{d}t} + \sigma_{h,i}\lambda_{h,i} + \sigma_{h,i}\psi_h(t) + \sum_{j=1}^{K} \widetilde{q}_{ij}\ell_j \right) \qquad \text{(A.21)}$$

with the terminal conditions $\bar{A}_i(T,T) = 1$, $i = 1, \cdots, K$.

Acknowledgements

The author is grateful to Naoto Hatakeyama, Junpei Igarashi, Prof. Toshinao Yoshiba and Prof. Kohta Takehara at Tokyo Metropolitan University for helpful discussions and various kinds of data analyses, and is also grateful to Rongzhu Liu, Toshiaki Seiwa and Masayuki Hamanaka at AFAS Corporation for giving me this interesting research theme and continuing the research project.

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Table 1: Parameters of default-free interest rate and default intensity processes under P.

regime k	m_k	σ_k	$m_{h,k}^{AAA}$	$\sigma_{h,k}^{AAA}$	$m_{h,k}^{BBB}$	$\sigma_{h,k}^{BBB}$	$m_{h,k}^{CCC}$	$\sigma_{h,k}^{CCC}$
0	0.33 %	0.46 %	0.14 %	0.18 %	0.34 %	0.40 %	1.70 %	2.43 %
1	2.73 %	1.08 %	0.28 %	0.33 %	0.66 %	0.79 %	3.24 %	3.90 %
2	-1.13 %	1.51 %	0.48 %	2.29 %	1.58 %	3.79 %	6.98 %	16.10 %

Table 2: Term structure of market prices of risk for default-free interest rate.

t	$(0,1)$	$(1,2)$	$(2,3)$	$(3,4)$	$(4,5)$	$(5,6)$	$(6,7)$	$(7,8)$	$(8,9)$	$(9,10)$
ψ_r	0.385	0.143	−0.253	−0.212	−0.226	−0.054	0.031	0.116	0.038	−0.192

Table 3: Basic statistics of portfolio's price distributions with different initial regimes.

	initial regime		
	0	1	2
average	14.57	14.29	14.04
standard deviation	0.33	0.40	0.56
percentile			
50.00%	14.68	14.54	14.10
10.00%	14.07	13.86	13.26
5.00%	13.91	13.60	12.99
1.00%	13.43	13.15	12.38
0.50%	13.26	12.94	12.15
0.10%	12.83	12.50	11.65
VaR = average $-$ $100(1-\alpha)$-percentile			
10.00%	0.49	0.53	0.78
5.00%	0.66	0.79	1.05
1.00%	1.14	1.24	1.66
0.50%	1.31	1.46	1.89
0.10%	1.73	1.90	2.39

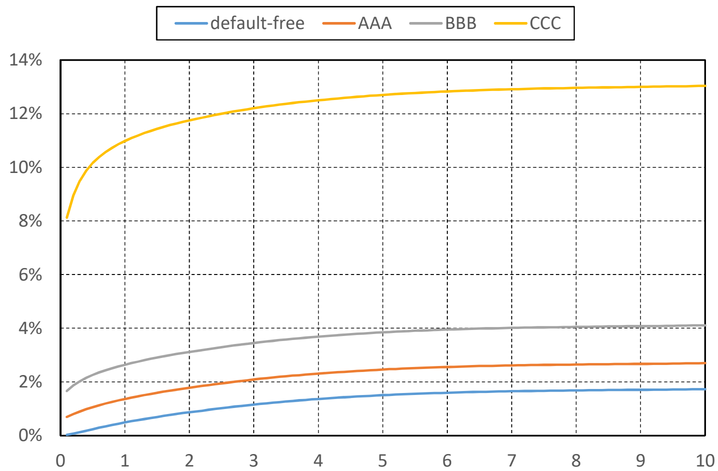

Figure 1: Term structures of zero rates (initial regime 0).

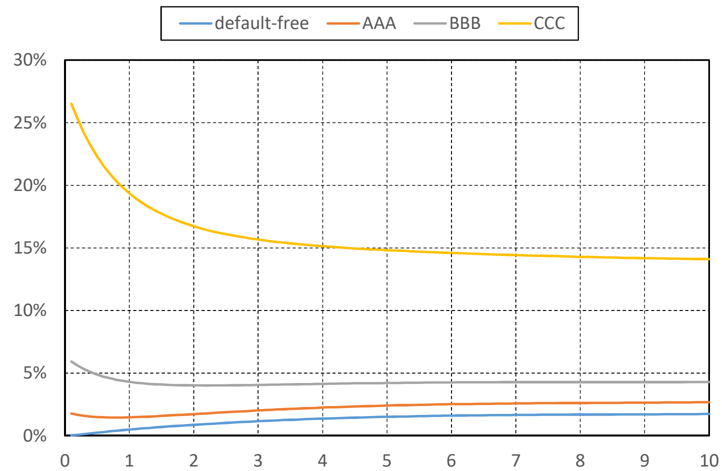

Figure 2: Term structures of zero rates (initial regime 2).

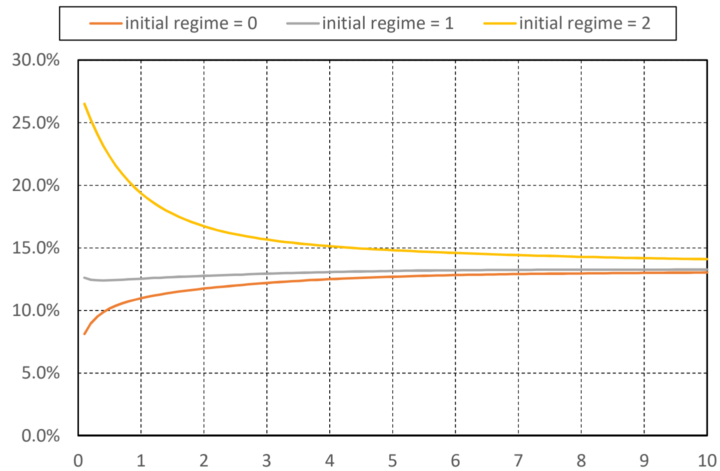

Figure 3: CCC zero rate curves with different initial regime.

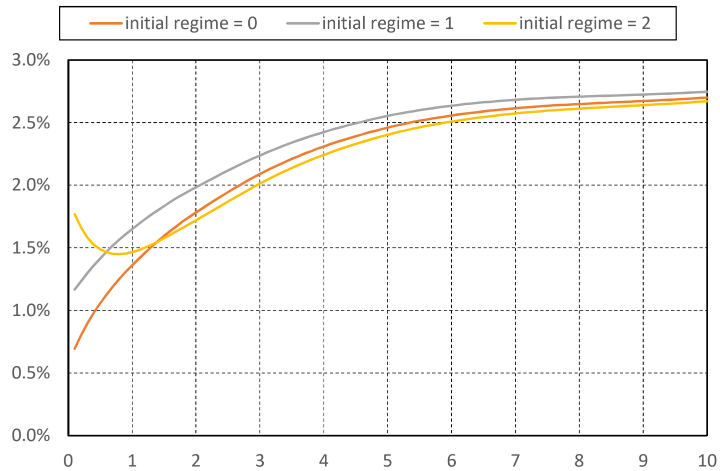

Figure 4: AAA zero rate curves with different initial regime.

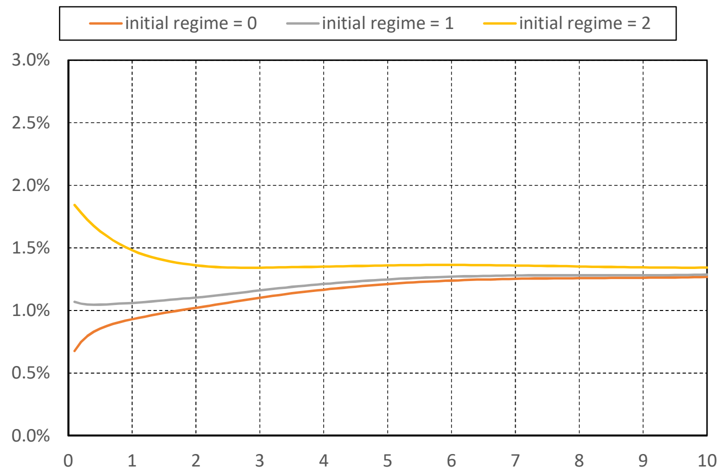

Figure 5: AAA zero rate curves with different initial regime ($r(t)$ is not regime-switching).

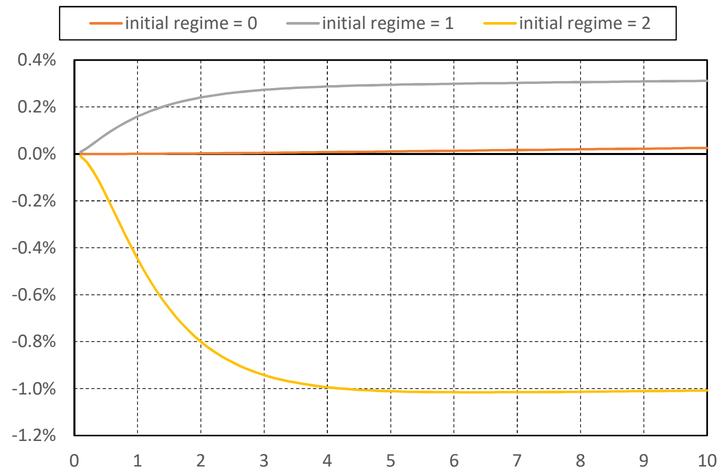

Figure 6: Term structure of relative difference $v(0, M, r(0), \boldsymbol{X}(0))\widetilde{P}\{\tau > M\}/p(0, M, r(0), \widetilde{h}(0), \boldsymbol{X}(0)) - 1$ (AAA bond).

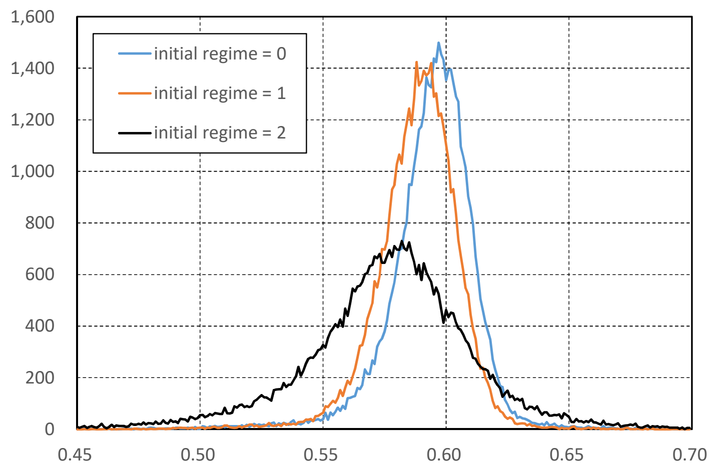

Figure 7: Distributions of CCC bond price with different initial regime after 1 year.

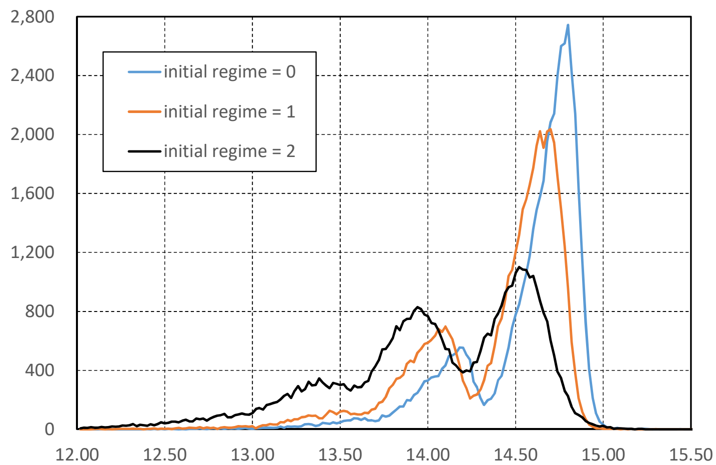

Figure 8: Distributions of portfolio price with different initial regime after 1 year.

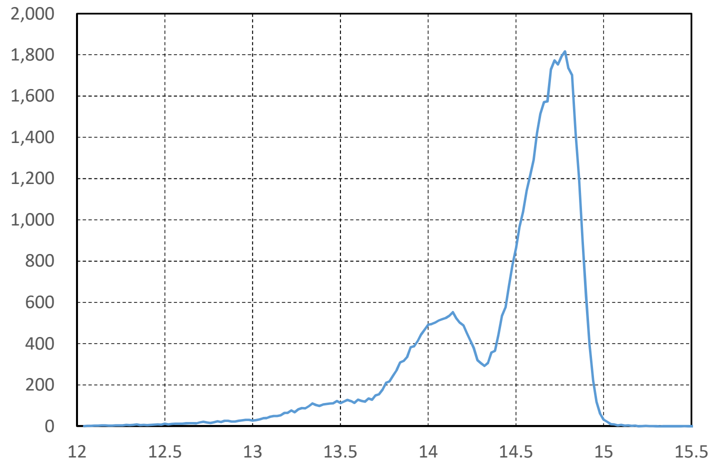

Figure 9: Hybrid distribution of portfolio price after 1 year.

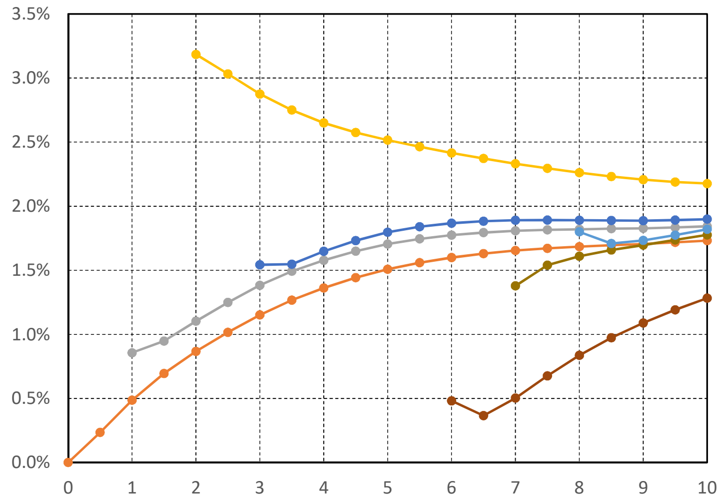

Figure 10: Evolution of default-free zero curve on a scenario.

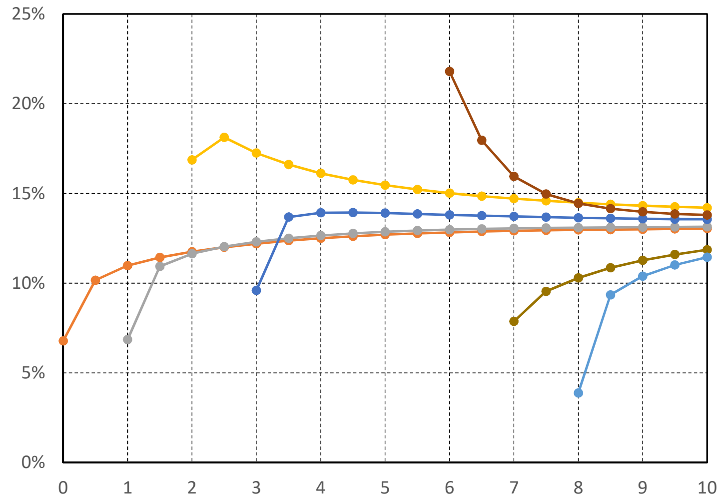

Figure 11: Evolution of CCC zero curve on a scenario.