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Group Strategy-proof Probabilistic Voting with Single-peaked Preferences

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Abstract

We study probabilistic voting rules in the case where agents have single-peaked preferences over alternatives. A probabilistic rule decides a probability distribution over the set of alternatives for each profile of agents' preferences. In this paper, we characterize the class of group strategy-proof and peak-only probabilistic rules.

Keywords: group strategy-proofness, probabilistic rule, single-peaked preferences

JEL Classification Numbers: D71

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1 Introduction

This paper studies non-manipulable probabilistic voting rules. Randomization is a widely accepted method to achieve a fair outcome. Agents have ordinal preferences over a finite set of alternatives. A *probabilistic rule* decides a probability distribution over the set of alternatives for each preference profile. *Strategy-proofness* is a key axiom related to incentive compatibility. It provides each individual agent a strong incentive to reveal his own preferences truthfully. *Group strategy-proofness* is an even stronger axiom than strategy-proofness. It prevents strategic misrepresentations of preferences via *groups* of agents as well as individual agents.

In voting contexts, one of the most widely studied preference domains is the domain of single-peaked preferences. In this paper, on the single-peaked domain, we study the class of probabilistic rules that satisfy the following notion of group strategy-proofness: whenever a group of agents jointly misrepresents their preferences, for at least one of them, the distribution chosen under truth-telling stochastically dominates the distribution chosen under joint misrepresentation.[1] This is a stronger requirement than strategy-proofness based on stochastic dominance relations, which is an extensively studied concept in probabilistic environments.

As is well known, on the single-peaked domain, any strategy-proof *deterministic* rule is also group strategy-proof.[2] In contrast to the deterministic case, the equivalence between strategy-proofness and group strategy-proofness does not extend to the probabilistic case. Thus, in probabilistic environments, some strategy-proof rules may be vulnerable to strategic misrepresentations via groups. In our probabilistic setting, the class of group strategy-proof rules is much smaller than that of strategy-proof rules.

In our main result, we characterize the class of group strategy-proof and peak-only probabilistic rules (Theorem 1).[3] We also characterize its three subclasses that additionally satisfy anonymity or unanimity or both.[4] Ehlers, Peters, and

[1] That is, the probability assigned to each upper contour set of the agent under truth-telling is no less than that under joint misrepresentation.

[2] See, for example, Barberà, Berga, and Moreno (2010).

[3] Peak-onlyness is the requirement that the rule should depend only on agents' peaks.

[4] More precisely, we characterize the class of group strategy-proof, peak-only, and anonymous rules (Corollary 1), the class of group strategy-proof and unanimous rules (Corollary 2), and the class of group strategy-proof, unanimous, and anonymous rules (Corollary 3). Anonymity is the requirement that the rule should not depend on the names of agents. Unanimity is the

Storcken (2002) characterize the class of strategy-proof and peak-only probabilistic rules, which they call "fixed-probabilistic-ballots rules." The class of rules we characterize in this paper is a subset of fixed-probabilistic-ballots rules. Our proofs also rely on their results.[5] The fixed-probabilistic-ballots rules are a probabilistic version of Moulin's (1980) well-known generalized median voter rules. Under the generalized median voter rules, a certain collection of alternatives associated with agents groups, called ballots, is fixed in advance, and then, the median of these prefixed ballots and agents' peaks is chosen as a final outcome. In the probabilistic version, a certain collection of probability distributions over alternatives is considered fixed ballots instead. Under the fixed-probabilistic-ballots rules, these prefixed probabilistic ballots behave in a similar manner to the generalized median voter rules.

What kind of fixed-probabilistic-ballots rules are group strategy-proof? To examine this, consider the case where there are two agents and three alternatives, say, 1, 2, and 3. Assume that agents have single-peaked preferences over these three alternatives. First, consider a unanimous fixed-probabilistic-ballots rule that assigns positive probabilities to *both* alternatives 1 and 3 when the reported peak of one agent is 1 and another agent's peak is 3.[6] Now, assume that the true peak of one agent is equal to 1 and another's true peak is equal to 3. Then, under this rule, because both alternatives 1 and 3 are chosen with positive probability, for each agent, the probability that the first- or second-ranked alternatives are chosen is less than one under truth-telling. However, if both agents jointly misreport their peaks as 2, because this rule is unanimous, alternative 2 is chosen with probability one. Thus, both agents can increase the probability that the first- or second-ranked alternatives are chosen by jointly misreporting their preferences. This is a violation of group strategy-proofness.

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requirement that if all agents' peaks coincide, then the rule should choose it with probability one.

[5]In a more precise sense, the model studied in their paper is slightly different from ours (in their model, the set of alternatives is the real line). However, the results obtained in their paper carry over to our setting.

[6]More precisely, we consider the fixed-probabilistic-ballots rule with the following probabilistic ballots: the probabilistic ballots associated with single-agent groups assign positive probabilities to both alternatives 1 and 3, the probabilistic ballot associated with the entire set of agents assigns probability one to alternative 1, and the probabilistic ballot associated with the empty set assigns probability one to alternative 3.

Next, consider a unanimous fixed-probabilistic-ballots rule that puts positive probabilities only on at most *two adjacent* alternatives when the reported peak of one agent is 1 and another's is 3. Then, under this rule, even if both agents jointly misreport their peaks as 2, it is impossible for at least one of them to increase the probability that the first- or second-ranked alternatives are chosen.[7] This is a key feature of the rules characterized in this paper. Our result says that group strategy-proof and unanimous rules are of this type of fixed-probabilistic-ballots rules, i.e., each associated probabilistic ballot must put positive probabilities only on at most *two adjacent* alternatives. Other classes we characterize in this paper also share a similar structure.

Many authors have analyzed strategy-proof probabilistic voting rules on the single-peaked domain. As mentioned above, Ehlers, Peters, and Storcken (2002) characterize the classes of strategy-proof probabilistic rules together with axioms such as peak-onlyness, anonymity, and unanimity. Peters, Roy, Sen, and Storcken (2014), Pycia and Ünver (2015), and Roy and Sadhukhan (2019) study extreme point characterizations. Peters, Roy, and Sadhukhan (2019) investigate strategy-proof and unanimous probabilistic rules on the domain of single-peaked preferences over graphs. Chatterji, Roy, Sadhukhan, Sen, and Zeng (2020) characterize a restricted class of fixed-probabilistic-ballots rules by strategy-proofness and unanimity on domains, which are hybrids of the single-peaked and the unrestricted domains. Characterizations of single-peaked domains and extensions to multi-dimensional settings have also been studied by authors such as Dutta, Peters, and Sen (2002), Chatterji, Sen, and Zeng (2016), and Chatterji and Zeng (2018, 2019).

By contrast, the literature on group strategy-proof probabilistic rules is still relatively small. Barberà (1979) characterizes the class of group strategy-proof probabilistic rules on the unrestricted domain.[8] Bogomolnaia, Moulin, and Stong (2005) examine group strategy-proof rules on the domain of dichotomous preferences and obtain some impossibility results. However, the class of group strategy-proof prob-

[7]For example, consider the fixed-probabilistic-ballots rule with the following probabilistic ballots: the probabilistic ballots associated with single-agent groups assign one-half to alternatives 1 and 2, the probabilistic ballot associated with the entire set of agents assigns probability one to alternative 1, and the probabilistic ballot associated with the empty set assigns probability one to alternative 3. Then, under this rule, for the agent whose true peak is equal to 1, the probability that the first- or second-ranked alternatives are chosen is equal to one under truth-telling.

[8]As is well known, Gibbard (1977) characterizes the class of strategy-proof probabilistic rules on this domain.

abilistic rules is still less well understood on many important domains. The present paper investigates group strategy-proof probabilistic rules on the single-peaked domain, which is one of the most important domains in the social choice literature.

Several authors have examined group strategy-proof rules in the random assignment and matching models. In the random assignment problem of non-identical indivisible goods, Bade (2016) shows that there is no group strategy-proof rule that satisfies ex-post efficiency and equal treatment of equals.[9] Bogomolnaia and Moulin (2004) construct a desirable group strategy-proof rule in the random matching model with dichotomous preferences.

In the multiple assignment problem of non-disposable identical indivisible goods, Hatsumi and Serizawa (2009) investigate group strategy-proof probabilistic rules defined over single-peaked and risk-averse utility functions.

This paper is organized as follows. In Section 2, we introduce the model and axioms. Our main result is provided in Section 3. Section 4 provides the proof of our main theorem. Omitted proofs are presented in the Appendix.

2 Model

Let $N = \{1, 2, \ldots, n\}$ be a finite set of agents. Let $M = \{1, 2, \ldots, m\}$ be a finite set of alternatives.

Each agent i has a single-peaked preference R_i over the set of alternatives. Let P_i denote the strict relation associated with R_i. A preference R_i is **single-peaked** if there is an alternative $p(R_i) \in M$ (called the **peak** of R_i) such that for each $x, y \in M$, if $y < x \leq p(R_i)$ or $p(R_i) \leq x < y$, then, $x \; P_i \; y$. Let $\mathcal{R}$ be the set of single-peaked preferences. Given a preference profile $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$ and a group S of agents, let $R_S = (R_i)_{i \in S}$ and $R_{-S} = (R_i)_{i \in N \setminus S}$.

A probability distribution over the set of alternatives is a list $W = (w_1, \ldots, w_m)$ of non-negative real numbers such that $\sum_{x \in M} w_x = 1$. Let $\Delta(M)$ denote the set of probability distributions over the set of alternatives. Given a probability distribution W and a subset Y of alternatives, let $W(Y)$ denote the probability that the chosen alternative under W is in Y, i.e., $W(Y) = \sum_{x \in Y} w_x$. A (probabilistic)

[9] See also Aziz and Kasajima (2017) and Zhang (2019, 2020) for further results on group strategy-proof rules in the random assignment problem.

rule is a function f from $\mathcal{R}^N$ to $\Delta(M)$.

Given a preference R_i and an alternative x, let $U(R_i, x)$ denote the **upper contour set of** R_i **at** x, i.e., $U(R_i, x) = \{y \in M : y \ R_i \ x\}$. Given two probability distributions W and W', W **stochastically dominates** W' **at** R_i if for each $x \in M$, $W(U(R_i, x)) \geq W'(U(R_i, x))$.

We next introduce the axioms. Our main axiom is group strategy-proofness. It requires that whenever a group of agents jointly misrepresents their preferences, for at least one of them, the distribution chosen under truth-telling stochastically dominates the distribution chosen under joint misrepresentation.

Group strategy-proofness: For each $R \in \mathcal{R}^N$, each $S \subseteq N$, and each $R'_S \in \mathcal{R}^S$, there is $i \in S$ such that $f(R)$ stochastically dominates $f(R'_S, R_{-S})$ at R_i.

Strategy-proofness prevents strategic misrepresentations of preferences via individual agents. Group strategy-proofness implies strategy-proofness.

Strategy-proofness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f(R)$ stochastically dominates $f(R'_i, R_{-i})$ at R_i.

Peak-onlyness requires that the rule should depend only on agents' peaks.

Peak-onlyness: For each $R, R' \in \mathcal{R}^N$, if for each $i \in N$, $p(R_i) = p(R'_i)$, then, $f(R) = f(R')$.

Unanimity requires that if all agents' peaks coincide, then the rule should choose it with probability one.

Unanimity: For each $R \in \mathcal{R}^N$ and each $x \in M$, if for each $i \in N$, $p(R_i) = x$, then, $f(R)(\{x\}) = 1$.

Finally, anonymity requires that the rule should not depend on the names of agents. Given a permutation π on N and $R \in \mathcal{R}^N$, let $R^\pi = (R_{\pi(i)})_{i \in N}$.

Anonymity: For each $R \in \mathcal{R}^N$ and each permutation π on N, $f(R) = f(R^\pi)$.

3 Main result

In this section, we provide our main characterization result. We first review the formal definition of fixed-probabilistic-ballots rules introduced in Ehlers, Peters, and Storcken (2002) and their characterization result of the class of strategy-proof and peak-only rules. We also use some basic definitions and notations used in their paper in describing our class.

Let $D = (D_S)_{S \subseteq N}$ be a collection of probability distributions over the set of alternatives satisfying the following property:

(P-1) for each $S \subseteq N$, each $S' \subseteq S$, and each $x \in M$, $D_S([1,x]) \geq D_{S'}([1,x])$.[10]

That is, for each pair S, S' of groups and each alternative x, if S contains S', the probability assigned to the interval $[1,x]$ under D_S is no less than the probability assigned to $[1,x]$ under $D_{S'}$.

Given a preference profile R, let $\bar{n}(R)$ denote the number of different peaks at R, and for each $\ell \in [1, \bar{n}(R)]$, let $p^\ell(R)$ denote the ℓ-th smallest peak at R. Let $S_\ell(R)$ denote the set of agents whose peak is less than, or equal to, the ℓ-th smallest peak at profile R, i.e., $S_\ell(R) = \{i \in N : p(R_i) \leq p^\ell(R)\}$. Let $p^0(R) = 1$, $p^{\bar{n}(R)+1}(R) = m$, and $S_0(R) = \emptyset$.

The fixed-probabilistic-ballots rule associated with $D = (D_S)_{S \subseteq N}$ determines the probability distribution over the set of alternatives as follows. For each preference profile R, the distribution over the alternatives between the ℓ-th and $\ell+1$-th smallest peaks coincides with $D_{S_\ell(R)}$, and the probability assigned to the ℓ-th smallest peak is equal to $D_{S_\ell(R)}([1, p^\ell(R)]) - D_{S_{\ell-1}(R)}([1, p^\ell(R) - 1])$. The following is the formal definition of fixed-probabilistic-ballots rules.

Fixed-probabilistic-ballots rule associated with $D = (D_S)_{S \subseteq N}$, f^D: For each $R \in \mathcal{R}^N$ and each $x \in M$,
(i) if for some $\ell \in [0, \bar{n}(R)]$, $p^\ell(R) < x < p^{\ell+1}(R)$, then,

$$f^D(R)(\{x\}) = D_{S_\ell(R)}(\{x\}),$$

(ii) if for some $\ell \in [0, \bar{n}(R)+1]$, $x = p^\ell(R)$, then,

$$f^D(R)(\{x\}) = D_{S_\ell(R)}([1,x]) - D_{S_{\ell-1}(R)}([1, x-1]).$$

The following is the characterization result of the class of strategy-proof and peak-only rules shown by Ehlers, Peters, and Storcken (2002).

[10] Given two integers x and y with $x \leq y$, let $[x,y] = \{x, x+1, \ldots, y\}$.

Theorem (Ehlers, Peters, and Storcken, 2002, Theorem 4.1). *A rule f is strategy-proof and peak-only if and only if it is a fixed-probabilistic-ballots rule, i.e., there is a collection $D = (D_S)_{S \subseteq N}$ satisfying (P-1) such that $f = f^D$.*

We next describe the class of "group" strategy-proof and peak-only rules. First, we introduce additional notations. Let $(x_S)_{S \subseteq N}$ be a collection of alternatives satisfying the following property: $x_\emptyset = m$, $x_N = 1$, and for each pair S, S' of groups, if S contains S', x_S is no greater than $x_{S'}$, i.e., for each $S \subseteq N$ and each $S' \subseteq S$, $x_S \leq x_{S'}$. Let $\mathcal{X}$ be the set of such collections.

Next, we introduce a condition on collections $D = (D_S)_{S \subseteq N}$ of probability distributions over the set of alternatives. Given a collection $(x_S)_{S \subseteq N}$ in $\mathcal{X}$, Condition 1 below says that (1) for each group S of agents, (1-a and 1-b) distribution D_S coincides with $D_\emptyset$ on the interval $[1, x_S - 1]$ and D_N on $[x_S + 2, m]$, (1-c) the probability assigned to alternative x_S under D_S is between $D_\emptyset(\{x_S\})$ and $D_N([1, x_S]) - D_\emptyset([1, x_S - 1])$, and (1-d) if S contains S' and x_S coincides with $x_{S'}$, the probability assigned to x_S under D_S is no less than the probability assigned to $x_{S'}$ under $D_{S'}$, and (2) for each alternative x, the probability assigned to the interval $[1, x]$ under D_N is no less than the probability assigned to $[1, x]$ under $D_\emptyset$.

Condition 1. Given a collection $(x_S)_{S \subseteq N}$ in $\mathcal{X}$, a collection $D = (D_S)_{S \subseteq N}$ of probability distributions over the set of alternatives satisfies Condition 1 if

(1) for each $S \subseteq N$ and each $x \in M$,

- (1-a) if $x < x_S$, $D_S(\{x\}) = D_\emptyset(\{x\})$,
- (1-b) if $x > x_S + 1$, $D_S(\{x\}) = D_N(\{x\})$,
- (1-c) if $x = x_S$, $D_\emptyset(\{x\}) \leq D_S(\{x\}) \leq D_N([1, x]) - D_\emptyset([1, x - 1])$,
- (1-d) for each $S' \subseteq S$, if $x_{S'} = x_S$, $D_{S'}(\{x_{S'}\}) \leq D_S(\{x_S\})$,

(2) for each $x \in M$, $D_N([1, x]) \geq D_\emptyset([1, x])$.

Note that if a collection D satisfies Condition 1, then it also satisfies (P-1).[11]

[11] To see this, let $S \subseteq N$ and $S' \subseteq S$. Then, $x_S \leq x_{S'}$. We show that for each $x \in M$, $D_S([1, x]) \geq D_{S'}([1, x])$. Let $x \in M$. If $x < x_S$, then, by (1-a), $D_S([1, x]) = D_\emptyset([1, x]) = D_{S'}([1, x])$. If $x = x_S < x_{S'}$, then, by (1-a) and (1-c), $D_S([1, x]) = D_\emptyset([1, x - 1]) + D_S(\{x\}) \geq D_\emptyset([1, x]) = D_{S'}([1, x])$. If $x = x_S = x_{S'}$, then, by (1-a) and (1-d), $D_S([1, x]) = D_\emptyset([1, x - 1]) + D_S(\{x\}) \geq D_\emptyset([1, x - 1]) + D_{S'}(\{x\}) = D_{S'}([1, x])$. If $x_S + 1 \leq x < x_{S'}$, then, by (1-a), (1-b), and (2), $D_S([1, x]) = D_N([1, x]) \geq D_\emptyset([1, x]) = D_{S'}([1, x])$. If $x_S + 1 \leq x = x_{S'}$, then, by (1-a), (1-b), and (1-c), $D_S([1, x]) = D_N([1, x]) \geq D_\emptyset([1, x - 1]) + D_{S'}(\{x\}) = D_{S'}([1, x])$. If $x_{S'} + 1 \leq x$, then, by (1-b), $D_S([1, x]) = D_N([1, x]) = D_{S'}([1, x])$.

Theorem 1 below is our main result. It says that a rule is group strategy-proof and peak-only if and only if it is a fixed-probabilistic-ballots rule satisfying Condition 1.

Theorem 1. *A rule is group strategy-proof and peak-only if and only if there are collections $(x_S)_{S \subseteq N}$ in $\mathcal{X}$ and $D = (D_S)_{S \subseteq N}$ satisfying Condition 1 such that it is the fixed-probabilistic-ballots rule associated with D.*

The proof of Theorem 1 is provided in the next section. The following are examples of rules characterized in Theorem 1.

Example 1. Let $N = \{1, 2, 3\}$ and $M = \{1, 2, 3, 4\}$. Let $R \in \mathcal{R}^N$ be a preference profile such that for each $i \in N$, $p(R_i) = i$.

(1) Let $x_N = 1$, $x_\emptyset = 4$, and for each $S \neq \emptyset, N$, $x_S = 2$. Then, $(x_S)_{S \subseteq N} \in \mathcal{X}$. For each $S, S' \subseteq N$ and each $x \in M$ with $|S| \leq 1 < |S'|$,[12] let

$$D_S(\{x\}) = \begin{cases} 0 & \text{if } x \in \{1,4\} \\ 1/3 & \text{if } x = 2 \\ 2/3 & \text{if } x = 3, \end{cases} \quad \text{and } D_{S'}(\{x\}) = \begin{cases} 0 & \text{if } x \in \{1,4\} \\ 2/3 & \text{if } x = 2 \\ 1/3 & \text{if } x = 3. \end{cases}$$

Then, the collection $D = (D_S)_{S \subseteq N}$ satisfies Condition 1. Let f be the fixed-probabilistic-ballots rule associated with D. Then, for each $x \in M$,

$$f(R)(\{x\}) = \begin{cases} 0 & \text{if } x \in \{1,4\} \\ 2/3 & \text{if } x = 2 \\ 1/3 & \text{if } x = 3. \end{cases}$$

(2) Let $x_{\{1,3\}} = x_N = 1$, $x_{\{1\}} = x_{\{1,2\}} = x_{\{2,3\}} = 2$, $x_{\{3\}} = 3$, and $x_\emptyset = x_{\{2\}} = 4$. Then, $(x_S)_{S \subseteq N} \in \mathcal{X}$. For each $x \in M$, let $D_\emptyset(\{x\}) = (x-1)/6 = D_{\{2\}}(\{x\})$,

[12] Given a set S, let $|S|$ denote the cardinality of S.

$D_N(\{x\}) = (5-x)/10 = D_{\{1,3\}}(\{x\})$,

$$D_{\{1\}}(\{x\}) = \begin{cases} 0 & \text{if } x = 1 \\ 1/2 & \text{if } x = 2 \\ 2/5 & \text{if } x = 3 \\ 1/10 & \text{if } x = 4, \end{cases} \text{ and } D_{\{3\}}(\{x\}) = \begin{cases} 0 & \text{if } x = 1 \\ 1/6 & \text{if } x = 2 \\ 1/2 & \text{if } x = 3 \\ 1/3 & \text{if } x = 4. \end{cases}$$

and for each $y \in \{1,4\}$, let $D_{\{1,2\}}(\{y\}) = D_{\{1\}}(\{y\}) = D_{\{2,3\}}(\{y\})$, $D_{\{1,2\}}(\{2\}) = 3/5 = D_{\{2,3\}}(\{3\})$, and $D_{\{1,2\}}(\{3\}) = 3/10 = D_{\{2,3\}}(\{2\})$.

Then, this collection $D = (D_S)_{S \subseteq N}$ also satisfies Condition 1. Let f be the fixed-probabilistic-ballots rule associated with this collection D. Then, for each $x \in M$,

$$f(R)(\{x\}) = \begin{cases} 0 & \text{if } x = 1 \\ 3/5 & \text{if } x = 2 \\ 3/10 & \text{if } x = 3 \\ 1/10 & \text{if } x = 4. \end{cases}$$

Note that peak-onlyness is indispensable in Theorem 1. The following example shows that there is a group strategy-proof rule which does not belong to the class of rules described in Theorem 1 when peak-onlyness is dropped.

Example 2. Let $N = \{1,2\}$ and $M = \{1,2,3\}$. Let f be a rule such that for each $R \in \mathcal{R}^N$ and each $x \in M$,

$$f(R)(\{x\}) = \begin{cases} 1/2 & \text{if } x \in \{p(R_1), \min \hat{p}(R_1)\} \\ 0 & \text{otherwise}, \end{cases}$$

where $\hat{p}(R_1)$ is the set of the second-ranked alternatives for R_1, i.e., $\hat{p}(R_1) = \{y \in M : y \neq p(R_1) \text{ and for each } x \neq p(R_1),\ y\ R_1\ x\}$.[13]

Then, this rule satisfies group strategy-proofness.[14] However, because the cho-

[13]Note that $\hat{p}(R_1)$ is a singleton unless alternative 1 is indifferent to 3 under R_1.

[14]To see this, let $R, R' \in \mathcal{R}^N$. We show that $f(R)$ stochastically dominates $f(R')$ at R_1. We first show that $f(R)$ stochastically dominates $f(R'_1, R_2)$ at R_1. First, since $f(R)(\{p(R_1)\}) = 1/2$ and $f(R'_1, R_2)(\{p(R_1)\}) = 0$ or $1/2$, we have $f(R)(U(R_1, p(R_1))) \geq f(R'_1, R_2)(U(R_1, p(R_1)))$. Next, let $y \in \hat{p}(R_1)$. Then, since $f(R)(U(R_1, y)) = 1$, we obtain $f(R)(U(R_1, y)) \geq$

sen probability under this rule changes when the second-ranked alternative for agent 1 is changed, f is not peak-only.

We next describe the class of group strategy-proof, peak-only, and anonymous rules. Given $(x_S)_{S\subseteq N}$ in $\mathcal{X}$ and k in $[0,n]$, let $x^{\min}(k)$ and $x^{\max}(k)$ denote the minimum and maximum of x_S among groups S with k agents, respectively, i.e., $x^{\min}(k)=\min\{x_S : |S|=k\}$ and $x^{\max}(k)=\max\{x_S : |S|=k\}$.

Condition 2 below says that for each $k\in[0,n]$ and each group S with k agents, (1) if $x^{\min}(k)$ coincides with $x^{\max}(k)$, then for each group S' with k agents, the probability assigned to alternative x_S under D_S coincides with the probability assigned to $x_{S'}$ under $D_{S'}$, (2) if the difference between $x^{\min}(k)$ and $x^{\max}(k)$ is equal to 1, then D_S coincides with $D_\emptyset$ at $x^{\min}(k)$ and D_N at $x^{\max}(k)+1$, (3) if the difference between $x^{\min}(k)$ and $x^{\max}(k)$ is more than 1, then D_S coincides with $D_\emptyset$ on the interval $[x^{\min}(k), x^{\max}(k)-1]$ and D_S coincides with D_N at $x^{\max}(k)$.

Condition 2. Given a collection $(x_S)_{S\subseteq N}$ in $\mathcal{X}$, a collection $D=(D_S)_{S\subseteq N}$ satisfies Condition 2 if for each $k\in[0,n]$ and each $S\subseteq N$ with $|S|=k$,
(1) if $x^{\min}(k)=x^{\max}(k)$, for each $S'\subseteq N$ with $|S'|=k$, $D_S(\{x_S\})=D_{S'}(\{x_{S'}\})$,
(2) if $x^{\min}(k)+1=x^{\max}(k)$,

$$D_S(\{x^{\min}(k)\})=D_\emptyset(\{x^{\min}(k)\}) \text{ and } D_S(\{x^{\max}(k)+1\})=D_N(\{x^{\max}(k)+1\}),$$

(3) if $x^{\min}(k)+1<x^{\max}(k)$, for each $x\in[x^{\min}(k), x^{\max}(k)]$,

$$D_S(\{x\})=\begin{cases} D_\emptyset(\{x\}) & \text{if } x^{\min}(k)\le x<x^{\max}(k)\\ D_N(\{x\}) & \text{if } x=x^{\max}(k).\end{cases}$$

Corollary 1 below says that a rule is group strategy-proof, peak-only, and anonymous if and only if it is a fixed-probabilistic-ballots rule satisfying Conditions 1 and 2.

Corollary 1. *A rule is group strategy-proof, peak-only, and anonymous if and*

$f(R_1',R_2)(U(R_1,y))$. Thus, for each $x\in M$, $f(R)(U(R_1,x))\ge f(R_1',R_2)(U(R_1,x))$, and so, f is strategy-proof for agent 1. Because f does not depend on agent 2's preferences, f is also strategy-proof for agent 2. Thus, it follows that $f(R')=f(R_1',R_2)$. This implies that, for each $x\in M$, $f(R)(U(R_1,x))\ge f(R_1',R_2)(U(R_1,x))=f(R')(U(R_1,x))$. Therefore, $f(R)$ stochastically dominates $f(R')$ at R_1.

only if there are collections $(x_S)_{S\subseteq N}$ in $\mathcal{X}$ and $D = (D_S)_{S\subseteq N}$ satisfying Conditions 1 and 2 such that it is the fixed-probabilistic-ballots rule associated with D.

We next describe the class of group strategy-proof and unanimous rules. Because unanimity together with strategy-proofness implies peak-onlyness in this model,[15] any group strategy-proof and unanimous rule is in the class of rules described in Theorem 1.

Let $(\alpha_S)_{S\subseteq N}$ be a collection of real numbers satisfying the following property: $\alpha_\emptyset = 1 = \alpha_N$, and for each $S \subseteq N$, $0 \leq \alpha_S \leq 1$, and if S contains S' and x_S coincides with $x_{S'}$, α_S is no less than $\alpha_{S'}$, i.e., for each $S' \subseteq S$, if $x_{S'} = x_S$, $\alpha_{S'} \leq \alpha_S$. Let $\mathcal{A}$ be the set of such collections.

Given $X = (x_S)_{S\subseteq N}$ in $\mathcal{X}$ and $A = (\alpha_S)_{S\subseteq N}$ in $\mathcal{A}$, let $D^{X,A} = (D_S^{X,A})_{S\subseteq N}$ be the collection of probability distributions defined as follows: for each $S \subseteq N$, $D_S^{X,A}$ puts probability α_S on alternative x_S and $1 - \alpha_S$ on $x_S + 1$. Then, consider the fixed-probabilistic-ballots rule associated with $D^{X,A}$. Corollary 2 below says that the class of group strategy-proof and unanimous rules coincides with the class of fixed-probabilistic-ballots rules defined in this way.

Corollary 2. *A rule is group strategy-proof and unanimous if and only if there is a pair (X, A) in $\mathcal{X} \times \mathcal{A}$ such that it is the fixed-probabilistic-ballots rule associated with $D^{X,A}$.*

Finally, we obtain the following characterization of the class of group strategy-proof, unanimous, and anonymous rules. Corollary 3 below says that a rule is group strategy-proof, unanimous, and anonymous if and only if it is the fixed-probabilistic-ballots rule associated with $D^{X,A}$ satisfying the following additional property: for each pair S, S' of groups with the same number of agents, x_S and α_S coincide with $x_{S'}$ and $\alpha_{S'}$, respectively.

Corollary 3. *A rule is group strategy-proof, unanimous, and anonymous if and only if there is a pair (X, A) in $\mathcal{X} \times \mathcal{A}$ satisfying for each $S, S' \subseteq N$ with $|S| = |S'|$, $x_S = x_{S'}$ and $\alpha_S = \alpha_{S'}$ such that it is the fixed-probabilistic-ballots rule associated with $D^{X,A}$.*

[15] See, for example, Ehlers, Peters, and Storcken (2002) and Chatterji and Zeng (2018).

4 Proof of Theorem 1

In this section, we provide the proof of our main characterization result (Theorem 1). The proofs of Corollaries 1, 2, and 3 are provided in the Appendix.

We first prove the "only if" part of Theorem 1. The next fact easily follows from the definition of fixed-probabilistic-ballots rules.

Fact 1. *Let f be the fixed-probabilistic-ballots rule associated with $D = (D_S)_{S\subseteq N}$. Let $x \in M$ and let R be a preference profile such that each agent's peak is equal to alternative x. Then, for each $y \in M$,*

$$f(R)(\{y\}) = \begin{cases} D_\emptyset(\{y\}) & \text{if } y \leq x-1 \\ D_N([1,x]) - D_\emptyset([1,x-1]) & \text{if } y = x \\ D_N(\{y\}) & \text{if } y \geq x+1. \end{cases}$$

Proof of Fact 1. Let $y \in M$. If $y < x$, then, by $y < p^1(R)$, $f(R)(\{y\}) = D_\emptyset(\{y\})$. If $y > x$, then, by $y > p^{\bar{n}(R)}(R)$, $f(R)(\{y\}) = D_N(\{y\})$. Finally, since $f(R)([1,x-1]) = D_\emptyset([1,x-1])$ and $f(R)([x+1,m]) = D_N([x+1,m])$, $f(R)(\{x\}) = 1 - D_\emptyset([1,x-1]) - D_N([x+1,m]) = D_N([1,x]) - D_\emptyset([1,x-1])$. □

We are now ready to provide the proof of the "only if" part of Theorem 1.

"Only if" part. Let f be a group strategy-proof and peak-only rule. Then, by Theorem 4.1 in Ehlers, Peters, and Storcken (2002), f is a fixed-probabilistic-ballots rule, i.e., there is a collection $D = (D_S)_{S\subseteq N}$ satisfying (P-1) such that f coincides with the fixed-probabilistic-ballots rule associated with D. Then, (2) of Condition 1 follows from (P-1).

In the next six steps, we prove (1) of Condition 1, i.e., we show that there is a collection $(x_S)_{S\subseteq N}$ in $\mathcal{X}$ such that for each $S \subseteq N$ and each $x \in M$,

(1-a) if $x < x_S$, $D_S(\{x\}) = D_\emptyset(\{x\})$,
(1-b) if $x > x_S + 1$, $D_S(\{x\}) = D_N(\{x\})$,
(1-c) if $x = x_S$, $D_\emptyset(\{x\}) \leq D_S(\{x\}) \leq D_N([1,x]) - D_\emptyset([1,x-1])$, and
(1-d) for each $S' \subseteq S$, if $x_{S'} = x_S$, $D_{S'}(\{x_{S'}\}) \leq D_S(\{x_S\})$.

For each $i \in N$, let R_i^1 and R_i^m be the preference of agent i whose peak is equal to alternative 1 and m, respectively. Then, by the definition of fixed-probabilistic-ballots rules, for each $S \subseteq N$, $D_S = f(R^1_S, R^m_{-S})$.

In Step 1, we construct a collection $(x_S)_{S \subseteq N}$. We define it as follows. Let $x_N = 1$, and for each $S \subsetneq N$, if there are alternatives at which distribution D_S differs from $D_\emptyset$, let x_S be the minimum of them, and if D_S coincides with $D_\emptyset$ at each alternative, let $x_S = m$. Then, (1-a) follows from the definition of $(x_S)_{S \subseteq N}$.

Step 1. *Construction of* $(x_S)_{S \subseteq N}$.

For each $S \subseteq N$, let $E_S = \{x \in M : D_S(\{x\}) \neq D_\emptyset(\{x\})\}$. Let $x_N = 1$, and for each $S \subsetneq N$, let

$$x_S = \begin{cases} \min E_S & \text{if } E_S \neq \emptyset \\ m & \text{if } E_S = \emptyset. \end{cases}$$

Then, $x_\emptyset = m$. Note also that, for each $S \subseteq N$, since $D_S(M) = 1 = D_\emptyset(M)$, if $E_S \neq \emptyset$, then, $x_S \leq m - 1$.

In Step 2, we prove that for each $S \subseteq N$, $D_S(\{x_S\})$ is no less than $D_\emptyset(\{x_S\})$. In the proof of Step 2, we show that if $D_S(\{x_S\})$ is less than $D_\emptyset(\{x_S\})$, then f is jointly manipulable via group S.

Step 2. *For each* $S \subseteq N$, $D_\emptyset(\{x_S\}) \leq D_S(\{x_S\})$.

Let $S \subseteq N$. Suppose by contradiction that $D_S(\{x_S\}) < D_\emptyset(\{x_S\})$. Then, $E_S \neq \emptyset$, and by (1-a), $f(R^1_S, R^m_{-S})([1, x_S - 1]) = D_S([1, x_S - 1]) = D_\emptyset([1, x_S - 1])$. Thus, for each $i \in S$,

$$\begin{aligned} f(R^1_S, R^m_{-S})(U(R^m_i, x_S + 1)) &= f(R^1_S, R^m_{-S})([x_S + 1, m]) \text{ (by } U(R^m_i, x_S) = [x_S + 1, m]) \\ &= 1 - D_\emptyset([1, x_S - 1]) - D_S(\{x_S\}) \\ &> 1 - D_\emptyset([1, x_S - 1]) - D_\emptyset(\{x_S\}) \text{ (by } D_S(\{x_S\}) < D_\emptyset(\{x_S\})) \\ &= f(R^m_N)([x_S + 1, m]) \text{ (by } f(R^m_N) = D_\emptyset) \\ &= f(R^m_N)(U(R^m_i, x_S + 1)). \end{aligned}$$

Thus, group S can manipulate $f(R^m_N)$ via R^1_S, which violates group strategy-proofness.

In Step 3, we prove (1-b). In the proof of Step 3, we first show that D_S coincides with D_N at m. As shown below, if $D_S(\{m\})$ is less than $D_N(\{m\})$, then

f is jointly manipulable via group $N \setminus S$, and if $D_S(\{m\})$ is greater than $D_N(\{m\})$, then f is jointly manipulable via the entire set of agents. By applying a similar argument inductively, we can also show that D_S coincides with D_N on the interval $[x_S + 2, m-1]$.

Step 3. *For each $S \subseteq N$ and each $x \in M$, if $x > x_S + 1$, $D_S(\{x\}) = D_N(\{x\})$.*

Let $S \subseteq N$ be such that $x_S + 1 < m$. If $S = N$, then, by $D_S = D_N$, Step 3 holds. Thus, assume that $S \neq N$. Since $x_S + 1 < m$, $E_S \neq \emptyset$. Then, by the definition of x_S, $D_\emptyset(\{x_S\}) \neq D_S(\{x_S\})$. Thus, by Step 2, $D_\emptyset(\{x_S\}) < D_S(\{x_S\})$.

First, we show $D_S(\{m\}) = D_N(\{m\})$. Suppose by contradiction that $D_S(\{m\}) < D_N(\{m\})$. Then, for each $i \in N \setminus S$,

$$\begin{aligned}
f(R_S^1, R_{-S}^m)(U(R_i^1, m-1)) &= f(R_S^1, R_{-S}^m)([1, m-1]) \text{ (by } U(R_i^1, m-1) = [1, m-1]) \\
&= 1 - D_S(\{m\}) \text{ (by } f(R_S^1, R_{-S}^m) = D_S) \\
&> 1 - D_N(\{m\}) \text{ (by } D_S(\{m\}) < D_N(\{m\})) \\
&= f(R_N^1)([1, m-1]) \text{ (by } f(R_N^1) = D_N) \\
&= f(R_N^1)(U(R_i^1, m-1)).
\end{aligned}$$

Thus, group $N \setminus S$ can manipulate $f(R_N^1)$ via R_{-S}^m, which violates group strategy-proofness.

Next, suppose that $D_S(\{m\}) > D_N(\{m\})$. For each $i \in N$, let R_i^{m-1} be a preference of agent i whose peak is equal to alternative $m-1$. Then, for each $i \in S$,

$$\begin{aligned}
f(R_S^1, R_{-S}^m)(U(R_i^1, m-1)) &= f(R_S^1, R_{-S}^m)([1, m-1]) \\
&= 1 - D_S(\{m\}) \text{ (by } f(R_S^1, R_{-S}^m) = D_S) \\
&< 1 - D_N(\{m\}) \text{ (by } D_S(\{m\}) > D_N(\{m\})) \\
&= f(R_N^{m-1})([1, m-1]) \text{ (by Fact 1)} \\
&= f(R_N^{m-1})(U(R_i^1, m-1)).
\end{aligned}$$

Also, for each $i \in N \setminus S$,

$$\begin{aligned} f(R^1_S, R^m_{-S})(U(R^m_i, x_S+1)) &= f(R^1_S, R^m_{-S})([x_S+1, m]) \\ &= 1 - D_\emptyset([1, x_S-1]) - D_S(\{x_S\}) \\ &< 1 - D_\emptyset([1, x_S-1]) - D_\emptyset(\{x_S\}) \text{ (by } D_S(\{x_S\}) > D_\emptyset(\{x_S\})) \\ &= f(R^{m-1}_N)([x_S+1, m]) \text{ (by Fact 1)} \\ &= f(R^{m-1}_N)(U(R^m_i, x_S+1)). \end{aligned}$$

Thus, group N can manipulate $f(R^1_S, R^m_{-S})$ via R^{m-1}_N, which violates group strategy-proofness.

Next, let $x \in [x_S+2, m-1]$. As the induction hypothesis, assume that for each $y \geq x+1$, $D_S(\{y\}) = D_N(\{y\})$. Then, $f(R^1_S, R^m_{-S})([x+1, m]) = D_N([x+1, m])$. We show $D_S(\{x\}) = D_N(\{x\})$.

Suppose by contradiction that $D_S(\{x\}) < D_N(\{x\})$. Then, for each $i \in N \setminus S$,

$$\begin{aligned} f(R^1_S, R^m_{-S})(U(R^1_i, x-1)) &= f(R^1_S, R^m_{-S})([1, x-1]) \\ &= 1 - D_S(\{x\}) - D_N([x+1, m]) \\ &> 1 - D_N(\{x\}) - D_N([x+1, m]) \text{ (by } D_S(\{x\}) < D_N(\{x\})) \\ &= f(R^1_N)([1, x-1]) \text{ (by } f(R^1_N) = D_N) \\ &= f(R^1_N)(U(R^1_i, x-1)). \end{aligned}$$

Thus, group $N \setminus S$ can manipulate $f(R^1_N)$ via R^m_{-S}, which violates group strategy-proofness.

Next, suppose that $D_S(\{x\}) > D_N(\{x\})$. For each $i \in N$, let R^{x-1}_i be a preference of agent i whose peak is equal to alternative $x-1$. Then, for each $i \in S$,

$$\begin{aligned} f(R^1_S, R^m_{-S})(U(R^1_i, x-1)) &= f(R^1_S, R^m_{-S})([1, x-1]) \\ &= 1 - D_S(\{x\}) - D_N([x+1, m]) \\ &< 1 - D_N(\{x\}) - D_N([x+1, m]) \text{ (by } D_S(\{x\}) > D_N(\{x\})) \\ &= f(R^{x-1}_N)([1, x-1]) \text{ (by Fact 1)} \\ &= f(R^{x-1}_N)(U(R^1_i, x-1)). \end{aligned}$$

Also, for each $i \in N \setminus S$,

$$\begin{aligned} f(R^1_S, R^m_{-S})(U(R^m_i, x_S+1)) &= f(R^1_S, R^m_{-S})([x_S+1, m]) \\ &= 1 - D_\emptyset([1, x_S-1]) - D_S(\{x_S\}) \\ &< 1 - D_\emptyset([1, x_S-1]) - D_\emptyset(\{x_S\}) \text{ (by } D_S(\{x_S\}) > D_\emptyset(\{x_S\})) \\ &= f(R^{x-1}_N)([x_S+1, m]) \text{ (by Fact 1)} \\ &= f(R^{x-1}_N)(U(R^m_i, x_S+1)). \end{aligned}$$

Thus, group N can manipulate $f(R^1_S, R^m_{-S})$ via R^{x-1}_N, which violates group strategy-proofness. Thus, we conclude that $D_S(\{x\}) = D_N(\{x\})$.

In Step 4, we prove that for each $S \subsetneq N$, $D_S(\{x_S\})$ is no greater than $D_N([1, x_S]) - D_\emptyset([1, x_S-1])$. In the proof of Step 4, we show that if $D_S(\{x_S\})$ is greater than $D_N([1, x_S]) - D_\emptyset([1, x_S-1])$, then f is jointly manipulable via the entire set of agents. Step 4 together with Step 2 implies (1-c).

Step 4. *For each $S \subseteq N$, $D_S(\{x_S\}) \leq D_N([1, x_S]) - D_\emptyset([1, x_S-1])$.*

Let $S \subsetneq N$ and $x = x_S$. If $S = \emptyset$ or $S = N$, then Step 4 holds. Thus, assume that $S \neq \emptyset, N$. For each $i \in N$, let R^x_i be a preference of agent i whose peak is equal to alternative x. By contradiction, suppose that $D_S(\{x\}) > D_N([1, x]) - D_\emptyset([1, x-1])$. Then, by Fact 1, $f(R^x_N)(\{x\}) = D_N([1, x]) - D_\emptyset([1, x-1]) < D_S(\{x\})$. Thus, for each $i \in N$, $f(R^1_S, R^m_{-S})(U(R^x_i, x)) = D_S(\{x\}) > f(R^x_N)(\{x\}) = f(R^x_N)(U(R^x_i, x))$. Hence, group N can manipulate $f(R^x_N)$ via (R^1_S, R^m_{-S}), which violates group strategy-proofness.

In Step 5, we prove that (1-d) follows from (P-1) and (1-a).

Step 5. *For each $S \subseteq N$ and each $S' \subseteq S$, if $x_{S'} = x_S$, $D_{S'}(\{x_{S'}\}) \leq D_S(\{x_S\})$.*

Let $S \subseteq N$ and $S' \subseteq S$ be such that $x_{S'} = x_S$. We show that $D_{S'}(\{x_{S'}\}) \leq D_S(\{x_S\})$. It follows from (P-1) that for each $x \in M$, $D_S([1, x]) \geq D_{S'}([1, x])$. If $x_S = 1$, then, $D_S(\{x_S\}) = D_S(\{1\}) \geq D_{S'}(\{1\}) = D_{S'}(\{x_{S'}\})$. If $x_S > 1$, then,

$$\begin{aligned} D_S(\{x_S\}) &= D_S([1, x_S]) - D_\emptyset([1, x_S-1]) \text{ (by (1-a))} \\ &\geq D_{S'}([1, x_{S'}]) - D_\emptyset([1, x_{S'}-1]) \\ &= D_{S'}(\{x_{S'}\}) \text{ (by (1-a))}. \end{aligned}$$

Finally, in Step 6, we show that the constructed collection $(x_S)_{S\subseteq N}$ is in $\mathcal{X}$, i.e., for each $S \subseteq N$ and each $S' \subseteq S$, $x_S \leq x_{S'}$. As shown below, if $x_S > x_{S'}$, then f is jointly manipulable via group $S \setminus S'$.

Step 6. *For each $S \subseteq N$ and each $S' \subseteq S$, $x_S \leq x_{S'}$.*

Let $S \subseteq N$ and $S' \subsetneq S$. If $E_{S'} = \emptyset$, then, $x_{S'} = m \geq x_S$. Thus, assume that $E_{S'} \neq \emptyset$. Then, $S' \neq \emptyset$. Since $S' \subsetneq S$, $S' \neq N$. By the definition of $x_{S'}$, $D_\emptyset(\{x_{S'}\}) \neq D_{S'}(\{x_{S'}\})$. Thus, by Step 2, $D_\emptyset(\{x_{S'}\}) < D_{S'}(\{x_{S'}\})$. Suppose by contradiction that $x_{S'} < x_S$. Then, for each $i \in S \setminus S'$,

$$\begin{aligned}
f(R^1_{S'}, R^m_{-S'})(U(R^m_i, x_{S'}+1)) &= f(R^1_{S'}, R^m_{-S'})([x_{S'}+1, m]) \\
&= 1 - D_\emptyset([1, x_{S'}-1]) - D_{S'}(\{x_{S'}\}) \\
&< 1 - D_\emptyset([1, x_{S'}-1]) - D_\emptyset(\{x_{S'}\}) \text{ (by } D_{S'}(\{x_{S'}\}) > D_\emptyset(\{x_{S'}\})) \\
&= 1 - D_\emptyset([1, x_{S'}]) \\
&= f(R^1_S, R^m_{-S})([x_{S'}+1, m]) \text{ (by (1-a) and } x_S > x_{S'}) \\
&= f(R^1_S, R^m_{-S})(U(R^m_i, x_{S'}+1)).
\end{aligned}$$

Thus, group $S \setminus S'$ can manipulate $f(R^1_{S'}, R^m_{-S'})$ via $R^1_{S\setminus S'}$, which violates group strategy-proofness.

□

We next prove the "if" part of Theorem 1. Before presenting the proof, we introduce one addition fact that is useful in our proof.

Let $(x_S)_{S\subseteq N} \in \mathcal{X}$. Then, for each $R \in \mathcal{R}^N$, there is a unique number $\ell^*(R) \in [1, \bar{n}(R)]$ such that $p^{\ell^*(R)}(R) \leq x_{S_{\ell^*(R)-1}(R)}$ and $x_{S_{\ell^*(R)}(R)} < p^{\ell^*(R)+1}(R)$.[16]

Let $R \in \mathcal{R}^N$ and $\ell = \ell^*(R)$. Let $D = (D_S)_{S\subseteq N}$ satisfy Condition 1. Fact 2 below says that the probability distribution chosen under the fixed-probabilistic-ballots rule f associated with D at the profile R is determined as follows: (a) if the ℓ-th smallest peak is less than, or equal to, $x_{S_\ell(R)}$, then the distribution coincides

[16]To see this, let $R \in \mathcal{R}^N$. Suppose that for each $\ell \in [1, \bar{n}(R)]$, $p^\ell(R) > x_{S_{\ell-1}(R)}$ or $p^{\ell+1}(R) \leq x_{S_\ell(R)}$. Since $x_\emptyset = m \geq p^1(R)$, $p^2(R) \leq x_{S_1(R)}$. This implies that $p^3(R) \leq x_{S_2(R)}$. Repeating this, $p^{\bar{n}(R)+1}(R) \leq x_{S_{\bar{n}(R)}(R)}$. However, since $x_{S_{\bar{n}(R)}(R)} = x_N = 1$ and $p^{\bar{n}(R)+1}(R) = m$, this is a contradiction. Thus, there is $\ell \in [1, \bar{n}(R)]$ such that $p^\ell(R) \leq x_{S_{\ell-1}(R)}$ and $x_{S_\ell(R)} < p^{\ell+1}(R)$.

Next, let $\ell' \in [1, \bar{n}(R)]$. If $\ell < \ell'$, then, $x_{S_{\ell'-1}(R)} \leq x_{S_\ell(R)} < p^{\ell+1}(R) \leq p^{\ell'}(R)$. If $\ell > \ell'$, then, $p^{\ell'+1}(R) \leq p^\ell(R) \leq x_{S_{\ell-1}(R)} \leq x_{S_{\ell'}(R)}$. Thus, for each $\ell' \in [1, \bar{n}(R)]$, if $\ell' \neq \ell$, $x_{S_{\ell'-1}(R)} < p^{\ell'}(R)$ or $p^{\ell'+1}(R) \leq x_{S_{\ell'}(R)}$.

with $D_\emptyset$ on the interval $[1, x_{S_\ell(R)} - 1]$ and D_N on $[x_{S_\ell(R)} + 2, m]$, the probability assigned to alternative $x_{S_\ell(R)}$ is equal to $D_{S_\ell(R)}(\{x_{S_\ell(R)}\})$, and the remaining probability is assigned to alternative $x_{S_\ell(R)} + 1$, and (b) if the ℓ-th smallest peak is greater than $x_{S_\ell(R)}$, then the distribution coincides with $D_\emptyset$ on $[1, p^\ell(R) - 1]$ and D_N on $[p^\ell(R)+1, m]$, and the remaining probability is assigned to the ℓ-th smallest peak.

Fact 2. *Let $(x_S)_{S\subseteq N} \in \mathcal{X}$ and let $D = (D_S)_{S\subseteq N}$ satisfy Condition 1. Let f be the fixed-probabilistic-ballots rule associated with D. Then, for each $R \in \mathcal{R}^N$,*

(a) if $p^\ell(R) \leq x_{S_\ell(R)}$, then,

$$f(R)(\{x\}) = \begin{cases} D_\emptyset(\{x\}) & \text{if } x < x_{S_\ell(R)} \\ D_{S_\ell(R)}(\{x\}) & \text{if } x = x_{S_\ell(R)} \\ D_N([1,x]) - D_\emptyset([1,x-2]) - D_{S_\ell(R)}(\{x-1\}) & \text{if } x = x_{S_\ell(R)} + 1 \\ D_N(\{x\}) & \text{if } x > x_{S_\ell(R)} + 1, \end{cases}$$

(b) if $x_{S_\ell(R)} < p^\ell(R)$, then,

$$f(R)(\{x\}) = \begin{cases} D_\emptyset(\{x\}) & \text{if } x < p^\ell(R) \\ D_N([1,x]) - D_\emptyset([1,x-1]) & \text{if } x = p^\ell(R) \\ D_N(\{x\}) & \text{if } x > p^\ell(R), \end{cases}$$

where $\ell = \ell^*(R)$.

The proof of Fact 2 is in the Appendix. We are now ready to provide the proof of the "if" part of Theorem 1.

"If" part. Let $(x_S)_{S\subseteq N} \in \mathcal{X}$ and let $D = (D_S)_{S\subseteq N}$ satisfy Condition 1. Let f be the fixed-probabilistic-ballots rule associated with D. By the definition of fixed-probabilistic-ballots rules, f is peak-only. We show that f is group strategy-proof.

Let $R, R' \in \mathcal{R}^N$ and $S \subseteq N$ be such that $R_{-S} = R'_{-S}$. In the next six steps, we show that for at least one agent in group S, $f(R)$ stochastically dominates $f(R')$. Let $\ell = \ell^*(R)$ and $\ell' = \ell^*(R')$. Then, $p^\ell(R) \leq x_{S_{\ell-1}(R)}$ and $x_{S_\ell(R)} < p^{\ell+1}(R)$, and

$p^{\ell'}(R') \leq x_{S_{\ell'-1}(R')}$ and $x_{S_{\ell'}(R')} < p^{\ell'+1}(R')$. Let $L(R) = \max\{p^\ell(R), x_{S_\ell(R)}\}$ and $L(R') = \max\{p^{\ell'}(R'), x_{S_{\ell'}(R')}\}$.

Step 1 says that (a) if $L(R)$ is no greater than $x_{S_{\ell'}(R')}$ and the peak of each agent in group S is greater than the ℓ-th smallest peak at R, then the peak of each agent in $S_\ell(R)$ is no greater than the ℓ'-th smallest peak at R', and (b) if $L(R')$ is no greater than $x_{S_\ell(R)}$ and the peak of each agent in group S is less than the $\ell+1$-th smallest peak at R, then the peak of each agent in $S_{\ell'}(R')$ is no greater than the ℓ-th smallest peak at R.

Step 1. *(a) If $L(R) \leq x_{S_{\ell'}(R')}$ and $S \cap S_\ell(R) = \emptyset$, then, $S_\ell(R) \subseteq S_{\ell'}(R')$.*
(b) If $L(R') \leq x_{S_\ell(R)}$ and $S \cap (N \setminus S_\ell(R)) = \emptyset$, then, $S_\ell(R) \supseteq S_{\ell'}(R')$.

(a) Assume that $L(R) \leq x_{S_{\ell'}(R')}$ and $S \cap S_\ell(R) = \emptyset$. Then, for each $i \in S_\ell(R)$, $R_i = R'_i$. Thus, if $p^{\ell'}(R') < p^\ell(R)$, then, $p^{\ell'+1}(R') \leq p^\ell(R) \leq L(R) \leq x_{S_{\ell'}(R')} < p^{\ell'+1}(R')$, which is a contradiction. Hence, $p^\ell(R) \leq p^{\ell'}(R')$. This implies that $S_\ell(R) \subseteq S_{\ell'}(R')$.

(b) Assume that $L(R') \leq x_{S_\ell(R)}$ and $S \cap (N \setminus S_\ell(R)) = \emptyset$. Then, for each $i \in N \setminus S_\ell(R)$, $R_i = R'_i$. Thus, if $p^{\ell+1}(R) < p^{\ell'+1}(R')$, then, $p^{\ell'}(R') \geq p^{\ell+1}(R) > x_{S_\ell(R)} \geq L(R') \geq p^{\ell'}(R')$, which is a contradiction. Hence, $p^{\ell'+1}(R') \leq p^{\ell+1}(R)$. This implies that $N \setminus S_\ell(R) \subseteq N \setminus S_{\ell'}(R')$, and so, $S_\ell(R) \supseteq S_{\ell'}(R')$.

Let

$$\hat{S}_\ell(R) = \begin{cases} N \setminus S_{\ell-1}(R) & \text{if } x_{S_\ell(R)} < p^\ell(R) \\ N \setminus S_\ell(R) & \text{if } x_{S_\ell(R)} \geq p^\ell(R). \end{cases}$$

Step 2 says that (a) if $L(R)$ is less than $L(R')$, then at least one agent in group S belongs to $S_\ell(R)$, and (b) if $L(R')$ is less than $L(R)$, then at least one agent in group S belongs to $\hat{S}_\ell(R)$.

Step 2. *(a) If $L(R) < L(R')$, then, $S \cap S_\ell(R) \neq \emptyset$.*
(b) If $L(R') < L(R)$, then, $S \cap \hat{S}_\ell(R) \neq \emptyset$.

(a) Assume that $L(R) < L(R')$. By contradiction, suppose that $S \cap S_\ell(R) = \emptyset$.

Case 1. $x_{S_{\ell'}(R')} < p^{\ell'}(R')$.

Since $L(R) < L(R')$, $p^\ell(R) \leq L(R) < L(R') = p^{\ell'}(R')$. Thus, $p^\ell(R) \leq p^{\ell'-1}(R')$. Then, $S_\ell(R) \subseteq S_{\ell'-1}(R')$. Thus, $x_{S_{\ell'-1}(R')} \leq x_{S_\ell(R)} \leq L(R) < L(R') = p^{\ell'}(R') \leq x_{S_{\ell'-1}(R')}$. This is a contradiction.

Case 2. $p^{\ell'}(R) \leq x_{S_{\ell'}(R')}$.

Then, $L(R) < L(R') = x_{S_{\ell'}(R')}$. Then, by Step 1-(a), $S_\ell(R) \subseteq S_{\ell'}(R')$. Thus, $x_{S_{\ell'}(R')} \leq x_{S_\ell(R)} \leq L(R) < L(R') = x_{S_{\ell'}(R')}$. This is a contradiction.

(b) Assume that $L(R') < L(R)$. By contradiction, suppose that $S \cap \hat{S}_\ell(R) = \emptyset$.

Case 1. $x_{S_\ell(R)} < p^\ell(R)$.

Then, $S \cap (N \setminus S_{\ell-1}(R)) = \emptyset$. Since $L(R') < L(R)$, $p^{\ell'}(R') \leq L(R') < L(R) = p^\ell(R)$. Thus, $p^{\ell'+1}(R) \leq p^\ell(R)$. Then, $N \setminus S_{\ell'}(R') \supseteq N \setminus S_{\ell-1}(R)$, and so, $S_{\ell'}(R') \subseteq S_{\ell-1}(R)$. Thus, $x_{S_{\ell-1}(R)} \leq x_{S_{\ell'}(R')} \leq L(R') < L(R) = p^\ell(R) \leq x_{S_{\ell-1}(R)}$. This is a contradiction.

Case 2. $p^\ell(R) \leq x_{S_\ell(R)}$.

Then, $S \cap (N \setminus S_\ell(R)) = \emptyset$. Since $L(R') < L(R) = x_{S_\ell(R)}$, by Step 1-(b), $S_\ell(R) \supseteq S_{\ell'}(R')$. Thus, $x_{S_\ell(R)} \leq x_{S_{\ell'}(R')} \leq L(R') < L(R) = x_{S_\ell(R)}$. This is a contradiction.

Step 3 says that when $L(R)$ coincides with $L(R')$, (a) if $x_{S_\ell(R)}$ is less than the ℓ-th smallest peak at R and $x_{S_{\ell'}(R')}$ is no less than the ℓ'-th smallest peak at R', then at least one agent in group S belongs to $S_\ell(R)$, and (b) if $x_{S_\ell(R)}$ is no less than the ℓ-th smallest peak at R and $x_{S_{\ell'}(R')}$ is less than the ℓ'-th smallest peak at R', then at least one agent in group S belongs to $\hat{S}_\ell(R)$.

Step 3. *Assume that* $L(R) = L(R')$.
(a) If $x_{S_\ell(R)} < p^\ell(R)$ *and* $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}$*, then,* $S \cap S_\ell(R) \neq \emptyset$.
(b) If $x_{S_\ell(R)} \geq p^\ell(R)$ *and* $p^{\ell'}(R') > x_{S_{\ell'}(R')}$*, then,* $S \cap \hat{S}_\ell(R) \neq \emptyset$.

(a) Assume that $x_{S_\ell(R)} < p^\ell(R)$ and $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}$. Then, $L(R) = L(R') = x_{S_{\ell'}(R')}$. Suppose that $S \cap S_\ell(R) = \emptyset$. Then, by Step 1-(a), $S_\ell(R) \subseteq S_{\ell'}(R')$. Thus, $x_{S_{\ell'}(R')} \leq x_{S_\ell(R)} < p^\ell(R) = L(R) = L(R') = x_{S_{\ell'}(R')}$. This is a contradiction.

(b) Assume that $x_{S_\ell(R)} \geq p^\ell(R)$ and $p^{\ell'}(R') > x_{S_{\ell'}(R')}$. Then, $L(R') = L(R) = x_{S_\ell(R)}$. Suppose that $S \cap (N \setminus S_\ell(R)) = \emptyset$. Then, by Step 1-(b), $S_\ell(R) \supseteq S_{\ell'}(R')$. Thus, $x_{S_\ell(R)} \leq x_{S_{\ell'}(R')} < p^{\ell'}(R') = L(R') = L(R) = x_{S_\ell(R)}$. This is a contradiction.

Step 4 says that if one of the following (4-a), (4-b), and (4-c) holds, for each agent in $S_\ell(R)$, $f(R)$ stochastically dominates $f(R')$.

Step 4. *Assume that (4-a)* $L(R) < L(R')$*, (4-b)* $L(R) = L(R')$*,* $x_{S_\ell(R)} < p^\ell(R)$*, and* $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}$*, or (4-c)* $L(R) = L(R')$*,* $p^\ell(R) \leq x_{S_\ell(R)}$*,* $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}$*,* *and* $D_{S_\ell(R)}(\{L(R)\}) \geq D_{S_{\ell'}(R')}(\{L(R')\})$. *Then, for each* $i \in S_\ell(R)$, $f(R)$ *stochastically dominates* $f(R')$ *at* R_i.

Let $i \in S_\ell(R)$, $x_i \in M$, and $[y_i, z_i] = U(R_i, x_i)$. We show that $f(R)([y_i, z_i]) \geq f(R')([y_i, z_i])$. Recall Fact 2.

Case 1. $z_i < L(R)$.

Then, for each $x \leq z_i$, $f(R)(\{x\}) = D_\emptyset(\{x\})$. Thus, $f(R)([y_i, z_i]) = D_\emptyset([y_i, z_i])$. Since $L(R) \leq L(R')$, it also follows that $f(R')([y_i, z_i]) = D_\emptyset([y_i, z_i])$. Thus, $f(R)([y_i, z_i]) = f(R')([y_i, z_i])$.

Case 2. $z_i = L(R)$ and $p^\ell(R) \leq x_{S_\ell(R)}$.

Then, for each $x \leq z_i - 1$, $f(R)(\{x\}) = D_\emptyset(\{x\})$ and $f(R)(\{z_i\}) = D_{S_\ell(R)}(\{z_i\})$. Thus, $f(R)([y_i, z_i]) = D_\emptyset([y_i, z_i - 1]) + D_{S_\ell(R)}(\{z_i\})$.

Subcase 2-1. $L(R) < L(R')$.

Then, $f(R')([y_i, z_i]) = D_\emptyset([y_i, z_i])$. By (1-c) of Condition 1, $D_{S_\ell(R)}(\{z_i\}) \geq D_\emptyset(\{z_i\})$. Thus, $f(R)([y_i, z_i]) \geq f(R')([y_i, z_i])$.

Subcase 2-2. $L(R) = L(R')$ and $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}$.

Then, $f(R')([y_i, z_i]) = D_\emptyset([y_i, z_i - 1]) + D_{S_{\ell'}(R')}(\{z_i\})$. Since $D_{S_\ell(R)}(\{z_i\}) \geq D_{S_{\ell'}(R')}(\{z_i\})$ by (4-c), it follows that $f(R)([y_i, z_i]) \geq f(R')([y_i, z_i])$.

Case 3. $z_i > L(R)$ or $[z_i = L(R)$ and $x_{S_\ell(R)} < p^\ell(R)]$.

Then, for each $x \leq y_i - 1$, $f(R)(\{x\}) = D_\emptyset(\{x\})$, and for each $x \geq z_i + 1$, $f(R)(\{x\}) = D_N(\{x\})$. Thus, $f(R)([y_i, z_i]) = 1 - D_N([z_i + 1, m]) - D_\emptyset([1, y_i - 1]) = D_N([1, z_i]) - D_\emptyset([1, y_i - 1])$.

Subcase 3-1. $z_i < L(R')$.

Then, $f(R')([y_i, z_i]) = D_\emptyset([y_i, z_i])$. By (2) of Condition 1, $D_N([1, z_i]) \geq D_\emptyset([1, z_i])$. This implies that $f(R)([y_i, z_i]) = D_N([1, z_i]) - D_\emptyset([1, y_i - 1]) \geq D_\emptyset([1, z_i]) - D_\emptyset([1, y_i - 1]) = f(R')([y_i, z_i])$.

Subcase 3-2. $z_i = L(R')$ and $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}$.

Then, $f(R')([y_i, z_i]) = D_\emptyset([y_i, z_i - 1]) + D_{S_{\ell'}(R')}(\{z_i\})$. By (1-c) of Condition 1, $D_{S_{\ell'}(R')}(\{z_i\}) \leq D_N([1, z_i]) - D_\emptyset([1, z_i - 1])$. This implies that $f(R)([y_i, z_i]) = D_N([1, z_i]) - D_\emptyset([1, y_i - 1]) \geq D_{S_{\ell'}(R')}(\{z_i\}) + D_\emptyset([1, z_i - 1]) - D_\emptyset([1, y_i - 1]) = D_\emptyset([y_i, z_i - 1]) + D_{S_{\ell'}(R')}(\{z_i\}) = f(R')([y_i, z_i])$.

Subcase 3-3. $z_i > L(R')$ or $[z_i = L(R')$ and $x_{S_{\ell'}(R')} < p^{\ell'}(R')]$.

Then, $f(R')([y_i, z_i]) = D_N([1, z_i]) - D_\emptyset([1, y_i - 1]) = f(R)([y_i, z_i])$.

Step 5 says that if one of the following (5-a), (5-b), and (5-c) holds, for each agent in $\hat{S}_\ell(R)$, $f(R)$ stochastically dominates $f(R')$.

Step 5. *Assume that (5-a)* $L(R') < L(R)$, *(5-b)* $L(R) = L(R')$, $x_{S_\ell(R)} \geq p^\ell(R)$, *and* $p^{\ell'}(R') > x_{S_{\ell'}(R')}$, *or (5-c)* $L(R) = L(R')$, $p^\ell(R) \leq x_{S_\ell(R)}$, $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}$, *and* $D_{S_\ell(R)}(\{L(R)\}) \leq D_{S_{\ell'}(R')}(\{L(R')\})$. *Then, for each* $i \in \hat{S}_\ell(R)$, $f(R)$ *stochastically dominates* $f(R')$ *at* R_i.

Let $i \in \hat{S}_\ell(R)$, $x_i \in M$, and $[y_i, z_i] = U(R_i, x_i)$. We show that $f(R)([y_i, z_i]) \geq f(R')([y_i, z_i])$. Recall Fact 2.

Case 1. $y_i > L(R) + 1$ or $[y_i = L(R) + 1$ and $x_{S_\ell(R)} < p^\ell(R)]$.

Then, for each $x \geq y_i$, $f(R)(\{x\}) = D_N(\{x\})$. Thus, $f(R)([y_i, z_i]) = D_N([y_i, z_i])$. Since $L(R') \leq L(R)$, it also follows that $f(R')([y_i, z_i]) = D_N([y_i, z_i])$. Thus, $f(R)([y_i, z_i]) = f(R')([y_i, z_i])$.

Case 2. $y_i = L(R) + 1$ and $p^\ell(R) \leq x_{S_\ell(R)}$.

Then, for each $x \geq y_i+1$, $f(R)(\{x\}) = D_N(\{x\})$ and $f(R)(\{y_i\}) = D_{S_\ell(R)}(\{y_i\})$. Thus, $f(R)([y_i, z_i]) = D_{S_\ell(R)}(\{y_i\}) + D_N([y_i + 1, z_i])$.

Subcase 2-1. $L(R') < L(R)$.

Then, $f(R')([y_i, z_i]) = D_N([y_i, z_i])$. By (1-c) of Condition 1, $D_{S_\ell(R)}(\{y_i - 1\}) \leq D_N([1, y_i - 1]) - D_\emptyset([1, y_i - 2])$. This implies that $D_{S_\ell(R)}(\{y_i\}) = 1 - D_\emptyset([1, y_i - 2]) - D_N([y_i + 1, m]) - D_{S_\ell(R)}(\{y_i - 1\}) \geq D_N(\{y_i\})$. Thus, $f(R)([y_i, z_i]) \geq f(R')([y_i, z_i])$.

Subcase 2-2. $L(R) = L(R')$ and $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}$.

Then, $f(R')([y_i, z_i]) = D_{S_{\ell'}(R')}(\{y_i\}) + D_N([y_i + 1, z_i])$. Note that $D_{S_\ell(R)}(\{y_i - 1, y_i\}) = D_{S_{\ell'}(R')}(\{y_i - 1, y_i\})$. Since $D_{S_\ell(R)}(\{y_i - 1\}) \leq D_{S_{\ell'}(R')}(\{y_i - 1\})$ by (5-c), it follows that $D_{S_\ell(R)}(\{y_i\}) \geq D_{S_{\ell'}(R')}(\{y_i\})$. Thus, $f(R)([y_i, z_i]) \geq f(R')([y_i, z_i])$.

Case 3. $y_i \leq L(R)$.

Then, for each $x \leq y_i - 1$, $f(R)(\{x\}) = D_\emptyset(\{x\})$, and for each $x \geq z_i + 1$, $f(R)(\{x\}) = D_N(\{x\})$. Thus, $f(R)([y_i, z_i]) = 1 - D_N([z_i + 1, m]) - D_\emptyset([1, y_i - 1]) = D_N([1, z_i]) - D_\emptyset([1, y_i - 1])$.

Subcase 3-1. $y_i > L(R') + 1$ or $[y_i = L(R') + 1$ and $x_{S_{\ell'}(R')} < p^{\ell'}(R')]$.

Then, $f(R')([y_i, z_i]) = D_N([y_i, z_i])$. By (2) of Condition 1, $D_N([1, y_i - 1]) \geq D_\emptyset([1, y_i - 1])$. This implies that $f(R)([y_i, z_i]) = D_N([1, z_i]) - D_\emptyset([1, y_i - 1]) \geq D_N([1, z_i]) - D_N([1, y_i - 1]) = D_N([y_i, z_i]) = f(R')([y_i, z_i])$.

Subcase 3-2. $y_i = L(R') + 1$ and $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}$.

Then, $f(R')([y_i, z_i]) = D_{S_{\ell'}(R')}(\{y_i\}) + D_N([y_i + 1, z_i])$. By (1-c) of Condition 1, $D_\emptyset(\{y_i - 1\}) \leq D_{S_{\ell'}(R')}(\{y_i - 1\})$. This implies that $D_{S_{\ell'}(R')}(\{y_i\}) = 1 - D_\emptyset([1, y_i -$

$2]) - D_N([y_i+1,m]) - D_{S_{\ell'}(R')}(\{y_i-1\}) \leq 1 - D_\emptyset([1,y_i-1]) - D_N([y_i+1,m]) = D_N([1,y_i]) - D_\emptyset([1,y_i-1])$. Thus, $f(R)([y_i,z_i]) = D_N([1,z_i]) - D_\emptyset([1,y_i-1]) = D_N([1,y_i]) - D_\emptyset([1,y_i-1]) + D_N([y_i+1,z_i]) \geq D_{S_{\ell'}(R')}(\{y_i\}) + D_N([y_i+1,z_i]) = f(R')([y_i,z_i])$.

Subcase 3-3. $y_i \leq L(R')$.

Then, $f(R')([y_i,z_i]) = D_N([1,z_i]) - D_\emptyset([1,y_i-1]) = f(R)([y_i,z_i])$.

In Step 6, we conclude that for at least one agent in group S, $f(R)$ stochastically dominates $f(R')$.

Step 6. *There is $i \in S$ such that $f(R)$ stochastically dominates $f(R')$ at R_i.*

Case 1. $L(R) < L(R')$ or $[L(R) = L(R')$, $x_{S_\ell(R)} < p^\ell(R)$, and $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}]$.

By Step 2-(a) or 3-(a), $S \cap S_\ell(R) \neq \emptyset$. Let $i \in S \cap S_\ell(R)$. Then, by Step 4, $f(R)$ stochastically dominates $f(R')$ at R_i.

Case 2. $L(R') < L(R)$ or $[L(R) = L(R')$, $x_{S_\ell(R)} \geq p^\ell(R)$, and $p^{\ell'}(R') > x_{S_{\ell'}(R')}]$.

By Step 2-(b) or 3-(b), $S \cap \hat{S}_\ell(R) \neq \emptyset$. Let $i \in S \cap \hat{S}_\ell(R)$. Then, by Step 5, $f(R)$ stochastically dominates $f(R')$ at R_i.

Case 3. $L(R) = L(R')$, $p^\ell(R) \leq x_{S_\ell(R)}$, and $p^{\ell'}(R') \leq x_{S_{\ell'}(R')}$.

Note that $S \cap S_\ell(R) \neq \emptyset$ or $S \cap (N \setminus S_\ell(R)) \neq \emptyset$.

Subcase 3-1. $S \cap S_\ell(R) \neq \emptyset$ and $S \cap (N \setminus S_\ell(R)) \neq \emptyset$.

Let $i \in S \cap S_\ell(R)$ and $j \in S \cap (N \setminus S_\ell(R))$. If $D_{S_\ell(R)}(\{L(R)\}) \geq D_{S_{\ell'}(R')}(\{L(R')\})$, then, by Step 4, $f(R)$ stochastically dominates $f(R')$ at R_i. If $D_{S_\ell(R)}(\{L(R)\}) < D_{S_{\ell'}(R')}(\{L(R')\})$, then, by Step 5, $f(R)$ stochastically dominates $f(R')$ at R_j.

Subcase 3-2. $S \cap S_\ell(R) = \emptyset$ and $S \cap (N \setminus S_\ell(R)) \neq \emptyset$.

Let $i \in S \cap (N \setminus S_\ell(R))$. By Step 1-(a), $S_\ell(R) \subseteq S_{\ell'}(R')$. Then, by (1-d) of Condition 1, $D_{S_\ell(R)}(\{L(R)\}) \leq D_{S_{\ell'}(R')}(\{L(R')\})$. Thus, by Step 5, $f(R)$ stochastically dominates $f(R')$ at R_i.

Subcase 3-3. $S \cap S_\ell(R) \neq \emptyset$ and $S \cap (N \setminus S_\ell(R)) = \emptyset$.

Let $i \in S \cap S_\ell(R)$. By Step 1-(b), $S_\ell(R) \supseteq S_{\ell'}(R')$. Then, by (1-d) of Condition 1, $D_{S_\ell(R)}(\{L(R)\}) \geq D_{S_{\ell'}(R')}(\{L(R')\})$. Thus, by Step 4, $f(R)$ stochastically dominates $f(R')$ at R_i.

Case 4. $L(R) = L(R')$, $p^\ell(R) > x_{S_\ell(R)}$, and $p^{\ell'}(R') > x_{S_{\ell'}(R')}$.

Let $i \in S$. Because $f(R)$ coincides with $f(R')$ (recall Fact 2-(b)), $f(R)$ stochastically dominates $f(R')$ at R_i.

$\square$

Appendix

Proof of Corollary 1. *"Only if" part.* Let f be a group strategy-proof, peak-only, and anonymous rule. By Theorem 1, there are collections $(x_S)_{S \subseteq N}$ in $\mathcal{X}$ and $D = (D_S)_{S \subseteq N}$ satisfying Condition 1 such that f is the fixed-probabilistic-ballots rule associated with D. We show that D satisfies Condition 2. First, by Corollary 4.1 in Ehlers, Peters, and Storcken (2002), anonymity implies that for each $S, S' \subseteq N$ with $|S| = |S'|$, $D_S = D_{S'}$.

Let $k \in [0, n]$, $S^{\min}(k) \in \operatorname{argmin}\{x_S : |S| = k\}$, and $S^{\max}(k) \in \operatorname{argmax}\{x_S : |S| = k\}$. Let $S \subseteq N$ be such that $|S| = k$.

(1) Assume that $x^{\min}(k) = x^{\max}(k)$. Let $S' \subseteq N$ be such that $|S'| = k$. Then, by $x^{\min}(k) = x^{\max}(k)$, $x_S = x_{S'}$. Thus, by $D_S = D_{S'}$, $D_S(\{x_S\}) = D_{S'}(\{x_{S'}\})$. Thus, (1) of Condition 2 holds.

(2) Assume that $x^{\min}(k)+1 = x^{\max}(k)$. Then, it follows from (1-a) of Condition 1, $x^{\min}(k) < x^{\max}(k)$, and $D_S = D_{S^{\max}(k)}$ that $D_S(\{x^{\min}(k)\}) = D_{S^{\max}(k)}(\{x^{\min}(k)\}) = D_\emptyset(\{x^{\min}(k)\})$. Similarly, it follows from (1-b) of Condition 1, $x^{\min}(k) + 1 < x^{\max}(k)+1$, and $D_S = D_{S^{\min}(k)}$ that $D_S(\{x^{\max}(k)+1\}) = D_{S^{\min}(k)}(\{x^{\max}(k)+1\}) = D_N(\{x^{\max}(k) + 1\})$.

(3) Assume that $x^{\min}(k) + 1 < x^{\max}(k)$. Let $x \in [x^{\min}(k), x^{\max}(k) - 1]$. Then, it follows from (1-a) of Condition 1, $x \leq x^{\max}(k) - 1$, and $D_S = D_{S^{\max}(k)}$ that $D_S(\{x\}) = D_{S^{\max}(k)}(\{x\}) = D_\emptyset(\{x\})$. Let $x = x^{\max}(k)$. Then, it follows from (1-b) of Condition 1, $x \geq x^{\min}(k)+2$, and $D_S = D_{S^{\min}(k)}$ that $D_S(\{x\}) = D_{S^{\min}(k)}(\{x\}) = D_N(\{x\})$.

"If" part. Let $(x_S)_{S \subseteq N} \in \mathcal{X}$ and let $D = (D_S)_{S \subseteq N}$ satisfy Conditions 1 and 2. Let f be the fixed-probabilistic-ballots rule associated with D. Then, by Theorem 1, f is group strategy-proof and peak-only. Thus, we only show that f satisfies anonymity. By Corollary 4.1 in Ehlers, Peters, and Storcken (2002), it suffices to show that for each $S, S' \subseteq N$ with $|S| = |S'|$, $D_S = D_{S'}$.

Let $S, S' \subseteq N$ be such that $|S| = |S'|$. Let $k = |S|$. By (1-a) of Condition 1, for each $x < x^{\min}(k)$, $D_S(\{x\}) = D_\emptyset(\{x\}) = D_{S'}(\{x\})$. Similarly, by (1-b) of Condition 1, for each $x > x^{\max}(k) + 1$, $D_S(\{x\}) = D_N(\{x\}) = D_{S'}(\{x\})$. Thus, in what follows, we show that for each $x \in [x^{\min}(k), x^{\max}(k)+1]$, $D_S(\{x\}) = D_{S'}(\{x\})$.

Case 1. $x^{\min}(k) = x^{\max}(k)$.

By (1) of Condition 2, $D_S(\{x_S\}) = D_{S'}(\{x_{S'}\})$. Since $x_S = x^{\min}(k) = x_{S'}$,

$D_S(\{x^{\min}(k)\}) = D_{S'}(\{x^{\min}(k)\})$. Thus, for each $x \in M \setminus \{x^{\max}(k)+1\}$, $D_S(\{x\}) = D_{S'}(\{x\})$. This implies that $D_S(\{x^{\min}(k)+1\}) = D_{S'}(\{x^{\min}(k)+1\})$.

Case 2. $x^{\min}(k)+1 = x^{\max}(k)$.

Then, by (2) of Condition 2, $D_S(\{x^{\min}(k)\}) = D_\emptyset(\{x^{\min}(k)\}) = D_{S'}(\{x^{\min}(k)\})$ and $D_S(\{x^{\max}(k)+1\}) = D_N(\{x^{\max}(k)+1\}) = D_{S'}(\{x^{\max}(k)+1\})$. Thus, for each $x \in M \setminus \{x^{\max}(k)\}$, $D_S(\{x\}) = D_{S'}(\{x\})$. This implies that $D_S(\{x^{\max}(k)\}) = D_{S'}(\{x^{\max}(k)\})$.

Case 3. $x^{\min}(k)+1 < x^{\max}(k)$.

By (3-a) of Condition 2, for each $x \in [x^{\min}(k), x^{\max}(k)-1]$, $D_S(\{x\}) = D_\emptyset(\{x\}) = D_{S'}(\{x\})$ and $D_S(\{x^{\max}(k)\}) = D_N(\{x^{\max}(k)\}) = D_{S'}(\{x^{\max}(k)\})$ Thus, for each $x \in M \setminus \{x^{\max}(k)+1\}$, $D_S(\{x\}) = D_{S'}(\{x\})$. This implies that $D_S(\{x^{\max}(k)+1\}) = D_{S'}(\{x^{\max}(k)+1\})$.

□

Proof of Corollary 2. *"Only if" part.* Let f be a group strategy-proof and unanimous rule. Then, by Proposition 5.2 in Ehlers, Peters, and Storcken (2002), f is peak-only. Thus, by Theorem 1, there are collections $X = (x_S)_{S \subseteq N}$ in $\mathcal{X}$ and $D = (D_S)_{S \subseteq N}$ satisfying Condition 1 such that f is the fixed-probabilistic-ballots rule associated with D. For each $S \subseteq N$, let $\alpha_S = D_S(\{x_S\})$. Let $A = (\alpha_S)_{S \subseteq N}$. Let $S \subseteq N$. Then, by (1-d) of Condition 1, for each $S' \subseteq S$, if $x_{S'} = x_S$, $\alpha_{S'} = D_{S'}(\{x_{S'}\}) \leq D_S(\{x_S\}) = \alpha_S$. By unanimity, $\alpha_\emptyset = D_\emptyset(\{m\}) = f(R_N^m)(\{m\}) = 1$ and $\alpha_N = D_N(\{1\}) = f(R_N^1)(\{1\}) = 1$. Finally, by (1-a) and (1-b) of Condition 1, $D_S(\{x_S, x_S+1\}) = 1$. Thus, $D_S(\{x_S+1\}) = 1 - \alpha_S$. Hence, $D = D^{X,A}$.

"If" part. Let $(X, A) \in \mathcal{X} \times \mathcal{A}$. Let f be the fixed-probabilistic-ballots rule associated with $D^{X,A}$. We first show $D^{X,A}$ satisfies Condition 1. Let $S \subseteq N$ and $x \in M$. If $x < x_S$, $D_S^{X,A}(\{x\}) = 0 = D_\emptyset^{X,A}(\{x\})$. If $x > x_S + 1$, $D_S^{X,A}(\{x\}) = 0 = D_N^{X,A}(\{x\})$. For each $S' \subseteq S$, if $x_{S'} = x_S$, $D_{S'}^{X,A}(\{x_{S'}\}) = \alpha_{S'} \leq \alpha_S = D_S^{X,A}(\{x_S\})$. Thus, if $x_S = x_\emptyset$, $D_\emptyset^{X,A}(\{x_S\}) \leq D_S^{X,A}(\{x_S\})$, and if $x_S < x_\emptyset$, $D_\emptyset^{X,A}(\{x_S\}) = 0 \leq D_S^{X,A}(\{x_S\})$. Finally, since $D_N^{X,A}([1,x]) = 1$, $D_N^{X,A}([1,x]) = 1 \geq D_\emptyset^{X,A}([1,x])$, and by $D_\emptyset^{X,A}([1, x_S - 1]) = 0$, $D_S^{X,A}(\{x_S\}) \leq 1 = D_N^{X,A}([1, x_S]) - D_\emptyset^{X,A}([1, x_S - 1])$. Thus, $D^{X,A}$ satisfies Condition 1. Then, by Theorem 1, f is group strategy-proof.

Let $R \in \mathcal{R}^N$ and $x \in M$. Assume that for each $i \in N$, $p(R_i) = x$. Then, by Fact 1, $f(R)(\{x\}) = D_N^{X,A}([1,x]) - D_\emptyset^{X,A}([1, x-1]) = 1$. Thus, f is unanimous.

□

Proof of Corollary 3. *"Only if" part.* Let f be a group strategy-proof, unanimous, and anonymous rule. Then, by Corollaries 1 and 2, there are $\hat{X} = (\hat{x}_S)_{S \subseteq N}$ in $\mathcal{X}$ and $\hat{A} = (\hat{\alpha}_S)_{S \subseteq N}$ in $\mathcal{A}$ such that f is the fixed-probabilistic-ballots rule associated with $D^{\hat{X},\hat{A}}$, and $D^{\hat{X},\hat{A}} = (D_S^{\hat{X},\hat{A}})_{S \subseteq N}$ satisfies Condition 2.

First, suppose that there is $k \in [0, n]$ such that $\hat{x}^{\min}(k) + 1 < \hat{x}^{\max}(k)$. Let $\bar{S} \in \operatorname{argmin}\{\hat{x}_S : |S| = k\}$. Then, $\hat{x}_{\bar{S}} = \hat{x}^{\min}(k)$. By $\hat{x}^{\min}(k) + 1 < \hat{x}^{\max}(k) \le m$, $D_\emptyset^{\hat{X},\hat{A}}(\{\hat{x}_{\bar{S}}, \hat{x}_{\bar{S}} + 1\}) = 0$. Thus, by (3) of Condition 2, $D_{\bar{S}}^{\hat{X},\hat{A}}(\{\hat{x}_{\bar{S}}, \hat{x}_{\bar{S}} + 1\}) = D_\emptyset^{\hat{X},\hat{A}}(\{\hat{x}_{\bar{S}}, \hat{x}_{\bar{S}} + 1\}) = 0$. However, by the definition of $D_{\bar{S}}^{\hat{X},\hat{A}}$, $D_{\bar{S}}^{\hat{X},\hat{A}}(\{\hat{x}_{\bar{S}}, \hat{x}_{\bar{S}} + 1\}) = 1$. This is a contradiction.

Thus, for each $k \in [0, n]$, $\hat{x}^{\min}(k) = \hat{x}^{\max}(k)$ or $\hat{x}^{\min}(k) + 1 = \hat{x}^{\max}(k)$. For each $S \subseteq N$, let $x_S = \hat{x}^{\max}(|S|)$ and $\alpha_S = D_S^{\hat{X},\hat{A}}(\{x_S\})$. Let $X = (x_S)_{S \subseteq N}$ and $A = (\alpha_S)_{S \subseteq N}$. Note that $x_\emptyset = \hat{x}_\emptyset = m$, $x_N = \hat{x}_N = 1$, $\alpha_\emptyset = \hat{\alpha}_\emptyset = 1$, and $\alpha_N = \hat{\alpha}_N = 1$. Note also that for each $k \in [1, n]$, $\hat{x}^{\max}(k) \le \hat{x}^{\max}(k-1)$. Thus, $X \in \mathcal{X}$.

Let $S, S' \subseteq N$ be such that $|S| = |S'|$. Then, $x_S = \hat{x}^{\max}(|S|) = \hat{x}^{\max}(|S'|) = x_{S'}$. Let $k = |S|$. If $\hat{x}^{\min}(k) = \hat{x}^{\max}(k)$, then, by (1) of Condition 2, $\alpha_S = \hat{\alpha}_S = \hat{\alpha}_{S'} = \alpha_{S'}$. If $\hat{x}^{\min}(k) + 1 = \hat{x}^{\max}(k)$, then it follows from $D_\emptyset^{\hat{X},\hat{A}}(\{\hat{x}^{\min}(k)\}) = 0$, $D_N^{\hat{X},\hat{A}}(\{\hat{x}^{\max}(k) + 1\}) = 0$, and (2) of Condition 2 that $\alpha_S = 1 = \alpha_{S'}$. Thus, for each $S, S' \subseteq N$ with $|S| = |S'|$, $x_S = x_{S'}$ and $\alpha_S = \alpha_{S'}$.

Next, let $S \subseteq N$ and $S' \subseteq S$ be such that $x_S = x_{S'}$. Let $\hat{S} \subseteq N$ be such that $|\hat{S}| = |S|$ and $\hat{x}_{\hat{S}} = \hat{x}^{\max}(|S|)$, and let $\hat{S}' \subseteq N$ be such that $|\hat{S}'| = |S'|$ and $\hat{x}_{\hat{S}'} = \hat{x}^{\max}(|S'|)$. Then, by $|\hat{S}| = |S|$ and $|\hat{S}'| = |S'|$, $\alpha_S = \alpha_{\hat{S}} = \hat{\alpha}_{\hat{S}}$ and $\alpha_{S'} = \alpha_{\hat{S}'} = \hat{\alpha}_{\hat{S}'}$. Also, by $x_S = x_{S'}$, $\hat{x}_{\hat{S}} = \hat{x}^{\max}(|S|) = x_S = x_{S'} = \hat{x}^{\max}(|S'|) = \hat{x}_{\hat{S}'}$. Then, by $\hat{A} \in \mathcal{A}$, $\hat{\alpha}_{\hat{S}} \ge \hat{\alpha}_{\hat{S}'}$. Hence, $\alpha_S = \hat{\alpha}_{\hat{S}} \ge \hat{\alpha}_{\hat{S}'} = \alpha_{S'}$. Thus, $A \in \mathcal{A}$.

Finally, because for each $S \subseteq N$, $D_S^{X,A} = D_S^{\hat{X},\hat{A}}$, f is the fixed-probabilistic-ballots rule associated with $D^{X,A}$.

"If" part. Let $(X, A) \in \mathcal{X} \times \mathcal{A}$ be such that for each $S, S' \subseteq N$ with $|S| = |S'|$, $x_S = x_{S'}$ and $\alpha_S = \alpha_{S'}$. Let f be the fixed-probabilistic-ballots rule associated with $D^{X,A}$. Then, by Corollary 2, f is group strategy-proof and unanimous. Because $D^{X,A}$ satisfies Conditions 1 and 2, it also follows from Corollary 1 that f is anonymous.

□

Proof of Fact 2. Let $(x_S)_{S \subseteq N} \in \mathcal{X}$ and let $D = (D_S)_{S \subseteq N}$ satisfy Condition 1.

Let f be the fixed-probabilistic-ballots rule associated with D. Let $R \in \mathcal{R}^N$ and $x \in M$. Let $\ell = \ell^*(R)$. Then, $p^\ell(R) \leq x_{S_{\ell-1}(R)}$ and $x_{S_\ell(R)} < p^{\ell+1}(R)$.

If $\ell' < \ell$, then, $p^\ell(R) \leq x_{S_{\ell-1}(R)} \leq x_{S_{\ell'}(R)}$, and so, for each $y < p^\ell(R)$, $D_{S_{\ell'}(R)}(\{y\}) = D_\emptyset(\{y\})$ and $D_{S_{\ell'}(R)}([1, y]) = D_\emptyset([1, y])$. Thus, if for some $\ell' < \ell$, $p^{\ell'}(R) < x < p^{\ell'+1}(R)$, then, $f(R)(\{x\}) = D_{S_{\ell'}(R)}(\{x\}) = D_\emptyset(\{x\})$, and if for some $\ell' < \ell$, $x = p^{\ell'}(R)$, then, $f(R)(\{x\}) = D_{S_{\ell'}(R)}([1, x]) - D_{S_{\ell'-1}(R)}([1, x-1]) = D_\emptyset([1, x]) - D_\emptyset([1, x-1]) = D_\emptyset(\{x\})$. Hence, if $x < p^\ell(R)$, $f(R)(\{x\}) = D_\emptyset(\{x\})$.

If $\ell' > \ell$, then, $p^{\ell+1}(R) > x_{S_\ell(R)} \geq x_{S_{\ell'}(R)}$, and so, for each $y > p^{\ell+1}(R)$, $D_{S_{\ell'}(R)}(\{y\}) = D_N(\{y\})$ and $D_{S_{\ell'}(R)}([y, m]) = D_N([y, m])$. Thus, if for some $\ell' > \ell$, $p^{\ell'}(R) < x < p^{\ell'+1}(R)$, then, $f(R)(\{x\}) = D_{S_{\ell'}(R)}(\{x\}) = D_N(\{x\})$, and if for some $\ell' > \ell + 1$, $x = p^{\ell'}(R)$, then, $f(R)(\{x\}) = D_{S_{\ell'}(R)}([1, x]) - D_{S_{\ell'-1}(R)}([1, x-1]) = 1 - D_{S_{\ell'}(R)}([x+1, m]) - (1 - D_{S_{\ell'-1}(R)}([x, m])) = D_N(\{x\})$. Hence, if $x > p^{\ell+1}(R)$, $f(R)(\{x\}) = D_N(\{x\})$.

Now, assume that $x = p^\ell(R)$. If $p^\ell(R) \leq x_{S_\ell(R)}$, $f(R)(\{x\}) = D_{S_\ell(R)}([1, x]) - D_{S_{\ell-1}(R)}([1, x-1]) = D_\emptyset([1, x-1]) + D_{S_\ell(R)}(\{x\}) - D_\emptyset([1, x-1]) = D_{S_\ell(R)}(\{x\})$. If $p^\ell(R) > x_{S_\ell(R)}$, $f(R)(\{x\}) = D_{S_\ell(R)}([1, x]) - D_{S_{\ell-1}(R)}([1, x-1]) = D_N([1, x]) - D_\emptyset([1, x-1])$. Hence, by the definition of $D_{S_\ell(R)}$,

$$f(R)(\{x\}) = \begin{cases} D_\emptyset(\{x\}) & \text{if } x < x_{S_\ell(R)} \\ D_{S_\ell(R)}(\{x\}) & \text{if } x = x_{S_\ell(R)} \\ D_N([1, x]) - D_\emptyset([1, x-1]) & \text{if } x > x_{S_\ell(R)}. \end{cases}$$

Next, assume that $p^\ell(R) < x < p^{\ell+1}(R)$. Then, $f(R)(\{x\}) = D_{S_\ell(R)}(\{x\})$. Hence, by the definition of $D_{S_\ell(R)}$,

$$f(R)(\{x\}) = \begin{cases} D_\emptyset(\{x\}) & \text{if } x < x_{S_\ell(R)} \\ D_{S_\ell(R)}(\{x\}) & \text{if } x = x_{S_\ell(R)} \\ D_N([1, x]) - D_\emptyset([1, x-2]) - D_{S_\ell(R)}(\{x-1\}) & \text{if } x = x_{S_\ell(R)} + 1 \\ D_N(\{x\}) & \text{if } x > x_{S_\ell(R)} + 1. \end{cases}$$

Finally, assume that $x = p^{\ell+1}(R)$. Then, $f(R)(\{x\}) = D_{S_{\ell+1}(R)}([1, x]) - D_{S_\ell(R)}([1, x-1]) = D_N([1, x]) - D_{S_\ell(R)}([1, x-1])$. Note that, by the definition of $D_{S_\ell(R)}$, if $x = x_{S_\ell(R)} + 1$, $D_{S_\ell(R)}([1, x-1]) = D_\emptyset([1, x-2]) + D_{S_\ell(R)}(\{x-1\})$, and if $x > x_{S_\ell(R)} + 1$, $D_{S_\ell(R)}([1, x-1]) = D_N([1, x-1])$. Hence,

$$f(R)(\{x\}) = \begin{cases} D_N([1,x]) - D_\emptyset([1,x-2]) - D_{S_\ell(R)}(\{x-1\}) & \text{if } x = x_{S_\ell(R)} + 1 \\ D_N(\{x\}) & \text{if } x > x_{S_\ell(R)} + 1. \end{cases}$$

□

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