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Strategy-proof Probabilistic Rules for the Division Problem of Identical Indivisible Goods with Single-Peaked Preferences

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Strategy-proof Probabilistic Rules for the Division Problem of Identical Indivisible Goods with Single-Peaked Preferences*

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Abstract

We consider the problem of allocating identical indivisible goods among a group of agents with single-peaked preferences. A probabilistic rule chooses a probability distribution over the set of allocations for each preference profile. We characterize the class of sd-strategy-proof and sd-efficient probabilistic rules. We also characterize the classes of sd-strategy-proof and sd-efficient probabilistic rules satisfying anonymity-type axioms.

Keywords: strategy-proofness, probabilistic rule, single-peaked preferences

JEL Classification Numbers: D71

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1 Introduction

Probabilistic rules are being extensively studied to achieve a fair allocation in the assignment of indivisible goods. In this paper, we study probabilistic rules in the allocation of identical indivisible goods with single-peaked preferences. For example, assume that a certain number of identical tasks are to be allocated among a group of workers. In this situation, workers are often assumed to have single-peaked preferences over the number of tasks (i.e., there is an ideal number of tasks for a worker, called his *peak*, such that his welfare monotonically decreases away from the peak). Tasks are usually available only in indivisible units. A *probabilistic rule* chooses a probability distribution over the set of allocations for each reported profile of agents' preferences. Our purpose is to investigate probabilistic rules that satisfy several desirable axioms.

The probabilistic rules studied in this paper only use information about the agents' ordinal preferences. Each agent is assumed to compare probability distributions over his own assignments based on the (first order) stochastic dominance relation derived from his own ordinal preferences.[1] Preferences are usually privately known. Agents may benefit from misreporting their own preferences. *Strategy-proofness* is an axiom that prevents such a strategic misrepresentation of preferences. We adopt the following notion of strategy-proofness, which is often imposed in probabilistic models. A probabilistic rule is *stochastic dominance (sd) strategy-proof* if for each agent, reporting his true preference stochastically dominates lying whatever his true and false preferences are, regardless of what the reported preferences of the other agents are.

The following is known as one of the central notions of *efficiency* in probabilistic models. Given a preference profile, a probability distribution over the set of allocations is *stochastic dominance (sd) efficient for the preference profile* if it is not Pareto-dominated by any other probability distribution according to the stochastic dominance relation derived from the preference profile. A probabilistic rule is *sd-efficient* if for each preference profile, it chooses an sd-efficient probability distribution.

[1] Given an agent's preference and two probability distributions f and g over his own assignments, f *stochastically dominates* g *at the preference* if the probability assigned to each upper contour set of the preference chosen under f is at least as high as the probability chosen under g.

Our first main result is a characterization of the class of sd-strategy-proof and sd-efficient probabilistic rules: *a probabilistic rule is sd-strategy-proof and sd-efficient if and only if it is a convex combination of Pareto-efficient deterministic rules satisfying a certain condition, which we call Condition 1* (Theorem 1). Condition 1 is the requirement that when the preference of an agent whose peak is zero (respectively, the total number of indivisible goods) is changed and there is excess supply (respectively, excess demand) at the new profile, the probability that the agent receives more (respectively, less) than his new peak should not change.

It is also important to investigate probabilistic rules that satisfy *fairness* axioms in addition to efficiency. We characterize the classes of sd-strategy-proof and sd-efficient probabilistic rules that satisfy *anonymity*-type axioms.[2] We introduce the following class of rules. Given a Pareto-efficient deterministic rule F and a permutation π on the set of agents $N = \{1, 2, \dots, n\}$,[3] let F^{π} denote the deterministic rule defined by setting for each preference profile $(R_1, \dots, R_n)$, the assignment of each agent i coincides with that of the $\pi^{-1}(i)$-th agent chosen under F at the preference profile $(R_{\pi(1)}, \dots, R_{\pi(n)})$. Then, the probabilistic rule that selects the allocation chosen by the deterministic rule F^{π} with equal probability for each permutation π satisfies a strong property of anonymity, which we call *anonymity in probabilistic allocation (APA)*.[4] Our second main result states that *a probabilistic rule satisfies sd-strategy-proofness, sd-efficiency, and APA if and only if it is a convex combination of the probabilistic rules described above satisfying a condition similar to Condition 1* (Theorem 2). We also characterize the class of sd-strategy-proof and sd-efficient rules satisfying either anonymity or sd-equal treatment of equals (Theorems 3 and 4 in the Appendix).[5]

For the division problem with an infinitely divisible good, Sprumont (1991) characterizes the uniform rule (Benassy, 1982) as the unique strategy-proof and

[2] Anonymity requires that rules should not depend on agents' names.

[3] A permutation on the set N is a one-to-one function from N to itself.

[4] An example of the probabilistic rule described here is the random priority rule. If F is the deterministic priority rule with priority order $1, 2, \dots, n$, then F^{π} coincides with the deterministic priority rule with priority order $\pi(1), \pi(2), \dots, \pi(n)$. Thus, the probabilistic rule selecting the allocation chosen by F^{π} with equal probability for each permutation π coincides with the well-known random priority rule.

[5] Sd-equal treatment of equals requires that two agents with the same preferences should receive indifferent marginal distributions in the stochastic dominance sense.

Pareto-efficient deterministic rule satisfying either anonymity or no-envy.[6] Sasaki (1997) considers a probabilistic version of the problem of allocating identical indivisible goods, but in a different setting from ours. In his setting, it is assumed that each agent has a single-peaked and risk-averse von Neumann–Morgenstern utility function and probability distributions are compared based on their expected utility. Sasaki (1997) introduces a randomized version of the uniform rule and characterizes the class of randomized uniform rules by strategy-proofness, Pareto-efficiency, and anonymity in his setting.

Ehlers and Klaus (2003) investigate probabilistic rules in the same setting as ours and characterize the class of randomized uniform rules by sd-strategy-proofness, sd-efficiency, and *sd-no-envy*.[7] Thus, Sprumont's (1991) characterization based on the no-envy axiom in the deterministic model extends to our probabilistic framework. On the other hand, as also mentioned in Ehlers and Klaus (2003), the randomized uniform rules are not the unique class of sd-strategy-proof, sd-efficient, and anonymous probabilistic rules. A contribution of our study is to characterize the entire classes of sd-strategy-proof and sd-efficient probabilistic rules satisfying anonymity-type axioms. As we will see in Example 2 of Section 5, these classes are quite large and contain several important rules. To our knowledge, no characterization of these classes has been obtained in existing studies. We also obtain a characterization of the entire class of sd-strategy-proof and sd-efficient rules.

This paper also contributes to the recent literature on extreme point characterizations of sd-strategy-proof probabilistic rules. In the one-dimensional and probabilistic public decision model with single-peaked preferences (Ehlers, Peters, and Storcken, 2002),[8] it is known that any sd-strategy-proof and sd-efficient probabilistic rule can be represented as a convex combination of strategy-proof and Pareto-efficient deterministic rules (Peters, Roy, Sen, and Storcken, 2014; Pycia and Ünver, 2015; Roy and Sadhukhan, 2018).[9] Similar results have also been ob-

[6] Ching (1994) also shows that the uniform rule is the only deterministic rule satisfying strategy-proofness, Pareto-efficiency, and equal treatment of equals.

[7] Sd-no-envy requires that each agent's marginal distribution should stochastically dominate that of anyone else.

[8] Ehlers, Peters, and Storcken (2002) extend Moulin's (1980) study for the deterministic model to the probabilistic model.

[9] More precisely, the extreme point characterization is shown for probabilistic rules satisfying sd-strategy-proofness and unanimity (if there are alternatives that are best for all agents, the

tained in several models with public goods.[10] On the other hand, it is also known that the extreme point property does not hold in some domains.[11] As mentioned in Remark 1 of Section 4, in our model, there are sd-strategy-proof and sd-efficient probabilistic rules that cannot be represented as a convex combination of strategy-proof and Pareto-efficient deterministic rules.

As mentioned above, Sasaki (1997) assumes that each agent compares probability distributions over his own assignments based on their expected utility. In Sasaki's (1997) setting, several authors have characterized the class of randomized uniform rules (Sasaki, 1997; Kureishi, 2000; Ehlers and Klaus, 2003; Kureishi and Mizukami, 2005, 2007; Hatsumi and Serizawa, 2009).[12] We remark that *sd-strategy-proofness* is quite different from *strategy-proofness based on expected utility evaluations*. As already mentioned, in that case, the randomized uniform rules are the only rules satisfying strategy-proofness, Pareto-efficiency, and anonymity.[13] On the other hand, in our setting, there are many sd-strategy-proof, sd-efficient, and anonymous probabilistic rules other than the randomized uniform rules. Thus, our result is independent of the results obtained in Sasaki's (1997) setting.

Another related model is the random assignment problem in which each agent is assigned exactly one indivisible good and the goods assigned to each agent differ from each other. Again, the results obtained in this model are quite different from ours. Bogomolnaia and Moulin (2001) show that sd-strategy-proofness, sd-efficiency, and equal treatment of equals are incompatible on the strict preference domain. Kasajima (2013) extends the impossibility to the single-peaked domain.[14]

This paper is organized as follows. In Sections 2 and 3, we provide the model and axioms. Section 4 states our main characterizations. Section 5 provides the proofs of our main results presented in Section 4. Other characterizations and omitted proofs are in the Appendix.

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outcome should be chosen from those alternatives). On the single-peaked domain, under sd-strategy-proofness, unanimity is equivalent to sd-efficiency (Ehlers, Peters, and Storcken, 2002).

[10]See, for example, Gibbard (1977), Chatterji, Roy, and Sen (2012), Picot and Sen (2012), Peters, Roy, Sadhukhan, and Storcken (2017), and Chatterji and Zeng (2018).

[11]See, for example, Chatterji, Sen, and Zeng (2014) and Gaurav, Picot, and Sen (2017).

[12]Ehlers and Klaus (2003) also include an analysis in Sasaki's (1997) setting.

[13]It is also known that only the randomized uniform rules satisfy strategy-proofness, Pareto-efficiency, and equal treatment of equals (Kureishi, 2000; Ehlers and Klaus, 2003).

[14]Chang and Chun (2017) further extend the impossibility to a more restricted domain.

2 Model

Let $N = \{1, \ldots, n\}$ be a finite set of agents with $n \geq 2$. Let $M = \{0, 1, \ldots, m\}$ be a finite set of identical indivisible goods to be allocated.

Each agent i has a complete, reflexive, and transitive preference relation R_i on M. We denote by P_i the strict preference relation associated with R_i. Given a preference R_i and an assignment $x \in M$, let $U(R_i, x) = \{y \in M : y \ R_i \ x\}$ be the **upper contour set of** R_i **at** x. A preference R_i on M is **single-peaked** if there is $p(R_i) \in M$ such that for each $x_i, x_i' \in M$, whenever $x_i' < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < x_i'$, we have $x_i \ P_i \ x_i'$. We refer to $p(R_i)$ as the **peak** of R_i. Let R_i^0 (respectively, R_i^m) denote the preference whose peak is equal to 0 (respectively, m). Let $\mathcal{R}$ denote the set of single-peaked preferences.

A **preference profile** is a list $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$. Let $p(R)$ denote the profile of the peaks of R, i.e., $p(R) = (p(R_1), \ldots, p(R_n))$. Given $i, j \in N$, let $R_{-i} = (R_k)_{k \in N \setminus \{i\}}$ and $R_{-i,j} = (R_k)_{k \in N \setminus \{i,j\}}$.

An **allocation** is a list $z = (z_1, \ldots, z_n) \in M^N$ such that $\sum_{i \in N} z_i = m$. Let Z be the set of allocations.

Given a nonempty finite set X, let $\Delta(X)$ denote the set of probability distributions over X, i.e., $\Delta(X) = \{(w_x)_{x \in X} \in \mathbb{R}_+^X : \sum_{x \in X} w_x = 1\}$.[15] Given $w = (w_x)_{x \in X} \in \Delta(X)$ and $\hat{X} \subseteq X$, let $w(\hat{X}) = \sum_{x \in \hat{X}} w_x$.

A probabilistic rule, or simply a **rule**, is a function $f : \mathcal{R}^N \to \Delta(Z)$. Given $Q \in \Delta(Z)$ and $i \in N$, let Q_i denote agent i's marginal distribution over M induced by Q, i.e., for each $X_i \subseteq M$, $Q_i(X_i) = Q(\{z \in Z : z_i \in X_i\})$. For each $i \in N$, let $f_i(R)$ denote agent i's marginal distribution over M induced by $f(R)$.

A **deterministic rule** is a function $F : \mathcal{R}^N \to Z$. Given a deterministic rule F, let $\bar{F}$ denote the probabilistic rule that places probability one on the allocation chosen by F for each preference profile, i.e., for each $R \in \mathcal{R}^N$, $\bar{F}(R)(\{F(R)\}) = 1$ and for each $z \in Z \setminus \{F(R)\}$, $\bar{F}(R)(\{z\}) = 0$.

Let $i \in N$, $R_i \in \mathcal{R}$, and $Q_i, Q_i' \in \Delta(M)$. We define the stochastic dominance relation associated with R_i as follows. First, Q_i **stochastically dominates** Q_i' **at** R_i if for each $x \in M$, $Q_i(U(R_i, x)) \geq Q_i'(U(R_i, x))$. Second, Q_i **strictly stochastically dominates** Q_i' **at** R_i if Q_i stochastically dominates Q_i' and for some $y \in M$, $Q_i(U(R_i, y)) > Q_i'(U(R_i, y))$.

[15] We denote by $\mathbb{R}_+$ the set of nonnegative real numbers.

3 Axioms

We introduce the axioms. Following Thomson (2018), we use the prefix "sd" for the axioms based on the stochastic dominance relation. Let f be a (probabilistic) rule.

The first axiom requires that for each agent, the marginal distribution chosen under truth-telling should stochastically dominate the marginal distribution chosen under lying whatever his true and false preferences are, regardless of what the reported preferences of the other agents are.

Sd-strategy-proofness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R)$ stochastically dominates $f_i(R'_i, R_{-i})$ at R_i.

The next two axioms are related to efficiency. A probability distribution $Q \in \Delta(Z)$ is **sd-efficient for** R if there is no $Q' \in \Delta(Z)$ such that for each $i \in N$, Q'_i stochastically dominates Q_i at R_i, and for some $j \in N$, Q'_j strictly stochastically dominates Q_j at R_j.

Sd-efficiency: For each $R \in \mathcal{R}^N$, $f(R)$ is sd-efficient for R.

Second, if there is excess supply (respectively, excess demand), no agent should receive less (respectively, more) than his own peak.

Same-sideness: For each $R \in \mathcal{R}^N$,
if $\sum_{j \in N} p(R_j) \leq m$ (excess supply), for each $i \in N$, $f_i(R)([p(R_i), m]) = 1$,
if $\sum_{j \in N} p(R_j) \geq m$ (excess demand), for each $i \in N$, $f_i(R)([0, p(R_i)]) = 1$.[16]

In our model, same-sideness (ex-post efficiency) is equivalent to sd-efficiency.

Fact 1 (Ehlers and Klaus, 2003, Lemma 1). *A rule satisfies sd-efficiency if and only if it satisfies same-sideness.*

Next, we introduce four standard axioms concerning fairness. First, each agent's marginal distribution should stochastically dominate the marginal distribution of anyone else.

Sd-no-envy: For each $R \in \mathcal{R}^N$ and each $i, j \in N$, $f_i(R)$ stochastically dominates $f_j(R)$ at R_i.

[16] Given $x, y \in M$ with $x \leq y$, we write $[x, y] = \{x, x+1, \ldots, y\}$.

Second, rules should not depend on agents' names. A permutation π on N is a one-to-one function from N to itself. Let Π be the set of permutations on N. Given $\pi \in \Pi$ and $R \in \mathcal{R}^N$, let $R^\pi = (R_{\pi(i)})_{i \in N}$.

Anonymity: For each $R \in \mathcal{R}^N$, each $\pi \in \Pi$, and each $i \in N$, $f_{\pi(i)}(R) = f_i(R^\pi)$.

Third, two agents with the same preferences should receive the same marginal distributions.

Equal treatment of equals: For each $R \in \mathcal{R}^N$ and each $i, j \in N$ with $R_i = R_j$, $f_i(R) = f_j(R)$.

Note that anonymity implies equal treatment of equals.

Fourth, two agents with the same preferences should receive indifferent marginal distributions in the stochastic dominance sense.

Sd-equal treatment of equals: For each $R \in \mathcal{R}^N$ and each $i, j \in N$ with $R_i = R_j$, for each $x \in M$, $f_i(R)(U(R_i, x)) = f_j(R)(U(R_i, x))$.

Sd-no-envy and equal treatment of equals both imply sd-equal treatment of equals. Thus, sd-equal treatment of equals is the weakest requirement of the four fairness axioms mentioned above. Fact 2 below says that under sd-efficiency, sd-equal treatment of equals is equivalent to equal treatment of equals in our probabilistic model.[17]

Fact 2. *If a rule satisfies sd-efficiency and sd-equal treatment of equals, then it satisfies equal treatment of equals.*

Proof of Fact 2. Let f be a rule satisfying sd-efficiency and sd-equal treatment of equals. Let $R \in \mathcal{R}^N$ and $i, j \in N$ be such that $R_i = R_j$. We show that for each $x \in M$, $f_i(R)(\{x\}) = f_j(R)(\{x\})$. Let $x \in M$. Assume that $\sum_{k \in N} p(R_k) \leq m$ (the other case can be treated symmetrically). Then, by sd-efficiency and Fact 1 (same-sideness), for each $k \in N$, $f_k(R)([p(R_k), m]) = 1$. Thus, if $x < p(R_i)$, $f_i(R)(\{x\}) = 0 = f_j(R)(\{x\})$. Next, we show that for each $y \in [p(R_i), m]$, $f_i(R)([p(R_i), y]) = f_j(R)([p(R_i), y])$. Let $y \in [p(R_i), m]$. By contradiction, suppose

[17] For the deterministic division problem with an infinitely divisible good, it is well known that a similar equivalence holds (Ching, 1994).

that $f_i(R)([p(R_i), y]) > f_j(R)([p(R_i), y])$ (the other case is similar). Then,

$$\begin{aligned} f_i(R)(U(R_i, y)) &= f_i(R)([p(R_i), y]) \text{ (by Fact 1)} \\ &> f_j(R)([p(R_i), y]) \\ &= f_j(R)(U(R_i, y)) \text{ (by Fact 1 and } p(R_i) = p(R_j)), \end{aligned}$$

which contradicts sd-equal treatment of equals.

Thus, if $x = p(R_i)$, $f_i(R)(\{x\}) = f_j(R)(\{x\})$. Finally, if $x > p(R_i)$, then,

$$\begin{aligned} f_i(R)(\{x\}) &= f_i(R)([p(R_i), x]) - f_i(R)([p(R_i), x-1]) \\ &= f_j(R)([p(R_i), x]) - f_j(R)([p(R_i), x-1]) \\ &= f_j(R)(\{x\}). \end{aligned}$$

Hence, for each $x \in M$, $f_i(R)(\{x\}) = f_j(R)(\{x\})$. □

Finally, we introduce one additional fairness axiom, which is key to our second main theorem. The next axiom says that the probability assigned to each allocation should not change when agents' names are shuffled. Given $\pi \in \Pi$ and $z \in Z$, let $z^\pi = (z_{\pi(i)})_{i \in N}$.

Anonymity in probabilistic allocation (APA): For each $R \in \mathcal{R}^N$, each $\pi \in \Pi$, and each $z \in Z$, $f(R)(\{z\}) = f(R^\pi)(\{z^\pi\})$.

As shown in Fact 3 below, APA implies anonymity, but the converse does not hold.[18]

Fact 3. *If a rule satisfies anonymity in probabilistic allocation, then it satisfies anonymity.*

Proof of Fact 3. Let f be a rule satisfying APA. Let $R \in \mathcal{R}^N$, $\pi \in \Pi$, $i \in N$, $x \in M$, and $j = \pi(i)$. Then, by APA, for each $z \in Z$ with $z_i = x$, $f(R^\pi)(\{z\}) =$

[18]To see this, let $n = m = 3$ and let g be the rule such that for each $R \in \mathcal{R}^N$, if $R = (R_1^0, R_2^0, R_3^0)$, $g(R)(\{(1,0,2)\}) = g(R)(\{(0,2,1)\}) = g(R)(\{(2,1,0)\}) = \frac{1}{3}$, and if $R \neq (R_1^0, R_2^0, R_3^0)$, $g(R)(\{(1,1,1)\}) = 1$. Then, g satisfies anonymity, but not APA. (Since $g(R_1^0, R_2^0, R_3^0)(\{(1,0,2)\}) = \frac{1}{3} \neq 0 = g(R_1^0, R_2^0, R_3^0)(\{(1,2,0)\})$, g violates APA at (R_1^0, R_2^0, R_3^0), $\pi = (\pi(1), \pi(2), \pi(3)) = (1,3,2)$, and $z = (1,0,2)$.)

$f(R)(\{(z_{\pi^{-1}(1)}, \ldots, z_{\pi^{-1}(j)}, \ldots, z_{\pi^{-1}(n)})\})$. Then,

$$\begin{aligned} f_i(R^\pi)(\{x\}) &= f(R^\pi)(\{z \in Z : z_i = x\}) \\ &= f(R)(\{z \in Z : z_j = x\}) \\ &= f_j(R)(\{x\}). \end{aligned}$$

Thus, $f_i(R^\pi)(\{x\}) = f_{\pi(i)}(R)(\{x\})$. $\square$

4 Main results

In this section, we provide our two main characterizations. We first characterize the class of sd-strategy-proof and sd-efficient rules.

Let $\mathcal{F}$ be the set of deterministic rules F satisfying Pareto-efficiency (same-sideness), i.e., for each $R \in \mathcal{R}^N$, if $\sum_{j \in N} p(R_j) \leq m$, for each $i \in N$, $F_i(R) \geq p(R_i)$, and if $\sum_{j \in N} p(R_j) \geq m$, for each $i \in N$, $F_i(R) \leq p(R_i)$.

Given a probability distribution $\alpha = (\alpha_F)_{F \in \mathcal{F}} \in \Delta(\mathcal{F})$ over $\mathcal{F}$, consider the rule f that selects the allocation chosen by the Pareto-efficient deterministic rule F with probability α_F. That is, f is a convex combination of Pareto-efficient deterministic rules, i.e., $f = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}$.[19] The next condition for a probability distribution $\alpha = (\alpha_F)_{F \in \mathcal{F}}$ over $\mathcal{F}$ guarantees that the convex combination $f = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}$ satisfies sd-strategy-proofness. It requires that when the preference of an agent whose peak is 0 (respectively, m) is changed and there is excess supply (respectively, excess demand) at the new profile, the probability that the agent receives more (respectively, less) than his new peak should not change.

Condition 1. For each $R \in \mathcal{R}^N$, each $i \in N$, and each $x \in M$,
(1-i) if $\sum_{j \in N} p(R_j) \leq m$ and $x > p(R_i)$,

$$\sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})(\{x\}),$$

[19]Given two rules f and g, $f = g$ means that for each $R \in \mathcal{R}^N$ and each $z \in Z$, $f(R)(\{z\}) = g(R)(\{z\})$.

(1-ii) if $\sum_{j\in N} p(R_j) \geq m$ and $x < p(R_i)$,

$$\sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^m, R_{-i})(\{x\}).$$

Theorem 1 below is our first main result. It says that a rule is sd-strategy-proof and sd-efficient if and only if it is a convex combination of Pareto-efficient deterministic rules satisfying Condition 1.

Theorem 1. *A rule f satisfies sd-strategy-proofness and sd-efficiency if and only if there is $\alpha = (\alpha_F)_{F\in\mathcal{F}} \in \Delta(\mathcal{F})$ satisfying Condition 1 such that $f = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}$.*

Example 1. We provide three examples of rules described in Theorem 1.

1-1. Let F be a Pareto-efficient deterministic rule satisfying the following property: when there is excess supply (respectively, excess demand), the assignment of each agent not receiving his own peak should coincide with his assignment chosen when his peak is 0 (respectively, m), i.e., for each $R \in \mathcal{R}^N$ and each $i \in N$,

$$F_i(R) = \begin{cases} \max\{p(R_i), F_i(R_i^0, R_{-i})\} & \text{if } \sum_{j\in N} p(R_j) \leq m \\ \min\{p(R_i), F_i(R_i^m, R_{-i})\} & \text{if } \sum_{j\in N} p(R_j) \geq m. \end{cases}$$

First, consider the rule $\bar{F}$ that places probability one on the allocation chosen by F for each preference profile, i.e., $\bar{F} = \sum_{F'\in\mathcal{F}} \alpha_{F'} \bar{F}'$, where $\alpha_F = 1$. In this simple case, we can easily check that Condition 1 holds. To see this, consider a preference profile R, an agent i, and an assignment x. Assume that there is excess supply and $x > p(R_i)$ (the other case is similar). Note that by the definition of F, if $F_i(R) > p(R_i)$, $F_i(R) = F_i(R_i^0, R_{-i})$. Thus, it follows that $F_i(R) = x$ if and only if $F_i(R_i^0, R_{-i}) = x$. Therefore, in this case, the probability that agent i receives x units under $\bar{F}$ does not change when his preferences are altered from R_i^0 to R_i, i.e., $\bar{F}_i(R)(\{x\}) = \bar{F}_i(R_i^0, R_{-i})(\{x\})$.[20] Hence, Condition 1 holds in this example. Thus, by Theorem 1, we conclude that $\bar{F}$ is sd-strategy-proof and sd-efficient.

From this observation, we can also show that any convex combination of deterministic rules $F^1, \ldots, F^K$ defined in this way satisfies sd-strategy-proofness and sd-efficiency. Let α be an arbitrary probability distribution over $\mathcal{F}$ that places

[20] Note that if $x = F_i(R)$, then $\bar{F}_i(R)(\{x\}) = 1 = \bar{F}_i(R_i^0, R_{-i})(\{x\})$, and if $x \neq F_i(R)$, then $\bar{F}_i(R)(\{x\}) = 0 = \bar{F}_i(R_i^0, R_{-i})(\{x\})$.

positive probabilities only on $F^1, \ldots, F^K$. Then, α also satisfies Condition 1.[21] Therefore, by Theorem 1, we can conclude that any convex combination of deterministic rules $F^1, \ldots, F^K$ is also sd-strategy-proof and sd-efficient.

As we will see in Corollary 1 in Section 5, the class of deterministic rules described in this example coincides with the entire class of strategy-proof and Pareto-efficient deterministic rules. Therefore, this example also illustrates the fact that any convex combination of strategy-proof and Pareto-efficient deterministic rules satisfies sd-strategy-proofness and sd-efficiency.

1-2. Let $F^1, \ldots, F^K$ be distinct Pareto-efficient deterministic rules satisfying the following property: for each $k \in \{1, \ldots, K\}$, when there is excess supply (respectively, excess demand), the assignment of each agent not receiving his own peak should coincide with his assignment chosen under F^{k+1} when his peak is 0 (respectively, m), i.e., for each $R \in \mathcal{R}^N$ and each $i \in N$,

$$F_i^k(R) = \begin{cases} \max\{p(R_i), F_i^{k+1}(R_i^0, R_{-i})\} & \text{if } \sum_{j \in N} p(R_j) \leq m \\ \min\{p(R_i), F_i^{k+1}(R_i^m, R_{-i})\} & \text{if } \sum_{j \in N} p(R_j) \geq m, \end{cases}$$

where $F^{K+1} = F^1$. Note that the deterministic rules studied in this example need not be strategy-proof. Consider the rule f that selects the allocation chosen by the deterministic rule F^k with equal probability for each $k \in \{1, \ldots, K\}$, i.e., $f = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}$, where for each $k \in \{1, \ldots, K\}$, $\alpha_{F^k} = \frac{1}{K}$. Then, we can show that this rule also satisfies sd-strategy-proofness and sd-efficiency.

To see this, consider a preference profile R, an agent i, and an assignment x. Assume that there is excess supply and $x > p(R_i)$ (the other case is similar). Let $k \in \{1, \ldots, K\}$. First, by the definition of F^k, if $F_i^k(R) > p(R_i)$, $F_i^k(R) = F_i^{k+1}(R_i^0, R_{-i})$. Thus, it follows that $F_i^k(R) = x$ if and only if $F_i^{k+1}(R_i^0, R_{-i}) = x$. This implies that $\bar{F}_i^k(R)(\{x\}) = \bar{F}_i^{k+1}(R_i^0, R_{-i})(\{x\})$. Hence, $\sum_{k=1}^K \bar{F}_i^k(R)(\{x\}) = \sum_{k=1}^K \bar{F}_i^{k+1}(R_i^0, R_{-i})(\{x\})$. Finally, since α assigns probability $\frac{1}{K}$ to each deterministic rule in $\{F^1, \ldots, F^K\}$, we have $\sum_{F \in \mathcal{F}} \alpha_F \bar{F} = \frac{1}{K} \sum_{k=1}^K \bar{F}^k$. Therefore,

[21] To see this, let $R \in \mathcal{R}^N$, $i \in N$, and $x \in M$ be such that $\sum_{j \in N} p(R_j) \leq m$ and $x > p(R_i)$ (the other case is similar). Then, for each $k \in \{1, \ldots, K\}$, $F_i^k(R) = x$ if and only if $F_i^k(R_i^0, R_{-i}) = x$. Thus, for each $k \in \{1, \ldots, K\}$, $\bar{F}_i^k(R)(\{x\}) = \bar{F}_i^k(R_i^0, R_{-i})(\{x\})$. Therefore, $\sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = \sum_{k=1}^K \alpha_{F^k} \bar{F}_i^k(R)(\{x\}) = \sum_{k=1}^K \alpha_{F^k} \bar{F}_i^k(R_i^0, R_{-i})(\{x\}) = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})(\{x\})$.

under $f = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}$, the probability that agent i receives x units does not change when his preferences are altered from R_i^0 to R_i, i.e., $\sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = \frac{1}{K}\sum_{k=1}^K \bar{F}_i^k(R)(\{x\}) = \frac{1}{K}\sum_{k=1}^K \bar{F}_i^{k+1}(R_i^0, R_{-i})(\{x\}) = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})(\{x\})$. Hence, Condition 1 also holds in this example. Thus, by Theorem 1, we conclude that f satisfies sd-strategy-proofness and sd-efficiency.

1-3. This is a variant of the previous example. Let $F^1, \dots, F^K$ be distinct deterministic rules satisfying the following property: for each $R \in \mathcal{R}^N$ and each $i \in N$, there is a one-to-one function $\lambda^{R,i} : \{1, \dots, K\} \to \{1, \dots, K\}$ such that for each $k \in \{1, \dots, K\}$,

$$F_i^k(R) = \begin{cases} \max\{p(R_i), F_i^{\lambda^{R,i}(k)}(R_i^0, R_{-i})\} & \text{if } \sum_{j \in N} p(R_j) \leq m \\ \min\{p(R_i), F_i^{\lambda^{R,i}(k)}(R_i^m, R_{-i})\} & \text{if } \sum_{j \in N} p(R_j) \geq m. \end{cases}$$

Note that $\lambda^{R,i}$ may depend on R and i. Consider the rule that selects the allocation chosen by the deterministic rule F^k with equal probability for each $k \in \{1, \dots, K\}$. Then, similarly to the previous example, we can also show that Condition 1 holds in this example.[22] Thus, by Theorem 1, this rule also satisfies sd-strategy-proofness and sd-efficiency.

We next provide a characterization of the class of sd-strategy-proof and sd-efficient rules satisfying anonymity in probabilistic allocation.[23] First, we introduce the following additional notations.

Given a deterministic rule F and a permutation π on N, let F^π denote the deterministic rule defined by setting for each preference profile R and each agent i, the assignment of agent i coincides with that of the $\pi^{-1}(i)$-th agent under F at the profile $(R_{\pi(1)}, \dots, R_{\pi(n)})$, i.e., $F_i^\pi(R) = F_{\pi^{-1}(i)}(R^\pi)$. Let π^0 be such that for each $i \in N$, $\pi^0(i) = i$. Note that $F^{\pi^0} = F$, and if F is Pareto-efficient, then for

[22] To see this, let $R \in \mathcal{R}^N$, $i \in N$, and $x \in M$ be such that $\sum_{j \in N} p(R_j) \leq m$ and $x > p(R_i)$ (the other case is similar). Let $k \in \{1, \dots, K\}$. Then, $F_i^k(R) = x$ if and only if $F_i^{\lambda^{R,i}(k)}(R_i^0, R_{-i}) = x$. Thus, $\bar{F}_i^k(R)(\{x\}) = \bar{F}_i^{\lambda^{R,i}(k)}(R_i^0, R_{-i})(\{x\})$. Since $\lambda^{R,i}$ is one-to-one, we have $\sum_{k=1}^K \bar{F}_i^k(R)(\{x\}) = \sum_{k=1}^K \bar{F}_i^{\lambda^{R,i}(k)}(R_i^0, R_{-i})(\{x\})$. Finally, since α assigns probability $\frac{1}{K}$ to each deterministic rule in $\{F^1, \dots, F^K\}$, we conclude that $\sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = \frac{1}{K}\sum_{k=1}^K \bar{F}_i^k(R)(\{x\}) = \frac{1}{K}\sum_{k=1}^K \bar{F}_i^{\lambda^{R,i}(k)}(R_i^0, R_{-i})(\{x\}) = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})(\{x\})$.

[23] Applying Theorem 1, we can also obtain characterizations of the classes of sd-strategy-proof and sd-efficient rules satisfying other fairness axioms (anonymity or sd-equal treatment of equals). See Theorems 3 and 4 in the Appendix.

each permutation π, F^π is also Pareto-efficient.[24]

Given a deterministic rule F, let $\hat{f}^F$ denote the rule that selects the allocation chosen by the deterministic rule F^π with probability $\frac{1}{n!}$ for each permutation π, i.e., $\hat{f}^F = \frac{1}{n!}\sum_{\pi\in\Pi}\bar{F}^\pi$. Let $\hat{\mathcal{F}}$ denote the set of rules $\hat{f}^F$ induced by all Pareto-efficient deterministic rules F, i.e., $\hat{\mathcal{F}} = \{\hat{f}^F : F \in \mathcal{F}\}$. Next, consider a convex combination f of rules in $\hat{\mathcal{F}}$, that is, $f = \sum_{\hat{f}\in\hat{\mathcal{F}}}\hat{\alpha}_{\hat{f}}\hat{f}$, where $\hat{\alpha} = (\hat{\alpha}_{\hat{f}})_{\hat{f}\in\hat{\mathcal{F}}}$ is a probability distribution over $\hat{\mathcal{F}}$. The next condition for a probability distribution $\hat{\alpha} = (\hat{\alpha}_{\hat{f}})_{\hat{f}\in\hat{\mathcal{F}}}$ over $\hat{\mathcal{F}}$ guarantees that the convex combination $f = \sum_{\hat{f}\in\hat{\mathcal{F}}}\hat{\alpha}_{\hat{f}}\hat{f}$ satisfies sd-strategy-proofness. It is a parallel requirement to Condition 1, which says that when the preference of an agent whose peak is 0 (respectively, m) is changed and there is excess supply (respectively, excess demand) at the new profile, the probability that the agent receives more (respectively, less) than his new peak should not change.

Condition 2. For each $R \in \mathcal{R}^N$, each $i \in N$, and each $x \in M$,
(2-i) if $\sum_{j\in N} p(R_j) \le m$ and $x > p(R_i)$,

$$\sum_{\hat{f}\in\hat{\mathcal{F}}}\hat{\alpha}_{\hat{f}}\hat{f}_i(R)(\{x\}) = \sum_{\hat{f}\in\hat{\mathcal{F}}}\hat{\alpha}_{\hat{f}}\hat{f}_i(R_i^0, R_{-i})(\{x\}),$$

(2-ii) if $\sum_{j\in N} p(R_j) \ge m$ and $x < p(R_i)$,

$$\sum_{\hat{f}\in\hat{\mathcal{F}}}\hat{\alpha}_{\hat{f}}\hat{f}_i(R)(\{x\}) = \sum_{\hat{f}\in\hat{\mathcal{F}}}\hat{\alpha}_{\hat{f}}\hat{f}_i(R_i^m, R_{-i})(\{x\}).$$

The following is our second main result. It says that a rule satisfies sd-strategy-proofness, sd-efficiency, and anonymity in probabilistic allocation if and only if it is a convex combination of rules in $\hat{\mathcal{F}}$ satisfying Condition 2.

Theorem 2. *A rule f satisfies sd-strategy-proofness, sd-efficiency, and anonymity in probabilistic allocation if and only if there is $\hat{\alpha} \in \Delta(\hat{\mathcal{F}})$ satisfying Condition 2 such that $f = \sum_{\hat{f}\in\hat{\mathcal{F}}}\hat{\alpha}_{\hat{f}}\hat{f}$.*

Example 2. The class of sd-strategy-proof and sd-efficient rules satisfying anonymity in probabilistic allocation is still large. We provide four examples of rules belonging

[24] To see this, let $F \in \mathcal{F}$ and $\pi \in \Pi$. Let $R \in \mathcal{R}^N$ and $i \in N$. Assume that $\sum_{j\in N} p(R_j) \le m$ (the other case is similar). By the definition of F^π, $F_i^\pi(R) = F_{\pi^{-1}(i)}(R_{\pi(1)}, \dots, R_{\pi(n)})$. Since $F \in \mathcal{F}$, $F_{\pi^{-1}(i)}(R_{\pi(1)}, \dots, R_{\pi(n)}) \ge p(R_{\pi(\pi^{-1}(i))})$. Thus, $F_i^\pi(R) \ge p(R_{\pi(\pi^{-1}(i))}) = p(R_i)$.

to this class.

2-1. Random priority rule. An example of rules described in Theorem 2 is the random priority rule, which is one of the central rules in the probabilistic model. We first introduce the (deterministic) priority rule Ψ with priority order $1, 2, \ldots, n$. This rule determines the assignment of each agent in the following way. First, agent 1 receives his most preferred assignment. Next, agent 2 receives his most preferred assignment from the remaining units, and so on. Finally, agent n receives the remaining units. That is, for each $R \in \mathcal{R}^N$ and each $i \in N$,
(i) if $\sum_{j \in N} p(R_j) \leq m$,

$$\Psi_i(R) = \begin{cases} p(R_i) & \text{if } i \neq n \\ m - \sum_{j \neq n} p(R_j) & \text{if } i = n, \end{cases}$$

(ii) if $\sum_{j \in N} p(R_j) \geq m$,

$$\Psi_i(R) = \begin{cases} p(R_i) & \text{if } i = 1 \\ \min\{p(R_i), m - \sum_{j=1}^{i-1} \Psi_j(R)\} & \text{if } i \neq 1. \end{cases}$$

The random priority rule is the rule that selects the allocation chosen by the priority rule with equal probability for each priority order. This rule can be obtained by setting $\hat{\alpha}$ as follows.

Let $\hat{\alpha}$ assign probability one to the rule $\hat{f}^{\Psi}$ induced by Ψ. Then, the rule $f = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f} = \frac{1}{n!} \sum_{\pi \in \Pi} \bar{\Psi}^{\pi}$ coincides with the random priority rule.[25]

2-2. Randomized uniform rule (Sasaki, 1997). The uniform rule is one of the extensively studied rules in the deterministic division problem with an infinitely divisible good. Let R be a preference profile. Under the uniform rule, the allocation is determined as follows. When there is excess supply (respectively, excess demand), each agent whose peak is greater (respectively, less) than $\lambda(R)$ receives his own peak amount and the remaining agents receive the common amount $\lambda(R)$, where $\lambda(R)$ satisfies $\sum_{i \in N} \max\{p(R_i), \lambda(R)\} = m$ (respectively, $\sum_{i \in N} \min\{p(R_i), \lambda(R)\} = m$).

We next define the class of randomized uniform rules. Let $\lambda^-(R)$ and $\lambda^+(R)$ de-

[25]Indeed, for each permutation π, Ψ^{π} coincides with the priority rule with priority order π.

note the closest integers to $\lambda(R)$ satisfying $\lambda^-(R) \leq \lambda(R) \leq \lambda^+(R)$. A randomized uniform rule is the rule that determines the marginal distribution of each agent as follows. Each agent assigned his own peak under the uniform rule receives his own peak amount with probability one, but the remaining agents receive $\lambda^-(R)$ units with probability $\lambda^+(R) - \lambda(R)$ and $\lambda^+(R)$ units with probability $\lambda(R) - \lambda^-(R)$.[26]

Our second example is a rule belonging to this class. Consider the rule that assigns equal probability to all possible allocations where each agent assigned his own peak under the uniform rule receives his own peak amount and the remaining agents receive $\lambda^-(R)$ or $\lambda^+(R)$ units. This rule can be obtained by setting $\hat{\alpha}$ as follows.

Let U denote the deterministic rule that assigns his own peak amount to each agent receiving his own peak under the uniform rule and determines the assignments of the remaining agents as follows. When there is excess supply, the first remaining $n^1(R)$ agents receive $\lambda^-(R)$ units and the other $n^2(R)$ agents receive $\lambda^+(R)$, where $n^1(R)$ and $n^2(R)$ satisfy $n^1(R)\lambda^-(R) + n^2(R)\lambda^+(R) = (n^1(R) + n^2(R))\lambda(R)$. Conversely, when there is excess demand, the first remaining $n^1(R)$ agents receive $\lambda^+(R)$ units and the other $n^2(R)$ agents receive $\lambda^-(R)$ units, where $n^1(R)$ and $n^2(R)$ satisfy $n^1(R)\lambda^+(R) + n^2(R)\lambda^-(R) = (n^1(R) + n^2(R))\lambda(R)$.

Let $\hat{\alpha}$ assign probability one to the rule $\hat{f}^U$ induced by U. Then, the rule $f = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f} = \frac{1}{n!} \sum_{\pi \in \Pi} \bar{U}^\pi$ is in the class of randomized uniform rules.

2-3. Let F be a deterministic rule studied in Example 1-1. Consider the rule $\hat{f}^F$ that selects the allocation chosen by F^π with equal probability for each permutation π, i.e., $\hat{\alpha}$ assigns probability one to $\hat{f}^F$. Then, we can show that this rule satisfies sd-strategy-proofness, sd-efficiency, and anonymity in probabilistic allocation. To see this, consider a preference profile R, an agent i, and an assignment x. Assume that there is excess supply and $x > p(R_i)$ (the other case is similar). Let $\pi \in \Pi$. Then, by the definitions of F and F^π, we can show that $F_i^\pi(R) = x$ if and only if $F_i^\pi(R_i^0, R_{-i}) = x$.[27] Thus, $\bar{F}_i^\pi(R)(\{x\}) = \bar{F}_i^\pi(R_i^0, R_{-i})(\{x\})$. Therefore, $\hat{f}^F(R)(\{x\}) = \frac{1}{n!} \sum_{\pi \in \Pi} \bar{F}_i^\pi(R)(\{x\}) = \frac{1}{n!} \sum_{\pi \in \Pi} \bar{F}_i^\pi(R_i^0, R_{-i})(\{x\}) =$

[26] When $\lambda(R)$ is an integer, the remaining agents receive $\lambda(R)$ units with probability one.

[27] To see this, let $j = \pi^{-1}(i)$ and $\bar{R} = (R_i^0, R_{-i})$. Assume that $F_i^\pi(R) = x$. Then, by the definition of F^π, $x = F_i^\pi(R) = F_j(R^\pi)$. Since $p(R_{\pi(j)}) = p(R_i) < x$, by the definition of F (Example 1-1), $F_j(R^\pi) = F_j(\bar{R}^\pi)$. Thus, by the definition of F^π, $x = F_j(\bar{R}^\pi) = F_i^\pi(\bar{R})$. Similarly, we can show that if $x = F_i^\pi(\bar{R})$, then $F_i^\pi(R) = x$.

$\hat{f}^F(R_i^0, R_{-i})(\{x\})$. Hence, Condition 2 holds in this example. Thus, by Theorem 2, $\hat{f}^F$ satisfies sd-strategy-proofness, sd-efficiency, and anonymity in probabilistic allocation.

As mentioned in Example 1-1, the class of deterministic rules studied in Example 1-1 coincides with the class of strategy-proof and Pareto-efficient deterministic rules. Thus, this example also shows that the rule $\hat{f}^F$ induced by any strategy-proof and Pareto-efficient deterministic rule F satisfies sd-strategy-proofness, sd-efficiency, and anonymity in probabilistic allocation.[28]

2-4. Let $F^1, \ldots, F^K$ be distinct deterministic rules studied in Example 1-2. For each $k \in \{1, 2, \ldots, K\}$, let $\hat{f}^{F^k}$ be the rule that selects the allocation chosen by $F^{k\pi}$ with equal probability for each permutation π. Consider the rule $f = \frac{1}{K}\sum_{k=1}^{K} \hat{f}^{F^k}$, i.e., $\hat{\alpha}$ assigns probability $\frac{1}{K}$ to each rule in $\{\hat{f}^{F^1}, \ldots, \hat{f}^{F^K}\}$. Then, the rule f also satisfies sd-strategy-proofness, sd-efficiency, and anonymity in probabilistic allocation. To see this, consider a preference profile R, an agent i, and an assignment x. Assume that there is excess supply and $x > p(R_i)$ (the other case is similar). Let $\pi \in \Pi$ and $k \in \{1, 2, \ldots, K\}$. Then, by the definitions of F^k, $F^{k\pi}$, and $F^{k+1\pi}$, we can show that $F_i^{k\pi}(R) = x$ if and only if $F_i^{k+1\pi}(R_i^0, R_{-i}) = x$.[29] Thus, $\bar{F}_i^{k\pi}(R)(\{x\}) = \bar{F}_i^{k+1\pi}(R_i^0, R_{-i})(\{x\})$. Therefore, $\frac{1}{K}\frac{1}{n!}\sum_{k=1}^{K}\sum_{\pi\in\Pi} \bar{F}_i^{k\pi}(R)(\{x\}) = \frac{1}{K}\frac{1}{n!}\sum_{k=1}^{K}\sum_{\pi\in\Pi} \bar{F}_i^{k+1\pi}(R_i^0, R_{-i})(\{x\})$. Hence, Condition 2 also holds in this example, i.e., $\frac{1}{K}\sum_{k=1}^{K} \hat{f}^{F^k}(R)(\{x\}) = \frac{1}{K}\sum_{k=1}^{K} \hat{f}^{F^k}(R_i^0, R_{-i})(\{x\})$. Thus, by Theorem 2, f satisfies sd-strategy-proofness, sd-efficiency, and anonymity in probabilistic allocation.

Remark 1. In the one-dimensional public decision model with single-peaked preferences, any sd-strategy-proof and sd-efficient probabilistic rule can be represented as a convex combination of strategy-proof and Pareto-efficient deterministic rules (Peters, Roy, Sen, and Storcken, 2014; Pycia and Ünver, 2015; Roy and Sadhukhan, 2018). As shown in Example 3 below, this fact does not extend to our model.

[28]By Fact 3, this example also shows that there are sd-strategy-proof, sd-efficient, and anonymous rules other than the convex combinations of the random priority rule and the randomized uniform rules. This answers the open question raised by Ehlers and Klaus (2003).

[29]To see this, let $j = \pi^{-1}(i)$ and $\bar{R} = (R_i^0, R_{-i})$. Assume that $F_i^{k\pi}(R) = x$. Then, by the definition of $F^{k\pi}$, $x = F_i^{k\pi}(R) = F_j^k(R^\pi)$. Since $p(R_{\pi(j)}) = p(R_i) < x$, by the definition of F^k (Example 1-2), $F_j^k(R^\pi) = F_j^{k+1}(\bar{R}^\pi)$. Thus, by the definition of $F^{k+1\pi}$, $x = F_j^{k+1}(\bar{R}^\pi) = F_i^{k+1\pi}(\bar{R})$. Similarly, we can show that if $x = F_i^{k+1\pi}(\bar{R})$, then $F_i^{k\pi}(R) = x$.

Example 3. Let $n = 3$ and $m = 4$. Let F^1 be the deterministic rule defined by setting for each $R \in \mathcal{R}^N$,[30]

$$F^1(R) = \begin{cases} \Psi^{321}(R) & \text{if } p(R) \in \{(0,1,0),(0,1,1),(0,1,2),(2,1,0)\} \\ \Psi^{312}(R) & \text{if } p(R_1) = 1 \text{ and } p(R_2) + p(R_3) \leq 2 \\ \Psi(R) & \text{otherwise.} \end{cases}$$

Let F^2 be the deterministic rule defined by setting for each $R \in \mathcal{R}^N$,

$$F^2(R) = \begin{cases} (0,2,2) & \text{if } p(R) = (0,1,0) \\ (2, 2 - p(R_3), p(R_3)) & \text{if } p(R) \in \{(0,0,0),(0,0,1),(0,0,2),(1,0,2)\} \\ (2 - p(R_2), p(R_2), 2) & \text{if } p(R) \in \{(0,1,1),(0,1,2),(1,0,0),(1,0,1)\} \\ \Psi^{321}(R) & \text{if } p(R) \in \{(0,0,3),(1,1,0),(1,1,1)\} \\ \Psi(R) & \text{otherwise.} \end{cases}$$

Let $f = \frac{1}{2}\bar{F}^1 + \frac{1}{2}\bar{F}^2$. Then, f is sd-strategy-proof and sd-efficient,[31] but cannot be represented as a convex combination of strategy-proof and Pareto-efficient deterministic rules.

To see this, suppose on the contrary that there is $\alpha \in \Delta(\mathcal{F})$ such that $f = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}$ and for each $F \in \mathcal{F}$ with $\alpha_F > 0$, F is strategy-proof. Let $F \in \mathcal{F}$ be such that $\alpha_F > 0$. First, since f chooses only the allocations $(0,0,4)$ and $(2,2,0)$ with positive probability at the profile R^0, F must choose the allocation $(0,0,4)$ or $(2,2,0)$ at R^0. Next, let R_i^1 be a preference of agent i such that $p(R_i^1) = 1$ and $0 \; P_i^1 \; 2$. Then, since f chooses only the allocations $(1,3,0)$ and $(2,0,2)$ with positive probability at the profile $(R_1^1, R_{2,3}^0)$, $F(R_1^1, R_{2,3}^0) = (1,3,0)$ or $(2,0,2)$. Similarly, $F(R_2^1, R_{1,3}^0) = (3,1,0)$ or $(0,2,2)$ and $F(R_{1,2}^1, R_3^0) = (1,3,0)$ or $(3,1,0)$. We now assume that $F(R^0) = (0,0,4)$. Then, since F is strategy-proof, $F(R_2^1, R_{1,3}^0) =$

[30] Let Ψ^{321} and Ψ^{312} denote the priority rule with priority order $3,2,1$ and $3,1,2$, respectively. Recall also that Ψ is the priority rule with priority order $1,2,3$.

[31] For each $R \in \mathcal{R}^N$, each $i \in N$, and each $k \in \{1,2\}$, let $\lambda^{R,i}(k) \neq k$ if $i = 1$ and $p(R) \in \{(1,1,0),(1,1,1)\}$, and let $\lambda^{R,i}(k) = k$ otherwise. Then, f is obtained as a special case of the rules described in Example 1-3.

$(3,1,0)$.[32] Then, strategy-proofness of F also implies that $F(R^1_{1,2}, R^0_3) = (3,1,0)$.[33] However, applying similar arguments, we can also show that $F(R^1_1, R^0_{2,3}) = (1,3,0)$ and $F(R^1_{1,2}, R^0_3) = (1,3,0)$. This is a contradiction. Thus, $F(R^0) = (2,2,0)$. However, in this case, we can also obtain a contradiction.[34]

5 Proofs

We provide the proofs of our main theorems presented in the previous section. First, we provide an alternative characterization of the class of sd-strategy-proof and sd-efficient rules, which plays a central role.

Given $i \in N$ and $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$ with $\sum_{j\neq i} p(R_j) \leq m$, let $e(R_{-i}) = m - \sum_{j\neq i} p(R_j)$. For each $i \in N$, let $\mathcal{A}_i$ denote the set of functions $a_i : \mathcal{R}^{N\setminus\{i\}} \to \Delta(M)$ such that for each $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$ with $\sum_{j\neq i} p(R_j) \leq m$, $a_i(R_{-i})([0, e(R_{-i})]) = 1$. That is, $a_i(R_{-i})$ is a probability distribution over M assigning probability 0 outside the interval $[0, e(R_{-i})]$ whenever $\sum_{j\neq i} p(R_j) \leq m$.

Given $a = (a_i)_{i\in N} \in \times_{i\in N}\mathcal{A}_i$ and $R \in \mathcal{R}^N$ with $\sum_{i\in N} p(R_i) \leq m$, let $\mathcal{Q}^a(R)$ denote the set of probability distributions Q over Z such that for each $i \in N$ and each $x \in M$,

$$Q_i(\{x\}) = \begin{cases} 0 & \text{if } x < p(R_i) \\ a_i(R_{-i})([0, p(R_i)]) & \text{if } x = p(R_i) \\ a_i(R_{-i})(\{x\}) & \text{if } x > p(R_i). \end{cases}$$

That is, under $Q \in \mathcal{Q}^a(R)$, agent i's marginal distribution coincides with $a_i(R_{-i})$ outside the interval $[0, p(R_i)]$ and the probability that agent i receives his own peak amount is equal to the probability assigned to the interval $[0, p(R_i)]$ under $a_i(R_{-i})$.

Let $\mathcal{A}$ denote the set of profiles of functions $a = (a_i)_{i\in N} \in \times_{i\in N}\mathcal{A}_i$ such that for each $R \in \mathcal{R}^N$ with $\sum_{i\in N} p(R_i) \leq m$, $\mathcal{Q}^a(R) \neq \emptyset$.

Similarly, for each $i \in N$, let $\mathcal{B}_i$ denote the set of functions $b_i : \mathcal{R}^{N\setminus\{i\}} \to \Delta(M)$

[32] To see this, suppose on the contrary that $F(R^1_2, R^0_{1,3}) = (0,2,2)$. Then, since $0 \; P^1_2 \; 2$, $F_2(R^0) = 0 \; P^1_2 \; 2 = F_2(R^1_2, R^0_{1,3})$, which contradicts strategy-proofness.

[33] To see this, suppose on the contrary that $F(R^1_{1,2}, R^0_3) = (1,3,0)$. Then, $F_1(R^1_{1,2}, R^0_3) = 1 \; P^0_1 \; 3 = F_1(R^1_2, R^0_{1,3})$, which contradicts strategy-proofness.

[34] By strategy-proofness, $F(R^1_1, R^0_{2,3}) = (2,0,2)$ and $F(R^1_2, R^0_{1,3}) = (0,2,2)$. However, strategy-proofness also implies that $F(R^1_{1,2}, R^0_3) = (3,1,0)$ and $F(R^1_{1,2}, R^0_3) = (1,3,0)$.

such that for each $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$ with $\sum_{j\neq i} p(R_j) \leq m$, $b_i(R_{-i})([e(R_{-i}), m]) = 1$. That is, $b_i(R_{-i})$ is a probability distribution over M assigning probability 0 outside the interval $[e(R_{-i}), m]$ whenever $\sum_{j\neq i} p(R_j) \leq m$.

Given $b = (b_i)_{i\in N} \in \times_{i\in N}\mathcal{B}_i$ and $R \in \mathcal{R}^N$ with $\sum_{i\in N} p(R_i) \geq m$, let $\mathcal{Q}^b(R)$ denote the set of probability distributions Q over Z such that for each $i \in N$ and each $x \in M$,

$$Q_i(\{x\}) = \begin{cases} 0 & \text{if } x > p(R_i) \\ b_i(R_{-i})([p(R_i), m]) & \text{if } x = p(R_i) \\ b_i(R_{-i})(\{x\}) & \text{if } x < p(R_i). \end{cases}$$

That is, under $Q \in \mathcal{Q}^b(R)$, agent i's marginal distribution coincides with $b_i(R_{-i})$ outside the interval $[p(R_i), m]$ and the probability that agent i receives his own peak amount is equal to the probability assigned to the interval $[p(R_i), m]$ under $b_i(R_{-i})$.

Let $\mathcal{B}$ denote the set of profiles of functions $b = (b_i)_{i\in N} \in \times_{i\in N}\mathcal{B}_i$ such that for each $R \in \mathcal{R}^N$ with $\sum_{i\in N} p(R_i) \geq m$, $\mathcal{Q}^b(R) \neq \emptyset$.

Given a pair $(a, b) \in \mathcal{A} \times \mathcal{B}$ and a preference profile $R \in \mathcal{R}^N$, let

$$\mathcal{Q}^{a,b}(R) = \begin{cases} \mathcal{Q}^a(R) & \text{if } \sum_{i\in N} p(R_i) \leq m \\ \mathcal{Q}^b(R) & \text{if } \sum_{i\in N} p(R_i) \geq m. \end{cases}$$

Proposition 1 below says that a rule is sd-strategy-proof and sd-efficient if and only if there is a pair $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that it is a selection from $\mathcal{Q}^{a,b}$.

Proposition 1. *A rule f is sd-strategy-proof and sd-efficient if and only if there is a pair $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that for each $R \in \mathcal{R}^N$, $f(R) \in \mathcal{Q}^{a,b}(R)$.*

For the deterministic model, it is well known that under strategy-proofness and Pareto-efficiency, each agent's assignment lies within predetermined upper and lower bounds (Sprumont, 1991; Barberà, Jackson, and Neme, 1997). Proposition 1 generalizes this fact to our probabilistic model. Indeed, the following characterization of the class of strategy-proof and Pareto-efficient deterministic rules is obtained as a corollary of Proposition 1.

Corollary 1. *A deterministic rule F is strategy-proof and Pareto-efficient if and only if for each $i \in N$, there are functions $A_i : \mathcal{R}^{N\setminus\{i\}} \to M$ and $B_i : \mathcal{R}^{N\setminus\{i\}} \to M$*

such that for each $R \in \mathcal{R}^N$,

$$F_i(R) = \begin{cases} \max\{p(R_i), A_i(R_{-i})\} & \text{if } \sum_{j\in N} p(R_j) \leq m \\ \min\{p(R_i), B_i(R_{-i})\} & \text{if } \sum_{j\in N} p(R_j) \geq m, \end{cases}$$

and for each $R \in \mathcal{R}^N$, $\sum_{j\in N} F_j(R) = m$.

The proof of Corollary 1 is in the Appendix.

We prove Proposition 1. First, we show two basic results. The next property says that if an agent's preference is changed but his peak amount remains the same, then the agent's marginal distribution should not change.

Own peak-onlyness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$ with $p(R_i) = p(R'_i)$, $f_i(R) = f_i(R'_i, R_{-i})$.

Lemma 1. *If a rule satisfies sd-strategy-proofness and sd-efficiency, then it satisfies own peak-onlyness.*

Proof of Lemma 1. Let f be a rule satisfying sd-strategy-proofness and sd-efficiency. Let $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$ be such that $p(R_i) = p(R'_i)$. We show that for each $x \in M$, $f_i(R)(\{x\}) = f_i(R'_i, R_{-i})(\{x\})$. Let $x \in M$. Assume that $\sum_{j\in N} p(R_j) \leq m$ (the other case can be treated symmetrically). Then, by Fact 1 (same-sideness), if $x < p(R_i)$, $f_i(R)(\{x\}) = 0 = f_i(R'_i, R_{-i})(\{x\})$. Next, we show that for each $y \in [p(R_i), m]$, $f_i(R)([p(R_i), y]) = f_i(R'_i, R_{-i})([p(R_i), y])$. Let $y \in [p(R_i), m]$. By contradiction, suppose that $f_i(R)([p(R_i), y]) < f_i(R'_i, R_{-i})([p(R_i), y])$ (the other case is similar). Then,

$$\begin{aligned} f_i(R'_i, R_{-i})(U(R_i, y)) &= f_i(R'_i, R_{-i})([p(R_i), y]) \text{ (by Fact 1 and } p(R_i) = p(R'_i)) \\ &> f_i(R)([p(R_i), y]) \\ &= f_i(R)(U(R_i, y)) \text{ (by Fact 1)}, \end{aligned}$$

which contradicts sd-strategy-proofness.

Thus, if $x = p(R_i)$, $f_i(R)(\{x\}) = f_i(R'_i, R_{-i})(\{x\})$. Finally, if $x > p(R_i)$, then,

$$\begin{aligned} f_i(R)(\{x\}) &= f_i(R)([p(R_i), x]) - f_i(R)([p(R_i), x-1]) \\ &= f_i(R'_i, R_{-i})([p(R_i), x]) - f_i(R'_i, R_{-i})([p(R_i), x-1]) \\ &= f_i(R'_i, R_{-i})(\{x\}). \end{aligned}$$

Hence, for each $x \in M$, $f_i(R)(\{x\}) = f_i(R'_i, R_{-i})(\{x\})$. □

For the one-dimensional and probabilistic public decision model with single-peaked preferences, the property called "uncompromisingness" is introduced by Ehlers, Peters, and Storcken (2002).[35] We extend this notion to our model. The next property says that when an agent's preference is changed, if there is excess supply (respectively, excess demand) at the initial profile and there is still excess supply (respectively, excess demand) at the new profile, then the agent's marginal distribution should not change outside the interval whose endpoints are the initial and new peaks of the agent.

Given $R_i, R'_i \in \mathcal{R}^N$, let $E(R_i, R'_i) = [\min\{p(R_i), p(R'_i)\}, \max\{p(R_i), p(R'_i)\}]$.

Uncompromisingness: For each $R \in \mathcal{R}^N$, each $i \in N$, each $R'_i \in \mathcal{R}$, and each $x \in M \setminus E(R_i, R'_i)$,
if $\sum_{j \in N} p(R_j) \leq m$ and $p(R'_i) + \sum_{j \neq i} p(R_j) \leq m$, $f_i(R)(\{x\}) = f_i(R'_i, R_{-i})(\{x\})$,
if $\sum_{j \in N} p(R_j) \geq m$ and $p(R'_i) + \sum_{j \neq i} p(R_j) \geq m$, $f_i(R)(\{x\}) = f_i(R'_i, R_{-i})(\{x\})$.

Lemma 2.[36] *If a rule satisfies sd-strategy-proofness and sd-efficiency, then it satisfies uncompromisingness.*

Proof of Lemma 2. Let f be a rule satisfying sd-strategy-proofness and sd-efficiency. Let $R \in \mathcal{R}^N$, $i \in N$, $R'_i \in \mathcal{R}$, and $x \in M \setminus E(R_i, R'_i)$. Assume that $\sum_{j \in N} p(R_j) \leq m$ and $p(R'_i) + \sum_{j \neq i} p(R_j) \leq m$ (the other case can be treated symmetrically). If $p(R_i) = p(R'_i)$, by Lemma 1 (own peak-onlyness), $f_i(R) = f_i(R'_i, R_{-i})$. Thus, we assume that $p(R_i) < p(R'_i)$ (the other case is similar).

Case 1. $x < p(R_i)$.

By Fact 1 (same-sideness), $f_i(R)(\{x\}) = 0 = f_i(R'_i, R_{-i})(\{x\})$.

Case 2. $x > p(R'_i)$.

[35] For the deterministic model, uncompromisingness is introduced by Border and Jordan (1983). Ehlers, Peters, and Storcken (2002) extend it to the probabilistic model.

[36] Similar results also hold in related models. For the deterministic model, Ching (1994) shows that uncompromisingness is implied by strategy-proofness and Pareto-efficiency. Kureishi (2000) and Kureishi and Mizukami (2005) introduce a different version of uncompromisingness and show that it is implied by strategy-proofness and Pareto-efficiency in Sasaki's (1997) probabilistic model.

Let $y \in \{x-1, x\}$. We show $f_i(R)([p(R_i), y]) = f_i(R'_i, R_{-i})([p(R_i), y])$. First,

$$\begin{aligned} f_i(R)([p(R_i), y]) &= f_i(R)(U(R_i, y)) \text{ (by Fact 1)} \\ &\geq f_i(R'_i, R_{-i})(U(R_i, y)) \text{ (by sd-strategy-proofness)} \\ &= f_i(R'_i, R_{-i})([p(R_i), y]) \text{ (by Fact 1).} \end{aligned}$$

Next, let $R''_i \in \mathcal{R}$ be such that $p(R''_i) = p(R'_i)$ and $U(R''_i, p(R_i)) = [p(R_i), y]$. Then, by Lemma 1 (own peak-onlyness), $f_i(R''_i, R_{-i}) = f_i(R'_i, R_{-i})$. Thus,

$$\begin{aligned} f_i(R)([p(R_i), y]) &= f_i(R)(U(R''_i, p(R_i))) \text{ (by } U(R''_i, p(R_i)) = [p(R_i), y]) \\ &\leq f_i(R''_i, R_{-i})(U(R''_i, p(R_i))) \text{ (by sd-strategy-proofness)} \\ &= f_i(R'_i, R_{-i})(U(R''_i, p(R_i))) \text{ (by Lemma 1)} \\ &= f_i(R'_i, R_{-i})([p(R_i), y]) \text{ (by } U(R''_i, p(R_i)) = [p(R_i), y]). \end{aligned}$$

Hence, for each $y \in \{x-1, x\}$, $f_i(R)([p(R_i), y]) = f_i(R'_i, R_{-i})([p(R_i), y])$. Then,

$$\begin{aligned} f_i(R)(\{x\}) &= f_i(R)([p(R_i), x]) - f_i(R)([p(R_i), x-1]) \\ &= f_i(R'_i, R_{-i})([p(R_i), x]) - f_i(R'_i, R_{-i})([p(R_i), x-1]) \\ &= f_i(R'_i, R_{-i})(\{x\}). \end{aligned}$$

Thus, $f_i(R)(\{x\}) = f_i(R'_i, R_{-i})(\{x\})$. □

We are now ready to provide the proof of Proposition 1.

Proof of Proposition 1.
***Only if* part.** Let f be a rule satisfying sd-strategy-proofness and sd-efficiency. The proof consists of two steps.

Step 1. *Construction of* $a = (a_i)_{i\in N} \in \times_{i\in N}\mathcal{A}_i$ *and* $b = (b_i)_{i\in N} \in \times_{i\in N}\mathcal{B}_i$.

Proof of Step 1. For each $i \in N$ and $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$, let $a_i(R_{-i}) = f_i(R^0_i, R_{-i})$ and $b_i(R_{-i}) = f_i(R^m_i, R_{-i})$.

Let $i \in N$. We show that $a_i \in \mathcal{A}_i$. Let $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$. Since $f_i(R^0_i, R_{-i}) \in \Delta(M)$, $a_i(R_{-i}) \in \Delta(M)$. Next, assume that $\sum_{j\neq i} p(R_j) \leq m$. We show that $a_i(R_{-i})([0, e(R_{-i})]) = 1$. By Fact 1 (same-sideness), if $e(R_{-i}) = 0$, $a_i(R_{-i})(\{0\}) = f_i(R^0_i, R_{-i})(\{0\}) = 1$. Thus, we assume that $e(R_{-i}) \geq 1$. By contradiction, suppose that

$$a_i(R_{-i})([0, e(R_{-i})]) = f_i(R_i^0, R_{-i})([0, e(R_{-i})]) < 1.$$

Let $\hat{R}_i \in \mathcal{R}$ be such that $p(\hat{R}_i) = e(R_{-i})$. Then, by Fact 1 (same-sideness), $f_i(\hat{R}_i, R_{-i})(\{e(R_{-i})\}) = 1$. Thus,

$$\begin{aligned} f_i(\hat{R}_i, R_{-i})(U(R_i^0, e(R_{-i}))) &= f_i(\hat{R}_i, R_{-i})([0, e(R_{-i})]) \\ &= 1 \\ &> f_i(R_i^0, R_{-i})([0, e(R_{-i})]) \\ &= f_i(R_i^0, R_{-i}))(U(R_i^0, e(R_{-i}))), \end{aligned}$$

which contradicts sd-strategy-proofness. Similarly, we can show that $b_i \in \mathcal{B}_i$. □

Step 2. *For each* $R \in \mathcal{R}^N$, $f(R) \in \mathcal{Q}^{a,b}(R)$, *and* $(a, b) \in \mathcal{A} \times \mathcal{B}$.

Proof of Step 2. Let $R \in \mathcal{R}^N$, $i \in N$, and $x \in M$. Assume that $\sum_{j \in N} p(R_j) \leq m$ (the other case is similar). Then, by Fact 1 (same-sideness), if $x < p(R_i)$, $f_i(R)(\{x\}) = 0$. Note that $p(R_i^0) + \sum_{j \neq i} p(R_j) \leq m$. Thus, by Lemma 2 (uncompromisingness), if $x > p(R_i)$, $f_i(R)(\{x\}) = f_i(R_i^0, R_{-i})(\{x\}) = a_i(R_{-i})(\{x\})$. Since $f_i(R)(M) = 1$,

$$f_i(R)([0, p(R_i) - 1]) + f_i(R)(\{p(R_i)\}) + f_i(R)([p(R_i) + 1, m]) = 1.$$

Thus, $f_i(R)(\{p(R_i)\}) = 1 - a_i(R_{-i})([p(R_i) + 1, m]) = a_i(R_{-i})([0, p(R_i)])$. Hence, $f(R) \in \mathcal{Q}^a(R)$. Since $f(R) \in \Delta(Z)$, $\mathcal{Q}^a(R) \neq \emptyset$. Thus, $a \in \mathcal{A}$. □

***If* part.** Let $(a, b) \in \mathcal{A} \times \mathcal{B}$. Let $f^{a,b}$ be a rule such that for each $R \in \mathcal{R}^N$, $f^{a,b}(R) \in \mathcal{Q}^{a,b}(R)$. Then, $f^{a,b}$ satisfies same-sideness. Thus, by Fact 1, $f^{a,b}$ is sd-efficient. We show that $f^{a,b}$ is sd-strategy-proof. Let $R \in \mathcal{R}^N$, $i \in N$, and $R_i' \in \mathcal{R}$. Assume that $\sum_{j \in N} p(R_j) \leq m$ (the other case is similar). Then, $f^{a,b}(R) \in \mathcal{Q}^a(R)$. Let $z \in M$. We show that $f_i^{a,b}(R)(U(R_i, z)) \geq f_i^{a,b}(R_i', R_{-i})(U(R_i, z))$. Since $f^{a,b}$ satisfies own peak-onlyness, we only consider the possibility of manipulation via moving i's own peak. Let $x \in [0, p(R_i)]$ and $y \in [p(R_i), m]$ be such that $U(R_i, z) = [x, y]$.

Case 1. $p(R_i') < p(R_i)$.

In this case, for each $y' \in [p(R_i) + 1, y]$, $f_i^{a,b}(R)(\{y'\}) = a_i(R_{-i})(\{y'\}) =$

$f_i^{a,b}(R_i', R_{-i})(\{y'\})$. Thus,

$$f_i^{a,b}(R)([p(R_i)+1, y]) = f_i^{a,b}(R_i', R_{-i})([p(R_i)+1, y]).$$

Note also that for each $x' \in [p(R_i')+1, p(R_i)]$, $f_i^{a,b}(R_i', R_{-i})(\{x'\}) = a_i(R_{-i})(\{x'\})$.

Subcase 1-1. $x \leq p(R_i')$.

Then, since $f_i^{a,b}(R_i', R_{-i})([x, p(R_i')]) = a_i(R_{-i})([0, p(R_i')])$,

$$f_i^{a,b}(R)([x, p(R_i)]) = a_i(R_{-i})([0, p(R_i)]) = f_i^{a,b}(R_i', R_{-i})([x, p(R_i)]).$$

Thus,

$$\begin{aligned} f_i^{a,b}(R)(U(R_i, z)) &= f_i^{a,b}(R)([x, p(R_i)]) + f_i^{a,b}(R)([p(R_i)+1, y]) \\ &= f_i^{a,b}(R_i', R_{-i})([x, p(R_i)]) + f_i^{a,b}(R_i', R_{-i})([p(R_i)+1, y]) \\ &= f_i^{a,b}(R_i', R_{-i})(U(R_i, z)). \end{aligned}$$

Subcase 1-2. $p(R_i') < x$.

Then,

$$\begin{aligned} f_i^{a,b}(R)([x, p(R_i)]) &= a_i(R_{-i})([0, p(R_i)]) \\ &\geq a_i(R_{-i})([x, p(R_i)]) \\ &= f_i^{a,b}(R_i', R_{-i})([x, p(R_i)]). \end{aligned}$$

Thus,

$$\begin{aligned} f_i^{a,b}(R)(U(R_i, z)) &= f_i^{a,b}(R)([x, p(R_i)]) + f_i^{a,b}(R)([p(R_i)+1, y]) \\ &\geq f_i^{a,b}(R_i', R_{-i})([x, p(R_i)]) + f_i^{a,b}(R_i', R_{-i})([p(R_i)+1, y]) \\ &= f_i^{a,b}(R_i', R_{-i})(U(R_i, z)). \end{aligned}$$

Case 2. $p(R_i) < p(R_i') < e(R_{-i})$.

In this case, since $e(R_{-i}) = m - \sum_{j \neq i} p(R_j)$, $p(R_i') + \sum_{j \neq i} p(R_j) < m$. Thus, by $f^{a,b}(R_i', R_{-i}) \in \mathcal{Q}^a(R_i', R_{-i})$, $f_i^{a,b}(R_i', R_{-i})([p(R_i'), e(R_{-i})]) = 1$. Note also that $f_i^{a,b}(R)([p(R_i), e(R_{-i})]) = 1$.

Subcase 2-1. $y < p(R_i')$.

Then, $f_i^{a,b}(R_i', R_{-i})(U(R_i, z)) = 0$. Thus,

$$f_i^{a,b}(R)(U(R_i, z)) \geq 0 = f_i^{a,b}(R_i', R_{-i})(U(R_i, z)).$$

Subcase 2-2. $p(R_i') \leq y$.

If $y \geq e(R_{-i})$, then $x \leq p(R_i) < p(R_i') < e(R_{-i}) \leq y$. Thus,

$$f_i^{a,b}(R)(U(R_i, z)) = 1 = f_i^{a,b}(R_i', R_{-i})(U(R_i, z)).$$

Next, assume that $y < e(p(R_{-i}))$. Then,

$$f_i^{a,b}(R)([x, p(R_i) - 1]) = 0 = f_i^{a,b}(R_i', R_{-i})([x, p(R_i) - 1]), \text{ and}$$

$$f_i^{a,b}(R)([p(R_i), y]) = a_i(R_{-i})([0, y]) = f_i^{a,b}(R_i', R_{-i})([p(R_i), y]).$$

Thus,

$$\begin{aligned} f_i^{a,b}(R)(U(R_i, z)) &= f_i^{a,b}(R)([x, p(R_i) - 1]) + f_i^{a,b}(R)([p(R_i), y]) \\ &= f_i^{a,b}(R_i', R_{-i})([x, p(R_i) - 1]) + f_i^{a,b}(R_i', R_{-i})([p(R_i), y]) \\ &= f_i^{a,b}(R_i', R_{-i})(U(R_i, z)). \end{aligned}$$

Case 3. $e(p(R_{-i})) \leq p(R_i')$.

In this case, $p(R_i') + \sum_{j \neq i} p(R_j) \geq m$. Thus, by $f^{a,b}(R_i', R_{-i}) \in \mathcal{Q}^b(R_i', R_{-i})$, $f_i^{a,b}(R_i', R_{-i})([e(R_{-i}), p(R_i')]) = 1$. Note also that $f_i^{a,b}(R)([p(R_i), e(R_{-i})]) = 1$.

Subcase 3-1. $y < e(p(R_{-i}))$.

Then, $f_i^{a,b}(R)(U(R_i, z)) \geq 0 = f_i^{a,b}(R_i', R_{-i})(U(R_i, z))$.

Subcase 3-2. $e(p(R_{-i})) \leq y$.

Then, $f_i^{a,b}(R)(U(R_i, z)) = 1 \geq f_i^{a,b}(R_i', R_{-i})(U(R_i, z))$. $\square$

We now turn to the proof of Theorem 1. First, we provide the basic idea of the proof. Let f be an sd-strategy-proof and sd-efficient rule. In the proof of the *only if* part of Theorem 1, we first inductively construct a probability distribution over Pareto-efficient deterministic rules decomposing f. Although our model is quite different from theirs, we owe some basic proof techniques in this step to Pycia and Ünver (2015). The proof of this step is as follows.

First, for each preference profile R, since $f(R)(Z) = 1 > 0$, there is an allocation

z_R^1 such that $f(R)(\{z_R^1\}) > 0$. Let $F^1 : \mathcal{R}^N \to Z$ denote the deterministic rule that chooses the allocation z_R^1 for each profile R. Next, let α_{F^1} be the minimum value of the probabilities assigned to the allocation z_R^1 under f for each profile R. If $\alpha_{F^1} = 1$, then f coincides with $\bar{F}^1$, and so, the proof of this step is complete. Thus, we next assume that $\alpha_{F^1} < 1$. Let $g^1 = f - \alpha_{F^1}\bar{F}^1$.

Then, for each preference profile R, since $f(R)(Z) = 1$ and $\bar{F}^1(R)(Z) = 1$, $g^1(R)(Z) = f(R)(Z) - \alpha_{F^1}\bar{F}^1(R)(Z) = 1 - \alpha_{F^1} > 0$. Thus, for each profile R, there is an allocation z_R^2 such that $g^1(R)(\{z_R^2\}) > 0$. Let $F^2 : \mathcal{R}^N \to Z$ denote the deterministic rule that chooses the allocation z_R^2 for each profile R. Next, let α_{F^2} be the minimum value of the probabilities assigned to the allocation z_R^2 under g^1 for each profile R. If $\alpha_{F^1} + \alpha_{F^2} = 1$, then $f = \alpha_{F^1}\bar{F}^1 + \alpha_{F^2}\bar{F}^2$, and so, the proof of this step is complete. Thus, we next assume that $\alpha_{F^1} + \alpha_{F^2} < 1$. Let $g^2 = g^1 - \alpha_{F^2}\bar{F}^2$.

Repeating this argument, we can construct deterministic rules $F^1, F^2, \ldots, F^K$ and positive real numbers $\alpha_{F^1}, \alpha_{F^2}, \ldots, \alpha_{F^K}$ such that $f = \sum_{k=1}^K \alpha_{F^k}\bar{F}^k$ and $\sum_{k=1}^K \alpha_{F^k} = 1$. Finally, applying Lemma 2 (uncompromisingness), we can easily show that the constructed probability distribution $(\alpha_{F^1}, \alpha_{F^2}, \ldots, \alpha_{F^K})$ satisfies Condition 1.

In the proof of the *if* part of Theorem 1, we apply Proposition 1. Let $\alpha = (\alpha_F)_{F\in\mathcal{F}}$ be a probability distribution over $\mathcal{F}$ satisfying Condition 1. Let $f = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}$. For each agent i and each preference profile R_{-i} of the other agents, let $a_i(R_{-i}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})$ and $b_i(R_{-i}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^m, R_{-i})$. Then, we can show that the constructed functions $a = (a_i)_{i\in N}$ and $b = (b_i)_{i\in N}$ belong to $\mathcal{A}$ and $\mathcal{B}$, respectively, and f is a selection from $\mathcal{Q}^{a,b}$. Then, applying Proposition 1, we conclude that f is sd-strategy-proof and sd-efficient.

We next provide the formal proof of Theorem 1. Given $\alpha \in \mathbb{R}_+$, let $\Delta(Z, \alpha)$ be the set of distributions over Z whose sum is equal to α, i.e., $\Delta(Z, \alpha) = \{(q_z)_{z\in Z} \in \mathbb{R}_+^Z : \sum_{z\in Z} q_z = \alpha\}$ and let $\mathcal{G}(\alpha)$ be the set of functions from $\mathcal{R}^N$ to $\Delta(Z, \alpha)$. That is, g in $\mathcal{G}(\alpha)$ is a rule that choses a distribution in $\Delta(Z, \alpha)$ for each preference profile.

Let $\alpha > 0$ and $g \in \mathcal{G}(\alpha)$. Lemma 3-(i) below says that there is a deterministic rule F such that g places a positive weight on the allocation chosen by F for each preference profile. Let α_F be the minimum value of the weights placed on the allocation $F(R)$ under g for each profile R. Let $\hat{g} = g - \alpha_F \bar{F}$. Then, Lemma 3-(ii) says that $\hat{g}$ is a rule that choses a distribution in $\Delta(Z, \alpha - \alpha_F)$ for each preference profile and $\alpha \geq \alpha_F > 0$. Lemma 3-(iii) says that for each preference profile R and each allocation z, if g places 0 on z at R, then $\hat{g}$ also places 0 on z at R, and there are preference profile $\hat{R}$ and allocation $\hat{z}$ such that $\hat{g}$ places 0 on $\hat{z}$ at $\hat{R}$, but not under g.

Lemma 3. *Let $\alpha > 0$ and $g \in \mathcal{G}(\alpha)$. Then,*
(i) there is a deterministic rule F such that for each $R \in \mathcal{R}^N$, $g(R)(\{F(R)\}) > 0$.
Let $\alpha_F = \min\{g(R)(\{F(R)\}) : R \in \mathcal{R}^N\}$ and $\hat{g} = g - \alpha_F \bar{F}$. Then,
(ii) for each $R \in \mathcal{R}^N$, $\hat{g}(R) \in \Delta(Z, \alpha - \alpha_F)$ and $\alpha \geq \alpha_F > 0$,
(iii) $\{(R, z) \in \mathcal{R}^N \times Z : g(R)(\{z\}) = 0\} \subsetneq \{(R, z) \in \mathcal{R}^N \times Z : \hat{g}(R)(\{z\}) = 0\}$.

Proof of Lemma 3.
Proof of (i). By $\alpha > 0$, for each $R \in \mathcal{R}^N$, there is $z_R \in Z$ such that $g(R)(\{z_R\}) > 0$. For each $R \in \mathcal{R}^N$, let $F(R) = z_R$. Then, $g(R)(\{F(R)\}) > 0$.
Proof of (ii). Let $R \in \mathcal{R}^N$. First, we show that for each $z \in Z$, $\hat{g}(R)(\{z\}) \geq 0$. Let $z \in Z$. If $z \neq F(R)$, then, by $\bar{F}(R)(\{z\}) = 0$, $\hat{g}(R)(\{z\}) = g(R)(\{z\}) \geq 0$. If $z = F(R)$, then, by $\bar{F}(R)(\{z\}) = 1$ and the definition of α_F, $\hat{g}(R)(\{z\}) = g(R)(\{z\}) - \alpha_F \geq 0$. Thus, $\hat{g}(R)(\{z\}) \geq 0$. Next, by $g(R)(Z) = \alpha$ and $\bar{F}(R)(Z) = 1$, $\hat{g}(R)(Z) = g(R)(Z) - \alpha_F \bar{F}(R)(Z) = \alpha - \alpha_F$. Third, since for each $z \in Z$, $\hat{g}(R)(\{z\}) \geq 0$, $\alpha \geq \alpha_F$. Finally, by (i), $\alpha_F > 0$.
Proof of (iii). Let $(R, z) \in \mathcal{R}^N \times Z$ be such that $g(R)(\{z\}) = 0$. By (ii), $\hat{g}(R)(\{z\}) \geq 0$. Thus, by $g(R)(\{z\}) \geq \hat{g}(R)(\{z\})$, $\hat{g}(R)(\{z\}) = 0$. Next, let $\hat{R} \in \operatorname{argmin}\{g(R)(\{F(R)\}) : R \in \mathcal{R}^N\}$. Then, $g(\hat{R})(\{F(\hat{R})\}) > 0$, but $\hat{g}(\hat{R})(\{F(\hat{R})\}) = g(\hat{R})(\{F(\hat{R})\}) - \alpha_F \bar{F}(\{F(\hat{R})\}) = 0$. $\square$

Proof of Theorem 1.
***Only if* part.** Let f be a rule satisfying sd-strategy-proofness and sd-efficiency. We first show that there is $\alpha = (\alpha_F)_{F \in \mathcal{F}} \in \Delta(\mathcal{F})$ such that $f = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}$.

Note that for each $R \in \mathcal{R}^N$, $f(R) \in \Delta(Z, 1)$. Then, by Lemma 3-(i), there is a deterministic rule $F^1 : \mathcal{R}^N \to Z$ such that for each $R \in \mathcal{R}^N$, $f(R)(\{F^1(R)\}) > 0$. Let $\alpha_{F^1} = \min\{f(R)(\{F^1(R)\}) : R \in \mathcal{R}^N\}$ and $g^1 = f - \alpha_{F^1} \bar{F}^1$. Then, by Lemma 3-(ii), for each $R \in \mathcal{R}^N$, $g^1(R) \in \Delta(Z, 1 - \alpha_{F^1})$ and $1 \geq \alpha_{F^1} > 0$. If $\alpha_{F^1} = 1$, the proof of this step is complete. Thus, we assume that $1 - \alpha_{F^1} > 0$.

Then, by Lemma 3-(i), there is a deterministic rule $F^2 : \mathcal{R}^N \to Z$ such that for each $R \in \mathcal{R}^N$, $g^1(R)(\{F^2(R)\}) > 0$. Let $\alpha_{F^2} = \min\{g^1(R)(\{F^2(R)\}) : R \in \mathcal{R}^N\}$ and $g^2 = g^1 - \alpha_{F^2}\bar{F}^2$. Then, by Lemma 3-(ii), for each $R \in \mathcal{R}^N$, $g^1(R) \in \Delta(Z, 1 - \alpha_{F^1} - \alpha_{F^2})$ and $1 - \alpha_{F^1} \geq \alpha_{F^2} > 0$. If $\alpha_{F^1} + \alpha_{F^2} = 1$, the proof of this step is complete. Thus, we assume that $1 - \alpha_{F^1} - \alpha_{F^2} > 0$.

Repeating this argument, there are deterministic rules $F^1, F^2, \dots, F^K$ and positive real numbers $\alpha_{F^1}, \alpha_{F^2}, \dots, \alpha_{F^K}$ such that (a) for each $R \in \mathcal{R}^N$, $f(R)(\{F^1(R)\}) > 0$, $\alpha_{F^1} = \min\{f(R)(\{F^1(R)\}) : R \in \mathcal{R}^N\}$, and $g^1 = f - \alpha_{F^1}\bar{F}^1$, (b) for each $k \in \{2, \dots, K\}$ and each $R \in \mathcal{R}^N$, $g^{k-1}(R)(\{F^k(R)\}) > 0$, $\alpha_{F^k} = \min\{g^{k-1}(R)(\{F^k(R)\}) : R \in \mathcal{R}^N\}$, and $g^k = g^{k-1} - \alpha_{F^k}\bar{F}^k$, and (c) for each $R \in \mathcal{R}^N$, $g^K(R) \in \Delta(Z, 0)$ ((c) follows from Lemma 3-(iii)).[37]

Then, for each $R \in \mathcal{R}^N$ and each $z \in Z$, $0 = g^K(R)(\{z\}) = f(R)(\{z\}) - \sum_{k=1}^K \alpha_{F^k}\bar{F}^k(R)(\{z\})$ and $0 = g^K(R)(Z) = f(R)(Z) - \sum_{k=1}^K \alpha_{F^k}\bar{F}^k(R)(Z) = 1 - \sum_{k=1}^K \alpha_{F^k}$. Thus, $f = \sum_{k=1}^K \alpha_{F^k}\bar{F}^k$ and $\sum_{k=1}^K \alpha_{F^k} = 1$.

Note that by Fact 1 (same-sideness), for each $k \in \{1, \dots, K\}$ and each $R \in \mathcal{R}^N$, if $\sum_{j \in N} p(R_j) \leq m$, for each $i \in N$, $F_i^k(R) \geq p(R_i)$, and if $\sum_{j \in N} p(R_j) \geq m$, for each $i \in N$, $F_i^k(R) \leq p(R_i)$. Thus, for each $k \in \{1, \dots, K\}$, $F^k \in \mathcal{F}$.

Finally, for each $F \in \mathcal{F} \setminus \{F^1, \dots, F^K\}$, let $\alpha_F = 0$. Then, we conclude that $\alpha \in \Delta(\mathcal{F})$ and $f = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}$.

We next show that α satisfies Condition 1. Let $R \in \mathcal{R}^N$, $i \in N$, and $x \in M$. Assume that $\sum_{j \in N} p(R_j) \leq m$ and $x > p(R_i)$ (the proof of (1-ii) of Condition 1 is similar). Then, by Lemma 2 (uncompromisingness), $f_i(R)(\{x\}) = f_i(R_i^0, R_{-i})(\{x\})$. Since $f = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}$, we conclude that $\sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = f_i(R)(\{x\}) = f_i(R_i^0, R_{-i})(\{x\}) = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})(\{x\})$. □

***If* part.** Let $\alpha = (\alpha_F)_{F \in \mathcal{F}} \in \Delta(\mathcal{F})$ satisfy Condition 1. Let $f = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}$. Note that for each $R \in \mathcal{R}^N$ and $z \in Z$, $f(R)(\{z\}) \geq 0$ and $f(R)(Z) = \sum_{F \in \mathcal{F}} \alpha_F = 1$. Thus, for each $R \in \mathcal{R}^N$, $f(R) \in \Delta(Z)$.

For each $i \in N$ and each $R_{-i} \in \mathcal{R}^{N \setminus \{i\}}$, let $a_i(R_{-i}) = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})$ and $b_i(R_{-i}) = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R_i^m, R_{-i})$. By Proposition 1, it suffices to show that $a = (a_i)_{i \in N} \in \mathcal{A}$, $b = (b_i)_{i \in N} \in \mathcal{B}$, and for each $R \in \mathcal{R}^N$, $f(R) \in \mathcal{Q}^{a,b}(R)$.

[37]Note that for each $k, k' \in \{1, \dots, K\}$ with $k \neq k'$, $F^k \neq F^{k'}$. To see this, suppose there are $k, k' \in \{1, \dots, K\}$ such that $k < k'$ and for each $R \in \mathcal{R}^N$, $F^k(R) = F^{k'}(R)$. Note that there is $\hat{R} \in \mathcal{R}^N$ such that $g^k(\hat{R})(\{F^k(\hat{R})\}) = 0$. Then, $g^{k'-1}(\hat{R})(\{F^k(\hat{R})\}) = 0$. Since $F^k = F^{k'}$, we have $g^{k'-1}(\hat{R})(\{F^{k'}(\hat{R})\}) = 0$. However, by $\alpha_{F^{k'}} > 0$, $g^{k'-1}(\hat{R})(\{F^{k'}(\hat{R})\}) > 0$. This is a contradiction.

Let $i \in N$. We show that $a_i \in \mathcal{A}_i$. Let $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$. Then, for each $x \in M$, $a_i(R_{-i})(\{x\}) \geq 0$ and $a_i(R_{-i})(M) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})(M) = \sum_{F\in\mathcal{F}} \alpha_F = 1$. Thus, $a_i(R_{-i}) \in \Delta(M)$. Next, assume that $\sum_{j\neq i} p(R_j) \leq m$. Let $F \in \mathcal{F}$. Then, for each $j \neq i$, $F_j(R_i^0, R_{-i}) \geq p(R_j)$. Thus, $F_i(R_i^0, R_{-i}) = m - \sum_{j\neq i} F_j(R_i^0, R_{-i}) \leq m - \sum_{j\neq i} p(R_j) = e(R_{-i})$. Hence, $a_i(R_{-i})([0, e(R_{-i})]) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})([0, e(R_{-i})]) = 1$. Similarly, we can show that $b_i \in \mathcal{B}_i$.

Next, let $R \in \mathcal{R}^N$. Assume that $\sum_{j\in N} p(R_j) \leq m$ (the other case is similar). We show that $f(R) \in \mathcal{Q}^a(R)$. Let $i \in N$ and $x \in M$.

Case 1. $x < p(R_i)$.

Since for each $F \in \mathcal{F}$, $F_i(R) \geq p(R_i)$, we have $f_i(R)(\{x\}) = 0$.

Case 2. $x > p(R_i)$.

It follows from (1-i) of Condition 1 that $f_i(R)(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})(\{x\}) = a_i(R_{-i})(\{x\})$.

Case 3. $x = p(R_i)$.

By Cases 1 and 2, $f_i(R)(\{x\}) = 1 - f_i(R)([x+1, m]) = 1 - a_i(R_{-i})([x+1, m]) = a_i(R_{-i})([0, x])$.

Thus, $f(R) \in \mathcal{Q}^a(R)$. Since $f(R) \in \Delta(Z)$, $\mathcal{Q}^a(R) \neq \emptyset$. Thus, $a \in \mathcal{A}$. □

We next prove Theorem 2. Its structure is similar to that of Theorem 1, but it is not directly implied by Theorem 1. Before presenting the formal proof, we provide the basic idea of the proof. First, we can prove the *if* part of Theorem 2 by applying the same arguments as in the proof of Theorem 1 and the following lemma.

Lemma 4. *Let F be a deterministic rule. Then, $\hat{f}^F = \frac{1}{n!}\sum_{\pi\in\Pi} \bar{F}^\pi$ satisfies anonymity in probabilistic allocation.*

The proof of Lemma 4 is in the Appendix. We next turn to the proof of the *only if* part of Theorem 2. Let f be a rule satisfying sd-strategy-proofness, sd-efficiency, and anonymity in probabilistic allocation (APA). As in the proof of Theorem 1, we first inductively construct a probability distribution over $\hat{\mathcal{F}}$ decomposing f. The proof of this step is as follows.

First, for each preference profile R, since $f(R)(Z) = 1 > 0$, there is an allocation z_R^1 such that $f(R)(\{z_R^1\}) > 0$. Let $F^1 : \mathcal{R}^N \to Z$ denote the deterministic rule that chooses the allocation z_R^1 for each profile R. Let $\hat{f}^1 = \frac{1}{n!}\sum_{\pi\in\Pi} \bar{F}^{1\pi}$. Next, let $\hat{\alpha}_1$ denote the minimum value of the ratios of $f(R)(\{F^{1\pi}(R)\})$ to $\hat{f}^1(R)(\{F^{1\pi}(R)\})$

for each profile R and each permutation π. As we will see in Lemma 5 below, we can show that $0 < \hat{\alpha}_1 \leq 1$. If $\hat{\alpha}_1 = 1$, then f coincides with $\hat{f}^1$, and so, the proof of this step is complete. Thus, we next assume that $\hat{\alpha}_1 < 1$. Let $g^1 = f - \hat{\alpha}_1 \hat{f}^1$.

Then, for each preference profile R, since $f(R)(Z) = 1$ and $\hat{f}^1(R)(Z) = 1$, $g^1(R)(Z) = f(R)(Z) - \hat{\alpha}_1 \hat{f}^1(R)(Z) = 1 - \hat{\alpha}_1 > 0$. Thus, for each profile R, there is an allocation z_R^2 such that $g^1(R)(\{z_R^2\}) > 0$. Let $F^2 : \mathcal{R}^N \to Z$ denote the deterministic rule that chooses the allocation z_R^2 for each profile R. Let $\hat{f}^2 = \frac{1}{n!} \sum_{\pi \in \Pi} \bar{F}^{2\pi}$. Next, let $\hat{\alpha}_2$ denote the minimum value of the ratios of $g^1(R)(\{F^{2\pi}(R)\})$ to $\hat{f}^2(R)(\{F^{2\pi}(R)\})$ for each profile R and each permutation π. Then, as we will also see in Lemma 5 below, we can show that $0 < \hat{\alpha}_2 \leq 1 - \hat{\alpha}_1$. If $\hat{\alpha}_1 + \hat{\alpha}_2 = 1$, then $f = \hat{\alpha}_1 \hat{f}^1 + \hat{\alpha}_2 \hat{f}^2$, and so, the proof of this step is complete. Thus, we next assume that $\hat{\alpha}_1 + \hat{\alpha}_2 < 1$. Let $g^2 = g^1 - \hat{\alpha}_2 \hat{f}^2$.

Repeating this argument, we can construct deterministic rules $F^1, F^2, \ldots, F^K$ and positive real numbers $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_K$ such that $f = \sum_{k=1}^K \hat{\alpha}_k \hat{f}^k$ and $\sum_{k=1}^K \hat{\alpha}_k = 1$, where for each $k \in \{1, \ldots, K\}$, $\hat{f}^k = \frac{1}{n!} \sum_{\pi \in \Pi} \bar{F}^{k\pi}$. Finally, applying Lemma 2 (uncompromisingness), we can easily show that the constructed probability distribution $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_K)$ satisfies Condition 2.

Next, we provide the formal proof of Theorem 2. Given $\alpha \in \mathbb{R}_+$, let $\mathcal{G}^{APA}(\alpha)$ be the set of functions g in $\mathcal{G}(\alpha)$ satisfying anonymity in probabilistic allocation (APA), i.e., for each $R \in \mathcal{R}^N$, each $\pi \in \Pi$, and each $z \in Z$, $g(R)(\{z\}) = g(R^\pi)(\{z^\pi\})$.

Lemma 5-(i) and 5-(iii) below are parallel results to Lemma 3-(ii) and 3-(iii), respectively. Let $\alpha > 0$ and $g \in \mathcal{G}^{APA}(\alpha)$. Then, by Lemma 3-(i), there is a deterministic rule F such that g places a positive weight on the allocation chosen by F for each preference profile. Let $\hat{f} = \hat{f}^F$ and let α_F denote the minimum value of the ratios of $g(R)(\{F^\pi(R)\})$ to $\hat{f}(R)(\{F^\pi(R)\})$ for each profile R and each permutation π. Let $\hat{g} = g - \alpha_F \hat{f}$. Then, Lemma 5-(i) says that $\hat{g}$ is a rule that choses a distribution in $\Delta(Z, \alpha - \alpha_F)$ for each preference profile and $\alpha \geq \alpha_F > 0$. Lemma 5-(iii) says that for each preference profile R and each allocation z, if g places 0 on z at R, then $\hat{g}$ also places 0 on z at R, and there are preference profile $\hat{R}$ and allocation $\hat{z}$ such that $\hat{g}$ places 0 on $\hat{z}$ at $\hat{R}$, but not under g. Lemma 5-(ii) easily follows from APA of g and Lemma 4. Thus, we omit the proof of Lemma 5 (see the Appendix for the formal proof).

Lemma 5. *Let* $\alpha > 0$ *and* $g \in \mathcal{G}^{APA}(\alpha)$.
Let F *be a deterministic rule such that for each* $R \in \mathcal{R}^N$, $g(R)(\{F(R)\}) > 0$.
Let $\hat{f} = \hat{f}^F$, $\alpha_F = \min\left\{\frac{g(R)(\{F^\pi(R)\})}{\hat{f}(R)(\{F^\pi(R)\})} : (R,\pi) \in \mathcal{R}^N \times \Pi\right\}$, *and* $\hat{g} = g - \alpha_F \hat{f}$. *Then,*
(i) for each $R \in \mathcal{R}^N$, $\hat{g}(R) \in \Delta(Z, \alpha - \alpha_F)$ *and* $\alpha \geq \alpha_F > 0$,
(ii) $\hat{g}$ *satisfies APA,*
(iii) $\{(R,z) \in \mathcal{R}^N \times Z : g(R)(\{z\}) = 0\} \subsetneq \{(R,z) \in \mathcal{R}^N \times Z : \hat{g}(R)(\{z\}) = 0\}$.

Proof of Theorem 2.

***Only if* part.** Let f be a rule satisfying sd-strategy-proofness, sd-efficiency, and anonymity in probabilistic allocation (APA). We first show that there is $\hat{\alpha} \in \Delta(\hat{\mathcal{F}})$ such that $f = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}$.

Since for each $R \in \mathcal{R}^N$, $f(R) \in \Delta(Z,1)$, by Lemma 3-(i), there is a deterministic rule $F^1 : \mathcal{R}^N \to Z$ such that for each $R \in \mathcal{R}^N$, $f(R)(\{F^1(R)\}) > 0$. Let $\hat{f}^1 = \hat{f}^{F^1}$, $\hat{\alpha}_1 = \min\left\{\frac{f(R)(\{F^{1\pi}(R)\})}{\hat{f}^1(R)(\{F^{1\pi}(R)\})} : (R,\pi) \in \mathcal{R}^N \times \Pi\right\}$, and $g^1 = f - \hat{\alpha}_1 \hat{f}^1$. Then, by Lemma 5-(i), for each $R \in \mathcal{R}^N$, $g^1(R) \in \Delta(Z, 1-\hat{\alpha}_1)$ and $1 \geq \hat{\alpha}_1 > 0$. If $\hat{\alpha}_1 = 1$, the proof of this step is complete. Thus, we assume that $1 - \hat{\alpha}_1 > 0$.

Then, by Lemma 3-(i), there is a deterministic rule $F^2 : \mathcal{R}^N \to Z$ such that for each $R \in \mathcal{R}^N$, $g^1(R)(\{F^2(R)\}) > 0$. By Lemma 5-(ii), g^1 satisfies APA. Let $\hat{f}^2 = \hat{f}^{F^2}$, $\hat{\alpha}_2 = \min\left\{\frac{g^1(R)(\{F^{2\pi}(R)\})}{\hat{f}^2(R)(\{F^{2\pi}(R)\})} : (R,\pi) \in \mathcal{R}^N \times \Pi\right\}$, and $g^2 = g^1 - \hat{\alpha}_2 \hat{f}^2$. Then, by Lemma 5-(i), for each $R \in \mathcal{R}^N$, $g^2(R) \in \Delta(Z, 1-\hat{\alpha}_1-\hat{\alpha}_2)$ and $1-\hat{\alpha}_1 \geq \hat{\alpha}_2 > 0$. If $\hat{\alpha}_1 + \hat{\alpha}_2 = 1$, the proof of this step is complete. Thus, we assume that $1 - \hat{\alpha}_1 - \hat{\alpha}_2 > 0$.

Repeating this argument, there are deterministic rules $F^1, F^2, \ldots, F^K$ and positive real numbers $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_K$ such that (a) for each $R \in \mathcal{R}^N$, $f(R)(\{F^1(R)\}) > 0$, $\hat{f}^1 = \hat{f}^{F^1}$, $\hat{\alpha}_1 = \min\left\{\frac{f(R)(\{F^{1\pi}(R)\})}{\hat{f}^1(R)(\{F^{1\pi}(R)\})} : (R,\pi) \in \mathcal{R}^N \times \Pi\right\}$, and $g^1 = f - \hat{\alpha}_1 \hat{f}^1$, (b) for each $k \in \{2, \ldots, K\}$ and each $R \in \mathcal{R}^N$, $g^{k-1}(R)(\{F^k(R)\}) > 0$, $\hat{f}^k = \hat{f}^{F^k}$, $\hat{\alpha}_k = \min\left\{\frac{g^{k-1}(R)(\{F^{k\pi}(R)\})}{\hat{f}^k(R)(\{F^{k\pi}(R)\})} : (R,\pi) \in \mathcal{R}^N \times \Pi\right\}$, and $g^k = g^{k-1} - \hat{\alpha}_k \hat{f}^k$, and (c) for each $R \in \mathcal{R}^N$, $g^K(R) \in \Delta(Z, 0)$ ((c) follows from Lemma 5-(iii)).[38]

[38]Note that for each $k, k' \in \{1, \ldots, K\}$ with $k \neq k'$, $\hat{f}^k \neq \hat{f}^{k'}$ and for each $\pi, \pi' \in \Pi$, $F^{k\pi} \neq F^{k'\pi'}$. To see this, let $k, k' \in \{1, \ldots, K\}$ be such that $k < k'$.

First, we show $\hat{f}^k \neq \hat{f}^{k'}$. Suppose that $\hat{f}^k = \hat{f}^{k'}$. Note that there is a pair $(\hat{R}, \hat{\pi}) \in \mathcal{R}^N \times \Pi$ such that $g^k(\hat{R})(\{F^{k\hat{\pi}}(\hat{R})\}) = 0$. Then, $g^{k'-1}(\hat{R})(\{F^{k\hat{\pi}}(\hat{R})\}) = 0$. However, since $\hat{f}^k = \hat{f}^{k'}$, we have $\hat{f}^{k'}(\hat{R})(\{F^{k\hat{\pi}}(\hat{R})\}) = \hat{f}^k(\hat{R})(\{F^{k\hat{\pi}}(\hat{R})\}) > 0$. Also, since $g^{k'-1}(\hat{R})(\{F^{k\hat{\pi}}(\hat{R})\}) - \hat{\alpha}_{k'} \hat{f}^{k'}(\hat{R})(\{F^{k\hat{\pi}}(\hat{R})\}) \geq 0$ and $\hat{\alpha}_{k'} > 0$, we have $g^{k'-1}(\hat{R})(\{F^{k\hat{\pi}}(\hat{R})\}) \geq \hat{\alpha}_{k'} \hat{f}^{k'}(\hat{R})(\{F^{k\hat{\pi}}(\hat{R})\}) > 0$. This is a contradiction.

Next, let $\pi, \pi' \in \Pi$. We show $F^{k\pi} \neq F^{k'\pi'}$. Suppose that $F^{k\pi} = F^{k'\pi'}$. Let $G = F^{k\pi}$ and

Then, for each $R \in \mathcal{R}^N$ and each $z \in Z$, $0 = g^K(R)(\{z\}) = f(R)(\{z\}) - \sum_{k=1}^K \hat{\alpha}_k \hat{f}^k(R)(\{z\})$ and $0 = g^K(R)(Z) = f(R)(Z) - \sum_{k=1}^K \hat{\alpha}_{F^k} \hat{f}^k(R)(Z) = 1 - \sum_{k=1}^K \hat{\alpha}_k$. Thus, $f = \sum_{k=1}^K \hat{\alpha}_k \hat{f}^k$ and $\sum_{k=1}^K \hat{\alpha}_k = 1$.

Since $f = \sum_{k=1}^K \hat{\alpha}_k \hat{f}^k = \sum_{k=1}^K \hat{\alpha}_k \frac{1}{n!} \sum_{\pi \in \Pi} \bar{F}^{k\pi}$, by Fact 1 (same-sideness), for each $k \in \{1, \ldots, K\}$, $F^k \in \mathcal{F}$. Thus, for each $k \in \{1, \ldots, K\}$, $\hat{f}^k \in \hat{\mathcal{F}}$.

Finally, for each $\hat{f} \in \hat{\mathcal{F}} \setminus \{\hat{f}^1, \ldots, \hat{f}^K\}$, let $\hat{\alpha}_{\hat{f}} = 0$. Then, we conclude that $\hat{\alpha} \in \Delta(\hat{\mathcal{F}})$ and $f = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}$.

We next show that $\hat{\alpha}$ satisfies Condition 2. Let $R \in \mathcal{R}^N$, $i \in N$, and $x \in M$. Assume that $\sum_{j \in N} p(R_j) \leq m$ and $x > p(R_i)$ (the proof of (2-ii) of Condition 2 is similar). Then, by Lemma 2 (uncompromisingness), $f_i(R)(\{x\}) = f_i(R_i^0, R_{-i})(\{x\})$. Since $f = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}$, we conclude that $\sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}_i(R)(\{x\}) = f_i(R)(\{x\}) = f_i(R_i^0, R_{-i})(\{x\}) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}_i(R_i^0, R_{-i})(\{x\})$. □

***If* part.** Let $\hat{\alpha} \in \Delta(\hat{\mathcal{F}})$ satisfy Condition 2. Let $f = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}$. Note that for each $R \in \mathcal{R}^N$ and $z \in Z$, $f(R)(\{z\}) \geq 0$ and $f(R)(Z) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} = 1$. Thus, for each $R \in \mathcal{R}^N$, $f(R) \in \Delta(Z)$.

We first show that f satisfies anonymity in probabilistic allocation (APA). Let $R \in \mathcal{R}^N$, $\pi \in \Pi$, and $z \in Z$. Then, by Lemma 4, for each $\hat{f} \in \hat{\mathcal{F}}$, $\hat{f}(R)(\{z\}) = \hat{f}(R^\pi)(\{z^\pi\})$. Thus, $f(R)(\{z\}) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}(R)(\{z\}) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}(R^\pi)(\{z^\pi\}) = f(R^\pi)(\{z^\pi\})$.

We next show that f satisfies sd-strategy-proofness and sd-efficiency. For each $i \in N$ and each $R_{-i} \in \mathcal{R}^{N \setminus \{i\}}$, let $a_i(R_{-i}) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}_i(R_i^0, R_{-i})$ and $b_i(R_{-i}) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}_i(R_i^m, R_{-i})$. By Proposition 1, it suffices to show that $a = (a_i)_{i \in N} \in \mathcal{A}$, $b = (b_i)_{i \in N} \in \mathcal{B}$, and for each $R \in \mathcal{R}^N$, $f(R) \in \mathcal{Q}^{a,b}(R)$.

Let $i \in N$. We show that $a_i \in \mathcal{A}_i$. Let $R_{-i} \in \mathcal{R}^{N \setminus \{i\}}$. Then, for each $x \in M$, $a_i(R_{-i})(\{x\}) \geq 0$ and $a_i(R_{-i})(M) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}_i(R_i^0, R_{-i})(M) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} = 1$. Thus, $a_i(R_{-i}) \in \Delta(M)$. Next, assume that $\sum_{j \neq i} p(R_j) \leq m$. Let $F \in \mathcal{F}$. Then, for each $j \neq i$, $F_j(R_i^0, R_{-i}) \geq p(R_j)$. Thus, $F_i(R_i^0, R_{-i}) = m - \sum_{j \neq i} F_j(R_i^0, R_{-i}) \leq m - \sum_{j \neq i} p(R_j) = e(R_{-i})$. Also, for each $\pi \in \Pi$, since $F^\pi \in \mathcal{F}$, $F_i^\pi(R_i^0, R_{-i}) \leq e(R_{-i})$. Thus, for each $F \in \mathcal{F}$ and each $\pi \in \Pi$, $F_i^\pi(R_i^0, R_{-i}) \leq e(R_{-i})$. Hence, $a_i(R_{-i})([0, e(R_{-i})]) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}_i(R_i^0, R_{-i})([0, e(R_{-i})]) = 1$. Similarly, we can show that $b_i \in \mathcal{B}_i$.

$H = F^{k'\pi'}$. Then, for each $\hat{\pi} \in \Pi$, $G^{\hat{\pi}} = H^{\hat{\pi}}$. Since $\hat{f}^k = \frac{1}{n!} \sum_{\hat{\pi} \in \Pi} \bar{G}^{\hat{\pi}}$ and $\hat{f}^{k'} = \frac{1}{n!} \sum_{\hat{\pi} \in \Pi} \bar{H}^{\hat{\pi}}$, we have $\hat{f}^k = \frac{1}{n!} \sum_{\hat{\pi} \in \Pi} \bar{G}^{\hat{\pi}} = \frac{1}{n!} \sum_{\hat{\pi} \in \Pi} \bar{H}^{\hat{\pi}} = \hat{f}^{k'}$. This contradicts $\hat{f}^k \neq \hat{f}^{k'}$.

Next, let $R \in \mathcal{R}^N$. Assume that $\sum_{j \in N} p(R_j) \leq m$ (the other case is similar). We show that $f(R) \in \mathcal{Q}^a(R)$. Let $i \in N$ and $x \in M$.

Case 1. $x < p(R_i)$.

Since for each $F \in \mathcal{F}$ and each $\pi \in \Pi$, $F^\pi \in \mathcal{F}$, we have $f_i(R)(\{x\}) = 0$.

Case 2. $x > p(R_i)$.

It follows from (2-i) of Condition 2 that $f_i(R)(\{x\}) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}_i(R)(\{x\}) = \sum_{\hat{f} \in \hat{\mathcal{F}}} \hat{\alpha}_{\hat{f}} \hat{f}_i(R_i^0, R_{-i})(\{x\}) = a_i(R_{-i})(\{x\})$.

Case 3. $x = p(R_i)$.

By Cases 1 and 2, $f_i(R)(\{x\}) = 1 - f_i(R)([x+1, m]) = 1 - a_i(R_{-i})([x+1, m]) = a_i(R_{-i})([0, x])$.

Thus, $f(R) \in \mathcal{Q}^a(R)$. Since $f(R) \in \Delta(Z)$, $\mathcal{Q}^a(R) \neq \emptyset$. Thus, $a \in \mathcal{A}$. $\square$

Appendix

A.1. Other characterizations

We provide characterizations of the classes of sd-strategy-proof and sd-efficient rules satisfying other fairness axioms (anonymity or sd-equal treatment of equals). We first introduce two additional conditions for $\alpha \in \Delta(\mathcal{F})$.[39]

Condition 3 below says that when there is excess supply (respectively, excess demand), the marginal distribution of each agent whose peak is 0 (respectively, m) should not change for any permutation of the other agents' preferences.

Condition 3. For each $i \in N$, each $x \in M$, each $\pi \in \Pi$, and each $R_{-i} \in \mathcal{R}^{N \setminus \{i\}}$ with $\pi(i) = i$,

(3-i) if $\sum_{j \neq i} p(R_j) \leq m$,

$$\sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})(\{x\}) = \sum_{F \in \mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i}^\pi)(\{x\}),$$

[39] By Proposition 1, if a rule f satisfies sd-strategy-proofness and sd-efficiency, then there is a pair $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that f is a selection from $\mathcal{Q}^{a,b}$. Thus, under sd-strategy-proofness and sd-efficiency, Conditions 3 and 4 are equivalent to requiring the following (i) and (ii), respectively: (i) for each $i \in N$, a_i and b_i are symmetric functions with respect to R_{-i}, and (ii) for each $i, j \in N$, each $x \in M$, and each $R_{-i} \in \mathcal{R}^{N \setminus \{i\}}$, if $\sum_{k \neq i} p(R_k) \leq m$ and $\bar{R}_i = R_j$, $a_i(R_{-i})$ coincides with $a_j(\bar{R}_i, R_{-i,j})$ outside the interval $[0, p(R_j)]$, and if $\sum_{k \neq i} p(R_k) \geq m$ and $\bar{R}_i = R_j$, $b_i(R_{-i})$ coincides with $b_j(\bar{R}_i, R_{-i,j})$ outside the interval $[p(R_j), m]$. For the deterministic model, similar facts are also shown in Sprumont (1991).

(3-ii) if $\sum_{j\neq i} p(R_j) \geq m$,

$$\sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^m, R_{-i})(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^m, R_{-i}^\pi)(\{x\}),$$

where $R_{-i}^\pi = (R_{\pi(k)})_{k\in N\setminus\{i\}}$.

Condition 4 below says that for each preference profile R and each pair i, j of agents, when there is excess supply (respectively, excess demand) and agent i's peak is 0 (respectively, m), agent j's marginal distribution chosen when the preferences of agents i and j are replaced should coincide with agent i's marginal distribution chosen at R outside the interval $[0, p(R_j)]$ (respectively, $[p(R_j), m]$).

Condition 4. For each $R \in \mathcal{R}^N$, each $i, j \in N$, and each $x \in M$,
(4-i) if $\sum_{k\in N} p(R_k) \leq m$, $p(R_i) = 0$, $x > p(R_j)$, and $\bar{R}_i = R_j$,

$$\sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(\bar{R}_i, R_j^0, R_{-i,j})(\{x\}),$$

(4-ii) if $\sum_{k\in N} p(R_k) \geq m$, $p(R_i) = m$, $x < p(R_j)$, and $\bar{R}_i = R_j$,

$$\sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(\bar{R}_i, R_j^m, R_{-i,j})(\{x\}).$$

The following is a characterization of the class of sd-strategy-proof and sd-efficient rules satisfying anonymity.

Theorem 3. *A rule f satisfies sd-strategy-proofness, sd-efficiency, and anonymity if and only if there is $\alpha = (\alpha_F)_{F\in\mathcal{F}} \in \Delta(\mathcal{F})$ satisfying Conditions 1, 3, and 4 such that $f = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}$.*

Proof of Theorem 3.
***Only if* part.** Let f be a rule satisfying sd-strategy-proofness, sd-efficiency, and anonymity. Then, by Theorem 1, there is $\alpha = (\alpha_F)_{F\in\mathcal{F}} \in \Delta(\mathcal{F})$ satisfying Condition 1 such that $f = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}$.

We first show that α satisfies Condition 3. Let $i \in N$, $x \in M$, $\pi \in \Pi$, and $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$ be such that $\pi(i) = i$. Assume that $\sum_{j\neq i} p(R_j) \leq m$ (the proof of (3-ii) of Condition 3 is similar). Then, by anonymity, $f_i(R_i^0, R_{-i}) = f_i(R_i^0, R_{-i}^\pi)$. Since $f = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}$, we have $\sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i})(\{x\}) = f_i(R_i^0, R_{-i})(\{x\}) =$

$f_i(R_i^0, R_{-i}^\pi)(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{-i}^\pi)(\{x\})$.

We next show that α satisfies Condition 4. Let $R \in \mathcal{R}^N$, $i, j \in N$, and $x \in M$. Assume that $\sum_{k\in N} p(R_k) \le m$, $p(R_i) = 0$, $x > p(R_j)$, and $\bar{R}_i = R_j$ (the proof of (4-ii) of Condition 4 is similar). It follows from anonymity that $f_i(R) = f_j(\bar{R}_i, R_j^0, R_{-i,j})$. Hence, $\sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = f_i(R)(\{x\}) = f_j(\bar{R}_i, R_j^0, R_{-i,j})(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(\bar{R}_i, R_j^0, R_{-i,j})(\{x\})$. □

***If* part.** Let $\alpha = (\alpha_F)_{F\in\mathcal{F}} \in \Delta(\mathcal{F})$ satisfy Conditions 1, 3, and 4. Let $f = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}$. By Theorem 1, it suffices to show that f satisfies anonymity. Let $R \in \mathcal{R}^N$, $\pi \in \Pi$, $i \in N$, and $x \in M$. Assume that $\sum_{k\in N} p(R_k) \le m$ (the other case is similar). Let $j = \pi(i)$ and $\bar{R}_i = R_{\pi(j)}$. We show $f_i(R^\pi)(\{x\}) = f_j(R)(\{x\})$. Since f satisfies same-sideness, $f_i(R^\pi)([p(R_j), m]) = 1 = f_j(R)([p(R_j), m])$. Thus, if $x < p(R_j)$, $f_i(R^\pi)(\{x\}) = 0 = f_j(R)(\{x\})$. Hence, we assume that $p(R_j) \le x$.

Case 1. $p(R_j) < x$.

Then,

$$\begin{aligned}
f_i(R^\pi)(\{x\}) &= \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R^\pi)(\{x\}) \\
&= \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R_i^0, R_{\pi(j)}, R_{-i,j}^\pi)(\{x\}) \text{ (by Condition 1)} \\
&= \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(\bar{R}_i, R_j^0, R_{-i,j}^\pi)(\{x\}) \text{ (by Condition 4)} \\
&= \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(R_j^0, R_{-j})(\{x\}) \text{ (by Condition 3)} \\
&= \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(R)(\{x\}) \text{ (by Condition 1)} \\
&= f_j(R)(\{x\}).
\end{aligned}$$

Case 2. $p(R_j) = x$.

By Case 1, for each $y \in [p(R_j) + 1, m]$, $f_i(R^\pi)(\{y\}) = f_j(R)(\{y\})$. Since $f_i(R^\pi)([p(R_j), m]) = 1 = f_j(R)([p(R_j), m])$, we have $f_i(R^\pi)(\{x\}) = 1 - f_i(R^\pi)([x + 1, m]) = 1 - f_j(R)([x + 1, m]) = f_j(R)(\{x\})$. □

We also obtain the following characterization of the class of sd-strategy-proof and sd-efficient rules satisfying sd-equal treatment of equals.

Theorem 4. *A rule f satisfies sd-strategy-proofness, sd-efficiency, and sd-equal* *treatment of equals if and only if there is $\alpha = (\alpha_F)_{F\in\mathcal{F}} \in \Delta(\mathcal{F})$ satisfying Conditions 1 and 4 such that $f = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}$.*

Proof of Theorem 4.

***Only if* part.** Let f be a rule satisfying sd-strategy-proofness, sd-efficiency, and sd-equal treatment of equals. Then, by Theorem 1, there is $\alpha = (\alpha_F)_{F\in\mathcal{F}} \in \Delta(\mathcal{F})$ satisfying Condition 1 such that $f = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}$. We show that α satisfies Condition 4. Let $R \in \mathcal{R}^N$, $i, j \in N$, and $x \in M$. Assume that $\sum_{k\in N} p(R_k) \leq m$, $p(R_i) = 0$, $x > p(R_j)$, and $\bar{R}_i = R_j$ (the proof of (4-ii) of Condition 4 is similar). By Lemma 2 (uncompromisingness), $f_i(R)(\{x\}) = f_i(\bar{R}_i, R_{-i})(\{x\})$ and $f_j(\bar{R}_i, R_{-i})(\{x\}) = f_j(\bar{R}_i, R^0_j, R_{-ij})(\{x\})$. Since $\bar{R}_i = R_j$, it follows from sd-equal treatment of equals and Fact 2 that $f_i(\bar{R}_i, R_{-i}) = f_j(\bar{R}_i, R_{-i})$. Therefore, $f_i(R)(\{x\}) = f_j(\bar{R}_i, R^0_j, R_{-ij})(\{x\})$. Hence, $\sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = f_i(R)(\{x\}) = f_j(\bar{R}_i, R^0_j, R_{-ij})(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(\bar{R}_i, R^0_j, R_{-ij})(\{x\})$. □

***If* part.** Let $\alpha = (\alpha_F)_{F\in\mathcal{F}} \in \Delta(\mathcal{F})$ satisfy Conditions 1 and 4. Let $f = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}$. By Theorem 1, it suffices to show that f satisfies equal treatment of equals. Let $i, j \in N$, $x \in M$, and $R \in \mathcal{R}^N$. Assume that $R_i = R_j$ and $\sum_{k\in N} p(R_k) \leq m$ (the other case is similar). We show $f_i(R)(\{x\}) = f_j(R)(\{x\})$. By same-sideness of f, $f_i(R)([p(R_i), m]) = 1 = f_j(R)([p(R_j), m])$. Thus, if $x < p(R_i)$, $f_i(R)(\{x\}) = 0 = f_j(R)(\{x\})$. Hence, we assume that $p(R_i) \leq x$.

Case 1. $p(R_i) < x$.

Since $R_i = R_j$, it follows from Condition 4 that $\sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R^0_i, R_{-i})(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(R^0_j, R_{-j})(\{x\})$. Therefore, by Condition 1, $\sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R^0_i, R_{-i})(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(R^0_j, R_{-j})(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(R)(\{x\})$. Hence, $f_i(R)(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_i(R)(\{x\}) = \sum_{F\in\mathcal{F}} \alpha_F \bar{F}_j(R)(\{x\}) = f_j(R)(\{x\})$.

Case 2. $p(R_i) = x$.

By Case 1, for each $y \in [p(R_i) + 1, m]$, $f_i(R)(\{y\}) = f_j(R)(\{y\})$. Since $f_i(R)([p(R_i), m]) = 1 = f_j(R)([p(R_j), m])$, we have $f_i(R)(\{x\}) = 1 - f_i(R)([x + 1, m]) = 1 - f_j(R)([x + 1, m]) = f_j(R)(\{x\})$. □

A.2. Proofs of Corollary 1 and Lemmas 4 and 5

Proof of Corollary 1.

***Only if* part.** Let F be a strategy-proof and Pareto-efficient deterministic rule.

Then, by Proposition 1, there is a pair $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that for each $R \in \mathcal{R}^N$, $\bar{F}(R) \in \mathcal{Q}^{a,b}(R)$. Let $i \in N$. For each $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$, let $A_i(R_{-i}) \in \{x \in M : a_i(R_{-i})(\{x\}) > 0\}$ and $B_i(R_{-i}) \in \{x \in M : b_i(R_{-i})(\{x\}) > 0\}$. Let $R \in \mathcal{R}^N$. Assume that $\sum_{j \in N} p(R_j) \leq m$ (the other case is similar). Then, by $\bar{F}(R) \in \mathcal{Q}^a(R)$, if $p(R_i) < A_i(R_{-i})$, $\bar{F}_i(R)(\{A_i(R_{-i})\}) = a_i(R_{-i})(\{A_i(R_{-i})\})$, and if $p(R_i) \geq A_i(R_{-i})$, $\bar{F}_i(R)(\{p(R_i)\}) = a_i(R_{-i})([0, p(R_i)]) \geq a_i(R_{-i})(\{A_i(R_{-i})\})$. Thus, by $a_i(R_{-i})(\{A_i(R_{-i})\}) > 0$, $\bar{F}_i(R)(\{\max\{p(R_i), A_i(R_{-i})\}\}) > 0$. Since $\bar{F}_i(R)(\{F_i(R)\}) = 1$, $F_i(R) = \max\{p(R_i), A_i(R_{-i})\}$. Finally, since $F(R) \in Z$, $\sum_{j \in N} F_j(R) = m$. $\square$

***If* part.** Assume that for each $i \in N$, there are functions $A_i : \mathcal{R}^{N\setminus\{i\}} \to M$ and $B_i : \mathcal{R}^{N\setminus\{i\}} \to M$ such that for each $R \in \mathcal{R}^N$,

$$F_i(R) = \begin{cases} \max\{p(R_i), A_i(R_{-i})\} & \text{if } \sum_{j \in N} p(R_j) \leq m \\ \min\{p(R_i), B_i(R_{-i})\} & \text{if } \sum_{j \in N} p(R_j) \geq m, \end{cases}$$

and for each $R \in \mathcal{R}^N$, $\sum_{j \in N} F_j(R) = m$.

For each $i \in N$, each $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$, and each $x \in M$, let

$$a_i(R_{-i})(\{x\}) = \begin{cases} 1 & \text{if } x = A_i(R_{-i}) \\ 0 & \text{if } x \neq A_i(R_{-i}) \end{cases} \text{ and } b_i(R_{-i})(\{x\}) = \begin{cases} 1 & \text{if } x = B_i(R_{-i}) \\ 0 & \text{if } x \neq B_i(R_{-i}). \end{cases}$$

By Proposition 1, it suffices to show that $a = (a_i)_{i \in N} \in \mathcal{A}$, $b = (b_i)_{i \in N} \in \mathcal{B}$, and for each $R \in \mathcal{R}^N$, $\bar{F}(R) \in \mathcal{Q}^{a,b}(R)$.

Let $i \in N$. We show that $a_i \in \mathcal{A}_i$. Let $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$. Then, for each $x \in M$, $a_i(R_{-i})(\{x\}) \geq 0$ and $a_i(R_{-i})(M) = 1$. Thus, $a_i(R_{-i}) \in \Delta(M)$. Next, assume that $\sum_{j \neq i} p(R_j) \leq m$. Then, $A_i(R_{-i}) = F_i(R_i^0, R_{-i}) = m - \sum_{j \neq i} F_j(R_i^0, R_{-i}) \leq m - \sum_{j \neq i} p(R_j) = e(R_{-i})$. Thus, $a_i(R_{-i})([0, e(R_{-i})]) = 1$. Similarly, we can show that $b_i \in \mathcal{B}_i$.

Let $i \in N$ and $R \in \mathcal{R}^N$. Assume that $\sum_{j \in N} p(R_j) \leq m$ (the other case is similar). Since $\bar{F}_i(R)(\{\max\{p(R_i), A_i(R_{-i})\}\}) = 1$, for each $x \in M$,

$$\bar{F}_i(R)(\{x\}) = \begin{cases} 0 & \text{if } x < p(R_i) \\ a_i(R_{-i})([0, p(R_i)]) & \text{if } x = p(R_i) \\ a_i(R_{-i})(\{x\}) & \text{if } x > p(R_i). \end{cases}$$

Thus, $\bar{F}(R) \in \mathcal{Q}^a(R)$. Since $F(R) \in Z$, $\mathcal{Q}^a(R) \neq \emptyset$. Thus, $a \in \mathcal{A}$. □

Proof of Lemma 4. Let $R \in \mathcal{R}^N$, $\pi \in \Pi$, and $z \in Z$. We show $\hat{f}^F(R)(\{z\}) = \hat{f}^F(R^\pi)(\{z^\pi\})$. Let $\Pi' = \{\tau \in \Pi : F^\tau(R) = z\}$ and $\Pi'' = \{\tau \in \Pi : F^\tau(R^\pi) = z^\pi\}$. Note that $\hat{f}^F(R)(\{z\}) = \frac{1}{n!}|\{\tau \in \Pi : F^\tau(R) = z\}| = \frac{1}{n!}|\Pi'|$ and $\hat{f}^F(R^\pi)(\{z^\pi\}) = \frac{1}{n!}|\{\tau \in \Pi : F^\tau(R^\pi) = z^\pi\}| = \frac{1}{n!}|\Pi''|$.[40] Thus, it suffices to show that $|\Pi'| = |\Pi''|$.

First, we show that if $|\Pi'| \geq 1$, then $|\Pi'| \leq |\Pi''|$. Assume that $|\Pi'| \geq 1$. Let $\tau \in \Pi'$. Then, $F^\tau(R) = z$. Let $\hat{\tau} = \pi^{-1} \circ \tau$.[41] Then, for each $i \in N$, $F_i^{\hat{\tau}}(R^\pi) = F_{\hat{\tau}^{-1}(i)}(R_{\pi(\hat{\tau}(1))}, \ldots, R_{\pi(\hat{\tau}(n))}) = F_{\tau^{-1}(\pi(i))}(R^\tau) = F^\tau_{\pi(i)}(R) = z_{\pi(i)}$.[42] Thus, $\hat{\tau} \in \Pi''$. Since for each $\tau', \tau'' \in \Pi'$ with $\tau' \neq \tau''$, $\pi^{-1} \circ \tau' \neq \pi^{-1} \circ \tau''$, we have $|\Pi'| \leq |\Pi''|$.

Next, we show that if $|\Pi''| \geq 1$, then $|\Pi''| \leq |\Pi'|$. Assume that $|\Pi''| \geq 1$. Let $\tau \in \Pi''$. Then, $F^\tau(R^\pi) = z^\pi$. Let $\hat{\tau} = \pi \circ \tau$. Then, for each $i \in N$, $F_i^{\hat{\tau}}(R) = F_{\hat{\tau}^{-1}(i)}(R^{\hat{\tau}}) = F_{\tau^{-1}(\pi^{-1}(i))}(R^{\pi \circ \tau}) = F^\tau_{\pi^{-1}(i)}(R^\pi) = z_i$. Thus, $\hat{\tau} \in \Pi'$. Since for each $\tau', \tau'' \in \Pi''$ with $\tau' \neq \tau''$, $\pi \circ \tau' \neq \pi \circ \tau''$, we have $|\Pi''| \leq |\Pi'|$.

Therefore, if $|\Pi'| \geq 1$, then $|\Pi'| \leq |\Pi''|$, which also implies $|\Pi''| \leq |\Pi'|$. This also implies that $|\Pi'| = 0$ if and only if $|\Pi''| = 0$. Hence, $|\Pi'| = |\Pi''|$. □

Proof of Lemma 5.

Proof of (i). Let $R \in \mathcal{R}^N$. First, we show that for each $z \in Z$, $\hat{g}(R)(\{z\}) \geq 0$. Let $z \in Z$. If for each $\pi \in \Pi$, $z \neq F^\pi(R)$, then, by $\hat{f}(R)(\{z\}) = 0$, $\hat{g}(R)(\{z\}) = g(R)(\{z\}) \geq 0$. If for some $\pi \in \Pi$, $z = F^\pi(R)$, then, by $\frac{g(R)(\{z\})}{\hat{f}(R)(\{z\})} \geq \alpha_F$, $\hat{g}(R)(\{z\}) = g(R)(\{z\}) - \alpha_F \hat{f}(R)(\{z\}) \geq 0$. Thus, $\hat{g}(R)(\{z\}) \geq 0$. Second, by $g(R)(Z) = \alpha$ and $\hat{f}(R)(Z) = 1$, $\hat{g}(R)(Z) = g(R)(Z) - \alpha_F \hat{f}(R)(Z) = \alpha - \alpha_F$. Third, since for each $z \in Z$, $\hat{g}(R)(\{z\}) \geq 0$, $\alpha \geq \alpha_F$. Finally, by APA of g, for each $R \in \mathcal{R}^N$ and each $\pi \in \Pi$, $0 < g(R^\pi)(\{F(R^\pi)\}) = g(R)(\{(F_{\pi^{-1}(1)}(R^\pi), \ldots, F_{\pi^{-1}(n)}(R^\pi))\}) =$

[40] Let $|X|$ denote the cardinality of a set X.

[41] Given $\tilde{\pi}, \tilde{\tau} \in \Pi$, $\tilde{\pi} \circ \tilde{\tau}$ denotes the composite function $\tilde{\pi}(\tilde{\tau}(\cdot))$, i.e., for each $j \in N$, $\tilde{\pi} \circ \tilde{\tau}(j) = \tilde{\pi}(\tilde{\tau}(j))$.

[42] Recall that for each $\hat{\pi} \in \Pi$ and each $j \in N$, $F_j^{\hat{\pi}}(R) = F_{\hat{\pi}^{-1}(j)}(R^{\hat{\pi}})$.

$g(R)(\{F^{\pi}(R)\})$. Also, for each $R \in \mathcal{R}^N$ and each $\pi \in \Pi$, $\hat{f}(R)(\{F^{\pi}(R)\}) > 0$. Thus, $\alpha_F > 0$.

Proof of (ii). Let $R \in \mathcal{R}^N$, $\pi \in \Pi$, and $z \in Z$. Since g and $\hat{f}$ satisfy APA (Lemma 4), $\hat{g}(R)(\{z\}) = g(R)(\{z\}) - \alpha_F \hat{f}(R)(\{z\}) = g(R^{\pi})(\{z^{\pi}\}) - \alpha_F \hat{f}(R^{\pi})(\{z^{\pi}\}) = \hat{g}(R^{\pi})(\{z^{\pi}\})$.

Proof of (iii). Let $(R, z) \in \mathcal{R}^N \times Z$ be such that $g(R)(\{z\}) = 0$. By (i), $\hat{g}(R)(\{z\}) \geq 0$. Thus, by $g(R)(\{z\}) \geq \hat{g}(R)(\{z\})$, $\hat{g}(R)(\{z\}) = 0$. Next, let $(\hat{R}, \hat{\pi}) \in \text{argmin}\left\{\frac{g(R)(\{F^{\pi}(R)\})}{\hat{f}(R)(\{F^{\pi}(R)\})} : (R, \pi) \in \mathcal{R}^N \times \Pi\right\}$. Then, $g(\hat{R})(\{F^{\hat{\pi}}(\hat{R})\}) > 0$, but $\hat{g}(\hat{R})(\{F^{\hat{\pi}}(\hat{R})\}) = g(\hat{R})(\{F^{\hat{\pi}}(\hat{R})\}) - \alpha_F \hat{f}(\hat{R})(\{F^{\hat{\pi}}(\hat{R})\}) = 0$. $\square$

References

Barberà, S., Jackson, M. O., and Neme, A. (1997) Strategy-proof allotment rules. *Games and Economic Behavior* 18: 1–21

Benassy, J.-P. (1982) *The Economics of Market Disequilibrium.* Academic Press, New York

Bogomolnaia, A. and Moulin, H. (2001) A new solution to the random assignment problem. *Journal of Economic Theory* 100: 295–328.

Border, K. C. and Jordan, J. S. (1983) Straightforward elections, unanimity and phantom voters. *Review of Economic Studies* 50: 153–170

Chang, H.-I. and Chun, Y. (2017) Probabilistic assignment of indivisible objects when agents have the same preferences except the ordinal ranking of one object. *Mathematical Social Sciences* 90: 80–92

Chatterji, S., Roy, S., and Sen, A. (2012) The structure of strategy-proof random social choice functions over product domains and lexicographically separable preferences. *Journal of Mathematical Economics* 48: 353–366

Chatterji, S., Sen, A., and Zeng, H. (2014) Random dictatorship domains. *Games and Economic Behavior* 86: 212–236

Chatterji, S. and Zeng, H. (2018) On random social choice functions with the tops-only property. *Games and Economic Behavior* 109: 413–435

Ching, S. (1994) An alternative characterization of the uniform rule. *Social Choice and Welfare* 11: 131–136

Ehlers, L. and Klaus, B. (2003) Probabilistic assignments of identical indivisible objects and uniform probabilistic rules. *Review of Economic Design* 8: 249–268

Ehlers, L., Peters, H., and Storcken, T. (2002) Strategy-proof probabilistic decision schemes for one-dimensional single-peaked preferences. *Journal of Economic Theory* 105: 408–434

Gaurav, A., Picot, J., and Sen, A. (2017) The decomposition of strategy-proof random social choice functions on dichotomous domains. *Mathematical Social Sciences* 90: 28–34

Gibbard, A. (1977) Manipulation of schemes that mix voting with chance. *Econometrica* 45: 665–681

Hatsumi, K. and Serizawa, S. (2009) Coalitionally strategy-proof rules in allotment economies with homogeneous indivisible goods. *Social Choice and Welfare* 33: 423–447

Kasajima, Y. (2013) Probabilistic assignment of indivisible goods with single-peaked preferences. *Social Choice and Welfare* 41: 203–215

Kureishi, W. (2000) A simple characterization of the randomized uniform rule. Working Paper, Osaka University

Kureishi, W. and Mizukami, H. (2005) A characterization of the randomized uniform rule. Discussion Papers in Economics and Business 05-20, Osaka University

Kureishi, W. and Mizukami, H. (2007) Equal probability for the best and the assignment of identical indivisible objects. *Economics Bulletin* 4: 1–10

Moulin, H. (1980) On strategy-proofness and single peakedness. *Public Choice* 35: 437–455.

Peters, H., Roy, S., Sadhukhan, S., and Storcken, T. (2017) An extreme point characterization of strategy-proof and unanimous probabilistic rules over binary restricted domains. *Journal of Mathematical Economics* 69: 84–90

Peters, H., Roy, S., Sen, A., and Storcken, T. (2014) Probabilistic strategy-proof rules over single-peaked domains. *Journal of Mathematical Economics* 52: 123–127

Picot, J. and Sen, A. (2012) An extreme point characterization of random strategy-proof social choice functions: the two alternative case. *Economics Letters* 115: 49–52

Pycia, M. and Ünver, M. U. (2015) Decomposing random mechanisms. *Journal of Mathematical Economics* 61: 21–33

Roy, S. and Sadhukhan, S. (2018) A characterization of random min–max domains and its applications. *Economic Theory*, forthcoming

Sasaki, H. (1997) Randomized uniform allocation mechanism and single-peaked preferences of indivisible good. Working Paper, Waseda University

Sprumont, Y. (1991) The division problem with single-peaked preferences: a characterization of the uniform allocation rule. *Econometrica* 59: 509–519

Thomson, W. (2018) On the terminology of economic design: a critical assessment and some proposals. *Review of Economic Design* 22: 67–99