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Best-reply potential for two-person one-dimensional pure location games*

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Abstract

We study two-person one-dimensional non-price pure location games à la Anderson et al. (1992) under the setting that the strategy set is either a compact real interval or a finite integer interval and demand as a continuous function of distance is constant (inelastic) or strictly decreasing (elastic). We show that on a finite integer interval, the game is a best-reply potential game (Voorneveld, 2000); on a compact real interval, it is a best-reply potential game if the demand is a sufficiently decreasing, strictly decreasing function of distance; otherwise a quasi-potential game (Schipper, 2004). We also show that, even if a best-reply potential exists, a generalized ordinal potential (Monderer and Shapley, 1996) need not exist. Thus, on a finite integer interval, the game generally lacks the finite improvement property (Monderer and Shapley, 1996) but has the finite best-reply property (Milchtaich, 1996); on a compact real interval, the existence of a pure Nash equilibrium is secured by the existence of some continuous potential function, which, as we shall show, is indeed the case.

Keywords: symmetric games, location games, best-reply potential games, pure Nash equilibrium existence

JEL Classification: C72 (Noncooperative game)

1 Introduction

This article is concerned with two active areas of research in game theory: location games and games having a pure Nash equilibrium. Although the former already has wide range of

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applications, and the latter is being studied in more general frameworks, location games are one of major areas in which we still do not have general results on the existence of equilibria. Our aim is to shed some new light on such a theoretical aspect of location games.

Location games date back to Hotelling (1929), originally as a two-person game of location and price choice by sellers in a market depicted by a line segment. He assumed that buyers are uniformly distributed, and each buyer demands a unit quantity of the product from a seller whose delivered price (price plus transportation cost that is increasing in distance) is cheaper. Thus, buyers are assumed to have completely inelastic demand. Smithies (1941) considered a situation where demand of buyers are strictly decreasing in delivered price, i.e., a situation where buyers have elastic demand. By setting sellers' prices and per distance transportation cost both fixed and equal across the sellers, we have a pure location game. Perhaps the most thorough study of such games are done by Anderson et al. (1992, Chapter 8.2), who analyzed equilibria of the games under the setting of inelastic as well as elastic demand;[1] see also Gabszewicz and Thisse (1992) for pure location games in general.

In this paper, we study such two-person location games à la Anderson et al. (1992) under the additional setting that the market, which is the strategy set of the sellers, is either a compact real interval or a finite integer interval. To be more precise, we consider two-person games as follows.

- Each player simultaneously chooses a point in the interval (seller's location) and receives a demand from each point in the interval (buyers' locations) that is closer to him than to his opponent, evenly splitting if equally distant.
- Buyers are uniformly distributed, and they have a common *demand function*, which is a constant or a strictly decreasing continuous function of distance.

This class of games includes the aforementioned pure location games of Anderson et al. (1992) on a line segment. Also, as those studied by Stevens (1961), pure location games on a discrete interval.[2]

[1]To be fair, we note that their analysis of two-person pure location games is just an opening of their comprehensive analyses of location games with more than two players, entry, and price competition; our analysis is limited in scope compared to theirs.

[2]Stevens (1961) considered matrix form pure location games, under the setting of inelastic as well as elastic demand. However, his interest was in the minimax solution of the game, which coincides with Nash equilibria for constant-sum inelastic demand case, but not in general for non-constant-sum elastic demand case—what he is actually analyzing is, interestingly enough, Nash equilibria of "relative payoff" game. To our knowledge, von Mouche and Pijnappel (2015) are the first who theoretically analyzed Nash equilibria of discrete pure location games under the setting of inelastic and elastic demand.

We show that, on a finite integer interval, our game is a *best-reply potential game* (Voorneveld, 2000) with constant as well as strictly decreasing demand function. On a compact real interval, it is a best-reply potential game if demand is a sufficiently decreasing strictly decreasing function; otherwise a *quasi-potential game* (Schipper, 2004). The potential functions therein are all continuous, by construction. We also show that even if a best-reply potential function exists, a *generalized ordinal potential function* (Monderer and Shapley, 1996) need not exist. Altogether, these imply the following things. First, if the game is on a finite integer interval, it generally lacks the *finite improvement property (FIP)* (Monderer and Shapley, 1996) but has the *finite best-reply property (FBRP)* (Milchtaich, 1996). That is, there could be an infinite improvement path, but every *best-reply* improvement path is finite, and so any maximal best-reply improvement path ends at a pure Nash equilibrium. Second, if the game is on a compact real interval, a continuous potential function exists in the class of our pure location games, despite the notorious discontinuity of payoff functions. This existence of a continuous potential function secures the existence of a pure Nash equilibrium.

The rest of the paper is organized as follows. In Section 2, we define the game and give some preliminaries. In Section 3, we show our main results. Some concluding remarks are given in Section 4.

2 Preliminaries

2.1 The game

Denote by S an integer interval $[0, L]_{\mathbb{Z}}$, where L is a positive integer, or a real interval $[0, L]$, where L is a positive real number. If $S = [0, L]_{\mathbb{Z}}$, then f denotes a real-valued positive function on $\mathbb{Z}_+$ that is constant or strictly decreasing; if $S = [0, L]$, then f is a similar function on $\mathbb{R}_+$ that is also continuous. We denote by $G = (S, f)$ a two-person game in strategic form such that the strategy set is S and the payoff functions $u_i \colon S \times S \to \mathbb{R}$ $(i = 1, 2)$ are defined as follows:

- if $S = [0, L]_{\mathbb{Z}}$:

$$u_i(x_1, x_2) := \sum_{y \in V_i(x_1,x_2)} f(|y - x_i|) + \frac{1}{2} \sum_{y \in V_0(x_1,x_2)} f(|y - x_i|) \tag{1}$$

- if $S = [0, L]$:

$$u_i(x_1, x_2) := \int_{V_i(x_1,x_2)} f(|y - x_i|)dy + \frac{1}{2} \int_{V_0(x_1,x_2)} f(|y - x_i|)dy \tag{2}$$

where, with $\{i,j\} = \{1,2\}$, $V_i(x_1,x_2) = \{y \in S \mid |y - x_i| < |y - x_j|\}$ and $V_0(x_1,x_2) = \{y \in S \mid |y - x_1| = |y - x_2|\}$.[3] We refer to f as a *demand function.*

This game is a symmetric game, i.e., the payoff functions satisfy

$$u_2(y,x) = u_1(x,y) \quad \forall x,y \in S, \tag{3}$$

which we will refer to as *player symmetry.* In addition, each u_i satisfies yet another symmetry, which we will call *location symmetry*:

$$u_i(x,y) = u_i(L-x, L-y) \quad \forall x,y \in S. \tag{4}$$

2.2 Potentials and path-acyclicities

There are several notions of potential functions (potentials, for short). To clarify the position of our games, we give some relevant definitions below. Let $N = \{1,\dots,n\}$ and let $((S_i)_{i\in N}, (u_i)_{i\in N})$ be a general n-person game, where S_i and $u_i\colon \prod_{j\in N} S_j \to \mathbb{R}$ are the strategy set and the payoff function of player $i \in N$, respectively. For $i \in N$, $s \in \prod_{j\in N} S_j$, and $s'_i \in S_i$, we denote by $s \setminus s'_i \in \prod_{j\in N} S_j$ a strategy profile obtained from s by replacing the ith element with s'_i. Let $P\colon \prod_{j\in N} S_j \to \mathbb{R}$. P is said to be a *generalized ordinal potential* of G (Monderer and Shapley, 1996) if for any i, s, s'_i

$$u_i(s \setminus s'_i) > u_i(s) \implies P(s \setminus s'_i) > P(s). \tag{5}$$

P is said to be a *best-reply potential* of G (Voorneveld, 2000) if for any i, s

$$\arg\max_{s'_i} u_i(s \setminus s'_i) = \arg\max_{s'_i} P(s \setminus s'_i). \tag{6}$$

If "=" in (6) is weakened to "$\supseteq$" then P is called a *pseudo-potential* of G (Dubey et al., 2006). P is called a *quasi-potential* of G (Schipper, 2004) if for any s

$$s_i \in \arg\max_{s'_i \in S_i} u_i(s \setminus s'_i)\ \forall i \iff s \in \arg\max_{s' \in S} P(s'). \tag{7}$$

A game having a generalized ordinal potential is called a generalized ordinal potential game, and so on. If a game G belongs to any one of these classes of potential games, and if the potential therein has a maximum, then any maximizer of the potential is a pure Nash equilibrium of G.

A path is a nonempty finite or infinite sequence of strategy profiles $(s^1, s^2, \dots)$ such that every s^k and s^{k+1} differ in exactly one, say the $i(k)$th, coordinate. A finite path

[3] We borrowed the notations $V_i(x_1,x_2)$ of Prisner (2011). In von Mouche and Pijnappel (2015), dealing with the discrete case, the function f was set to an exponential function $f(z) = w^z$ with $0 < w \leq 1$. Because of its workability, we will make use of this f in the proof of Proposition 3.5 below.

$(s^1, s^2, \ldots, s^m)$ is called *cyclic* if $s^m = s^1$; *trivial* if $m = 1$. A path is an *improvement path* if $u_{i(k)}(s^{k+1}) > u_{i(k)}(s^k)$ for every k; *nondeteriorating* if $u_{i(k)}(s^{k+1}) \geq u_{i(k)}(s^k)$ for every k. If in each step k the deviator $i(k)$ is choosing a best-reply, an improvement path is a *best-reply improvement path*. A *(best-reply) improvement cycle* is a cyclic (best-reply) improvement path. A *weak best-reply improvement cycle* is a cyclic nondeteriorating path $(s^1, \ldots, s^m)$ such that $u_{i(k)}(s^{k+1}) > u_{i(k)}(s^k)$ for some $k \in \{1, \ldots, m-1\}$. It is known that there is no non-trivial improvement cycle in a generalized ordinal potential game (Monderer and Shapley, 1996). Also, there is no weak best-reply improvement cycle in a best-reply potential game (Voorneveld, 2000).[4] Since non-trivial best-reply improvement cycles are weak best-reply improvement cycles, it follows that there is no non-trivial best-reply improvement cycle in a best-reply potential game.

A game is said to have the *finite improvement property* (FIP) (Monderer and Shapley, 1996) if every improvement path is finite; *finite best-reply property* (FBRP) (Milchtaich, 1996) if every best-reply improvement path is finite. Any game with the FIP has the FBRP but not vice versa. A finite game, i.e., a game where each strategy set is finite, has the FIP if and only if it is a generalized ordinal potential game (Monderer and Shapley, 1996). A finite best-reply potential game has the FBRP because it has no non-trivial best-reply improvement cycle.

3 The results

3.1 Games on $S = [0, L]_{\mathbb{Z}}$

In this subsection, we show that $G = (S, f)$ with $S = [0, L]_{\mathbb{Z}}$ is a best-reply potential game. We first introduce some expressions. Let $a, b \in S$. We let:

$$\mathbf{L}(a) := \sum_{z=0}^{a} f(z), \quad \mathbf{R}(b) := \sum_{z=0}^{L-b} f(z), \quad \mathbf{M}(a,b) := \sum_{z=1}^{\lfloor \frac{|a-b|}{2} \rfloor} f(z) + \sum_{z=1}^{\lfloor \frac{|a-b|-1}{2} \rfloor} f(z).$$

Assuming $a \leq b$, $\mathbf{L}(a)$ is the sum of payoffs from the left of a (a inclusive); $\mathbf{R}(b)$ that from the right of b (b inclusive); and $\mathbf{M}(a,b)$ that from the middle of a and b (a and b exclusive), where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. It is to be understood that the sum in the definition of $\mathbf{M}(a,b)$ is zero if the upper bound of the summation is less than unity. $\mathbf{M}(a,b)$ is a function of the absolute value $|a-b|$, so $\mathbf{M}(a,b) = \mathbf{M}(b,a)$ for any $a, b \in S$.

[4] Voorneveld (2000) calls our weak best-reply improvement cycle simply a best-response cycle.

Using these expressions, the payoff functions given by (1) are rewritten as follows:

$$u_i(x_1,x_2) = \begin{cases} \mathbf{L}(x_i) + \frac{1}{2}\mathbf{M}(x_1,x_2) & \text{if } x_i < x_j \\ \mathbf{R}(x_i) + \frac{1}{2}\mathbf{M}(x_1,x_2) & \text{if } x_i > x_j \\ \frac{1}{2}\big(\mathbf{L}(x_i) + \mathbf{R}(x_i) - f(0)\big) & \text{if } x_i = x_j. \end{cases}$$

Define $\hat{P}\colon S^2 \to \mathbb{R}$ by

$$\hat{P}(x_1,x_2) := \begin{cases} \mathbf{L}(\min\{x_1,x_2\}) + \mathbf{R}(\max\{x_1,x_2\}) + \frac{1}{2}\mathbf{M}(x_1,x_2) & \text{if } x_1 \neq x_2 \\ \mathbf{L}(x_1) + \mathbf{R}(x_1) - f(0) & \text{if } x_1 = x_2. \end{cases} \tag{8}$$

Proposition 3.1. *$G = (S, f)$ with $S = [0, L]_{\mathbb{Z}}$ and a strictly decreasing f is a best-reply potential game, with a best-reply potential $P^\bullet\colon S^2 \to \mathbb{R}$ defined by*

$$P^\bullet(x_1,x_2) := \begin{cases} \hat{P}(x_1,x_2) & \text{if } (x_1,x_2) \neq (\frac{L}{2},\frac{L}{2}) \\ \hat{P}(x_1,x_2) + \frac{1}{2}f(0) & \text{if } (x_1,x_2) = (\frac{L}{2},\frac{L}{2}). \end{cases} \tag{9}$$

Proof. By player symmetry (3) and $P^\bullet(x_1,x_2) = P^\bullet(x_2,x_1)$, it suffices to show (6) for $i = 1$. Let

$$B(x_2) := \arg\max_{x_1 \in S} u_1(x_1,x_2) \quad \text{and} \quad M^\bullet(x_2) := \arg\max_{x_1 \in S} P^\bullet(x_1,x_2).$$

By location symmetry (4) and $P^\bullet(x_1,x_2) = P^\bullet(L-x_1, L-x_2)$, it suffices to show $B(x_2) = M^\bullet(x_2)$ for $x_2 \leq \frac{L}{2}$. We distinguish two cases.

1. Case $x_2 < \frac{L}{2}$. First, if $x_1 < x_2$, then for $x_1' = 2x_2 - x_1$, $\mathbf{M}(x_1,x_2) = \mathbf{M}(x_2,x_1')$. Noting that $P^\bullet(x_1,x_2) = \hat{P}(x_1,x_2)$ for any $x_1 \in S$,

$$\begin{aligned} u_1(x_1',x_2) - u_1(x_1,x_2) &= \mathbf{R}(x_1') - \mathbf{L}(x_1) \\ &> P^\bullet(x_1',x_2) - P^\bullet(x_1,x_2) = \big(\mathbf{L}(x_2) + \mathbf{R}(x_1')\big) - \big(\mathbf{L}(x_1) + \mathbf{R}(x_2)\big) \\ &= \left(\sum_{z=0}^{x_2} f(z) - \sum_{z=0}^{x_1} f(z)\right) + \left(\sum_{z=0}^{L-x_1'} f(z) - \sum_{z=0}^{L-x_2} f(z)\right) = \sum_{z=x_1+1}^{x_2} f(z) - \sum_{z=L-x_1'+1}^{L-x_2} f(z) > 0, \end{aligned}$$

where the first inequality is by $\mathbf{L}(x_2) < \mathbf{R}(x_2)$, and the last by $x_1 + 1 < L - x_1' + 1$ and $x_2 - (x_1+1) = (L - x_2) - (L - x_1' + 1)$. Hence $x_1 \geq x_2$ if $x_1 \in B(x_2) \cup M^\bullet(x_2)$. Second, if $x_1 = x_2$, then $\mathbf{M}(x_2,x_1) = \mathbf{M}(x_2,x_1+1) = 0$, and

$$\begin{aligned} 2\big(u_1(x_1+1,x_2) - u_1(x_1,x_2)\big) &= 2\mathbf{R}(x_2+1) - \big(\mathbf{L}(x_2) + \mathbf{R}(x_2) - f(0)\big) \\ &\geq P^\bullet(x_1+1,x_2) - P^\bullet(x_1,x_2) = \mathbf{L}(x_2) + \mathbf{R}(x_2+1) - \big(\mathbf{L}(x_2) + \mathbf{R}(x_2) + f(0)\big) \\ &= \sum_{z=0}^{L-x_2-1} f(z) - \sum_{z=0}^{L-x_2} f(z) + f(0) = -f(L-x_1) + f(0) > 0, \end{aligned}$$

where the first weak inequality is by $\mathbf{R}(x_2+1)-\mathbf{L}(x_2)=\sum_{z=0}^{L-x_2-1} f(z)-\sum_{z=0}^{x_2} f(z)\geq 0$ due to $L-x_2-1\geq x_2$. Hence $x_1>x_2$ if $x_1\in B(x_2)\cup M^\bullet(x_2)$. Then $B(x_2)$ equals the set of maximizers of $u_1(\cdot,x_2)=\mathbf{R}(\cdot)+\frac{1}{2}\mathbf{M}(\cdot,x_2)$ over $x_1>x_2$, and $M^\bullet(x_2)$ equals the set of maximizers of $P^\bullet(\cdot,x_2)=\mathbf{L}(x_2)+\mathbf{R}(\cdot)+\frac{1}{2}\mathbf{M}(\cdot,x_2)$ over $x_1>x_2$. Since $\mathbf{L}(x_2)$ is a constant, $B(x_2)=M^\bullet(x_2)$.

2. Case $x_2=\frac{L}{2}$. In this case $B(x_2)$ and $M^\bullet(x_2)$ are symmetric in that $y\in B(x_2) \iff L-y\in B(x_2)$ and $y\in M^\bullet(x_2) \iff L-y\in M^\bullet(x_2)$. Hence $B(x_2)\cap[x_2,L]\neq\emptyset$ and $M^\bullet(x_2)\cap[x_2,L]\neq\emptyset$. We are done if $B(x_2)\cap[x_2,L]=M^\bullet(x_2)\cap[x_2,L]$ is shown. Now, observe that if $x_1=x_2$ then

$$u_1(x_1,x_2)=\mathbf{R}(x_1)-\frac{1}{2}f(0)\quad\text{and}\quad P^\bullet(x_1,x_2)=\mathbf{L}(x_2)+\mathbf{R}(x_1)-\frac{1}{2}f(0);$$

if $x_1>x_2$ then

$$u_1(x_1,x_2)=\mathbf{R}(x_1)+\frac{1}{2}\mathbf{M}(x_1,x_2)\quad\text{and}\quad P^\bullet(x_1,x_2)=\mathbf{L}(x_2)+\mathbf{R}(x_1)+\frac{1}{2}\mathbf{M}(x_1,x_2).$$

Thus $P^\bullet(x_1,x_2)=u_1(x_1,x_2)+\mathbf{L}(x_2)$ for all $x_1\geq x_2$, and $\arg\max_{x_1\geq x_2}P^\bullet(x_1,x_2)=\arg\max_{x_1\geq x_2}u_1(x_1,x_2)$ since $\mathbf{L}(x_2)$ is constant. Hence $B(x_1,x_2)\cap[x_2,L]=M^\bullet(x_1,x_2)\cap[x_2,L]$. □

Proposition 3.2. *$G=(S,f)$ with $S=[0,L]_{\mathbb{Z}}$ and a constant $f=c$ is a best-reply potential game, with a best-reply potential $P^\circ\colon S^2\to\mathbb{R}$ defined by*

$$P^\circ(x_1,x_2):=\hat{P}(x_1,x_2)-r\min\left\{\left|\frac{L}{2}-x_1\right|,\left|\frac{L}{2}-x_2\right|\right\},\tag{10}$$

where r is a constant such that $0<r<\frac{c}{2}$.

Proof. By player symmetry (3) and $P^\circ(x_1,x_2)=P^\circ(x_2,x_1)$, it suffices to show (6) for $i=1$. Let

$$B(x_2):=\arg\max_{x_1\in S}u_1(x_1,x_2)\quad\text{and}\quad M^\circ(x_2):=\arg\max_{x_1\in S}P^\circ(x_1,x_2).$$

By location symmetry (4) and $P^\circ(x_1,x_2)=P^\circ(L-x_1,L-x_2)$, it suffices to show $B(x_2)=M^\circ(x_2)$ for $x_2\leq\frac{L}{2}$. We distinguish three cases. Note that G is $c(L+1)$-sum and

$$\hat{P}(x_1,x_2)=\begin{cases}c(L+1-\frac{1}{2}(|x_2-x_1|-1)) & \text{if } x_1\neq x_2\\ c(L+1) & \text{if } x_1=x_2.\end{cases}$$

1. Case $x_2\leq\frac{L}{2}-1$. In this case we can verify that $B(x_2)=\{x_2+1\}$. Notice that $\hat{P}(x_2,x_2)=\hat{P}(x_2+1,x_2)$ and $r\min\{\frac{L}{2}-x_2,\frac{L}{2}-x_2\}>r\min\{\frac{L}{2}-(x_2+1),\frac{L}{2}-x_2\}$, i.e., $P^\circ(x_2,x_2)<P^\circ(x_2+1,x_2)$; and $\hat{P}(x_1,x_2)\leq\hat{P}(x_2+1,x_2)$ and $r\min\{\frac{L}{2}-x_1,\frac{L}{2}-x_2\}>$

$r\min\{\frac{L}{2}-(x_2+1),\frac{L}{2}-x_2\}$ for all $x_1<x_2$, i.e., $P^\circ(x_1,x_2)<P^\circ(x_2+1,x_2)$ for all $x_1<x_2+1$. Also, $P^\circ(x_1,x_2)<P^\circ(x_2+1,x_2)$ for all $x_1>x_2+1$. To see this, note that $\hat{P}(x_1,x_2)-\hat{P}(x_2+1,x_2)=-\frac{c}{2}(x_1-x_2-1)$ for all $x_1>x_2+1$, and $r\min\{|\frac{L}{2}-x_1|,\frac{L}{2}-x_2\}-r\min\{\frac{L}{2}-(x_2+1),\frac{L}{2}-x_2\}=-r(x_1-x_2-1)$ if $x_1\le L-x_2$, and zero if $x_1>L-x_2$. In either case their sum $P^0(x_1,x_2)-P^0(x_2+1,x_2)$ is negative (note that $-\frac{c}{2}+r<0$). Hence $M^\circ(x_2)=B(x_2)=\{x_2+1\}$ if $x_2\le\frac{L}{2}-1$.

2. Case $\frac{L}{2}-1<x_2<\frac{L}{2}$. In this case L is odd and $x_2=\frac{L-1}{2}$. We can see that $B(x_2)=\{x_2,x_2+1\}$. Since $\hat{P}(x_2,x_2)=\hat{P}(x_2+1,x_2)>\hat{P}(x_1,x_2)$ for all $x_1\notin\{x_2,x_2+1\}$ and $r\min\{\frac{L}{2}-x_2,\frac{L}{2}-x_2\}=r\min\{\frac{L}{2}-(x_2+1),\frac{L}{2}-x_2\}=r\min\{|\frac{L}{2}-x_1|,\frac{L}{2}-x_2\}$ for all $x_1\notin\{x_2,x_2+1\}$, we have $M^\circ(x_2)=B(x_2)=\{x_2,x_2+1\}$ if $\frac{L}{2}-1<x_2<\frac{L}{2}$.

3. Case $x_2=\frac{L}{2}$. In this case we can see that $B(x_2)=\{x_2\}$. Since $\hat{P}(x_2,x_2)>\hat{P}(x_1,x_2)$ and $r\min\{\frac{L}{2}-x_2,\frac{L}{2}-x_2\}=r\min\{|\frac{L}{2}-x_1|,\frac{L}{2}-x_2\}$ for all $x_1\ne x_2$, we have $M^\circ(x_2)=B(x_2)=\{x_2\}$ if $x_2=\frac{L}{2}$. □

3.2 Games on $S=[0,L]$

In this subsection we consider $G=(S,f)$ with $S=[0,L]$. Let $u_i\colon S^2\to\mathbb{R}$ be defined by (2), $i=1,2$. In this case the best reply may be empty if, as is well-known, the demand function f is constant ($f=c$): Indeed, since

$$u_1(x_1,x_2)=\begin{cases}\frac{x_1+x_2}{2}c & \text{if } x_1<x_2\\ \frac{L}{2}c & \text{if } x_1=x_2\\ (L-\frac{x_1+x_2}{2})c & \text{if } x_1>x_2,\end{cases}$$

we have

$$\arg\max_{x_1\in S}u_1(x_1,x_2)=\begin{cases}\emptyset & \text{if } x_2\ne L/2\\ \{L/2\} & \text{if } x_2=L/2.\end{cases}$$

As we shall see in Remark 3.1 below, the situation is not so different if f is strictly decreasing and $\frac{1}{2}f(0)<f(\frac{L}{2})$. We shall first consider the case $\frac{1}{2}f(0)\ge f(\frac{L}{2})$.

Let $a,b\in S$. We now let

$$\mathbf{L}(a):=\int_0^a f(z)dz,\quad \mathbf{R}(b):=\int_0^{L-b} f(z)dz,\quad \mathbf{M}(a,b):=2\int_0^{\frac{|a-b|}{2}} f(z)dz,$$

and express the payoff functions given by (2) as

$$u_i(x_1,x_2)=\begin{cases}\mathbf{L}(x_i)+\frac{1}{2}\mathbf{M}(x_1,x_2) & \text{if } x_i<x_j\\ \mathbf{R}(x_i)+\frac{1}{2}\mathbf{M}(x_1,x_2) & \text{if } x_i>x_j\\ \frac{1}{2}\big(\mathbf{L}(x_i)+\mathbf{R}(x_i)\big) & \text{if } x_i=x_j.\end{cases}$$

Define $P\colon S^2 \to \mathbb{R}$ by

$$P(x_1, x_2) := \mathbf{L}(\min\{x_1, x_2\}) + \mathbf{R}(\max\{x_1, x_2\}) + \frac{1}{2}\mathbf{M}(x_1, x_2). \tag{11}$$

Note that P is continuous even if f were not continuous.

Proposition 3.3. *$G = (S, f)$ with $S = [0, L]$ and a strictly decreasing f satisfying*

$$\frac{1}{2}f(0) \geq f(\frac{L}{2})$$

is a best-reply potential game, with a best-reply potential $P\colon S^2 \to \mathbb{R}$ defined by (11).

Proof. By player symmetry (3) and $P(x_1, x_2) = P(x_2, x_1)$, it suffices to show (6) for $i = 1$. Let

$$B(x_2) := \arg\max_{x_1 \in S} u_1(x_1, x_2) \quad \text{and} \quad M(x_2) := \arg\max_{x_1 \in S} P(x_1, x_2).$$

By location symmetry (4) and $P(x_1, x_2) = P(L - x_1, L - x_2)$, it suffices to show $B(x_2) = M(x_2)$ for $x_2 \leq \frac{L}{2}$. We distinguish two cases.

1. Case $x_2 < \frac{L}{2}$. First, if $x_1 < x_2$, then for $x_1' = 2x_2 - x_1$, $\mathbf{M}(x_1, x_2) = \mathbf{M}(x_2, x_1')$, and

$$\begin{aligned} u_1(x_1', x_2) - u_1(x_1, x_2) &= \mathbf{R}(x_1') - \mathbf{L}(x_1) \\ &> P(x_1', x_2) - P(x_1, x_2) = \big(\mathbf{L}(x_2) + \mathbf{R}(x_1')\big) - \big(\mathbf{L}(x_1) + \mathbf{R}(x_2)\big) \\ &= \left(\int_0^{x_2} f(z)dz - \int_0^{x_1} f(z)dz\right) + \left(\int_0^{L-x_1'} f(z)dz - \int_0^{L-x_2} f(z)dz\right) \\ &= \int_{x_1}^{x_2} f(z)dz - \int_{L-x_1'}^{L-x_2} f(z)dz > 0, \end{aligned} \tag{12}$$

where the last inequality is by $x_1 < L - x_1'$ and $x_2 - x_1 = (L - x_2) - (L - x_1')$. Hence $x_1 \geq x_2$ if $x_1 \in B(x_2) \cup M(x_2)$. Second, if $x_1 = x_2$, note that $\frac{1}{2}f(0) - f(L - x_2) > \frac{1}{2}f(0) - f(\frac{L}{2}) \geq 0$, i.e., $\frac{1}{2}f(0) > f(L - x_2)$. By the continuity of f, we can choose $\epsilon > 0$ such that $\epsilon < L - x_2 - \epsilon$ and $\frac{1}{2}f(\epsilon) > f(L - x_2 - \epsilon)$. Then

$$\begin{aligned} u_1(x_1 + \epsilon, x_2) - u_1(x_1, x_2) &= \frac{1}{2}\mathbf{M}(x_2, x_2 + \epsilon) + \mathbf{R}(x_2 + \epsilon) - \frac{1}{2}(\mathbf{R}(x_2) + \mathbf{L}(x_2)) \\ &> P(x_1 + \epsilon, x_2) - P(x_1, x_2) = \frac{1}{2}\mathbf{M}(x_2, x_2 + \epsilon) + \mathbf{R}(x_2 + \epsilon) - \mathbf{R}(x_2) \\ &= \int_0^{\frac{\epsilon}{2}} f(z)dz - \int_{L-x_2-\epsilon}^{L-x_2} f(z)dz > \int_0^{\epsilon} \frac{1}{2}f(z)dz - \int_{L-x_2-\epsilon}^{L-x_2} f(z)dz > 0, \end{aligned}$$

where the first inequality is by $\mathbf{L}(x_2) < \mathbf{R}(x_2)$. Hence $x_1 > x_2$ if $x_1 \in B(x_2) \cup M(x_2)$. Thus $B(x_2) = \arg\max_{x_1 > x_2}\big(\mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)\big)$ and $M(x_2) = \arg\max_{x_1 > x_2}\big(\mathbf{L}(x_2) + \mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)\big)$. Since $\mathbf{L}(x_2)$ is a constant, we have $B(x_2) = M(x_2)$.

2. Case $x_2 = \frac{L}{2}$. In this case, $B(x_2)$ and $M(x_2)$ are symmetric in that $y \in B(x_2) \iff L - y \in B(x_2)$ and $y \in M(x_2) \iff L - y \in M(x_2)$. Also, both $u_1(\cdot, x_2)$ and $P(\cdot, x_2)$ are continuous. Hence $B(x_2) \cap [x_2, L] \neq \emptyset$ and $M(x_2) \cap [x_2, L] \neq \emptyset$. We are done if $B(x_2) \cap [x_2, L] = M(x_2) \cap [x_2, L]$ is shown. Now, as $x_2 = \frac{L}{2}$, we have $u_1(x_1, x_2) = \mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)$ for any $x_1 \in [x_2, L]$. Observe that $B(x_2) \cap [x_2, L] = \arg\max_{x_1 \geq x_2} \big(\mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)\big)$ and $M(x_2) \cap [x_2, L] = \arg\max_{x_1 \geq x_2} \big(\mathbf{L}(x_2) + \mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)\big)$, which, by the constancy of $\mathbf{L}(x_2)$, imply $B(x_2) \cap [x_2, L] = M(x_2) \cap [x_2, L]$. □

Remark 3.1. Let $x_2 < \frac{L}{2}$. Then, with $x_1 = x_2$,

$$u_1(x_1, x_2) = \frac{1}{2}\big(\mathbf{L}(x_2) + \mathbf{R}(x_2)\big) < \mathbf{R}(x_2) = \lim_{y_1 \downarrow x_1 = x_2} \big(\mathbf{R}(y_1) + \frac{1}{2}\mathbf{M}(x_2, y_1)\big) = \lim_{y_1 \downarrow x_1 = x_2} u_1(y_1, x_2). \tag{13}$$

That is, $u_1(\cdot, x_2)$ is not continuous at $x_1 = x_2$, and $u_1(x_1, x_2) < u_1(x_1 + \epsilon, x_2)$ for a small $\epsilon > 0$. Nevertheless, the condition $\frac{1}{2}f(0) \geq f(\frac{L}{2})$ ensures that $B(x_2) \neq \emptyset$ for every $x_2 \in [0, L]$ as Proposition 3.3 suggests (in fact $B(x_2) = M(x_2) \neq \emptyset$ for every $x_2 \in [0, L]$). If $\frac{1}{2}f(0) < f(\frac{L}{2})$, on the other hand, $B(x_2)$ may be empty for some x_2. To see this, observe that for $y_1 > x_2$ with $x_2 < \frac{L}{2}$, $u_1(\cdot, x_2)$ is continuously differentiable at y_1 and

$$\begin{aligned} \frac{\partial}{\partial y_1} u_1(y_1, x_2) &= \frac{\partial}{\partial y_1} \left(\mathbf{R}(y_1) + \frac{1}{2}\mathbf{M}(x_2, y_1) \right) \\ &= \frac{\partial}{\partial y_1} \left(\int_0^{L - y_1} f(z)dz + \int_0^{\frac{y_1 - x_2}{2}} f(z)dz \right) = -f(L - y_1) + \frac{1}{2} f\Big(\frac{y_1 - x_2}{2}\Big). \end{aligned} \tag{14}$$

If $\frac{1}{2}f(0) < f(\frac{L}{2})$, and if $x_2 < \frac{L}{2}$ is sufficiently close to $\frac{L}{2}$, then, by the continuity of f, (14) converges to $-f(L - x_2) + \frac{1}{2}f(0) < 0$ as $y_1 \downarrow x_2$. That is, we have $\frac{\partial}{\partial x_1} u_1(x_1, x_2) < 0$ for all $x_1 > x_2$, since $-f(L - x_1) + \frac{1}{2}f(\frac{x_1 - x_2}{2})$ is decreasing in x_1. With (13), this says that $u_1(\cdot, x_2)$ has no maximum on $[x_2, L]$. Also, with (12), which also holds here, it has no maximum on $[0, L]$, namely, $B(x_2) = \emptyset$, while $M(x_2) \neq \emptyset$ since P is continuous. Hence in this case P defined by (11) cannot be a best-reply potential, nor even a pseudo-potential since $B(x_2) \not\supseteq M(x_2)$. ∎

Before we proceed, we note that if (x_1, x_2) is an equilibrium of G with a strictly decreasing f, then $x_1 + x_2 = L$, and the equilibrium is unique up to player symmetry.[5] To see this, assume $x_2 \leq x_1$, without loss of generality, and suppose $x_1 + x_2 < L$. If $x_2 = x_1$, then, by (13) that also holds here, we have $u_1(x_1, x_2) < u_1(x_1 + \epsilon, x_2)$ for a small $\epsilon > 0$, a contradiction.

[5] These points are also shown by Anderson et al. (1992). However, we include our proofs for the sake of completeness.

If $x_2 < x_1$, note that

$$\frac{\partial}{\partial x_1}u_1(x_1,x_2) = -f(L-x_1) + \frac{1}{2}f(\frac{x_1-x_2}{2}),\ \frac{\partial}{\partial x_2}u_2(x_1,x_2) = f(x_2) - \frac{1}{2}f(\frac{x_1-x_2}{2}),$$

and $x_2 \neq 0$ since $\frac{\partial}{\partial x_2}u_2(x_1,0) > 0$. Then $\frac{\partial}{\partial x_1}u_1(x_1,x_2) = \frac{\partial}{\partial x_2}u_2(x_1,x_2) = 0$ by the first order condition, and $f(L-x_2) = f(x_2)$, contradicting the strict decreasingness of f. Hence $x_1 + x_2 \geq L$. Note that $(L-x_1, L-x_2)$ is also an equilibrium by location symmetry (4), and the above argument also implies $(L-x_1)+(L-x_2) \geq L$. Hence $x_1 + x_2 = L$. For the uniqueness, suppose that (x_1,x_2) and (x'_1,x'_2) are two equilibria such that $x_2 \leq x_1$, $x'_2 \leq x'_1$, and $x'_1 < x_1$. Then $u_1(x_1,x_2) \geq u_1(x'_1,x_2)$ implies $\int_0^{\frac{x_1-x_2}{2}} f(z)dz + \int_0^{L-x_1} f(z)dz \geq \int_0^{\frac{x'_1-x_2}{2}} f(z)dz + \int_0^{L-x'_1} f(z)dz$, and $u_1(x_1,x'_2) \leq u_1(x'_1,x'_2)$ implies $\int_0^{\frac{x_1-x'_2}{2}} f(z)dz + \int_0^{L-x_1} f(z)dz \leq \int_0^{\frac{x'_1-x'_2}{2}} f(z)dz + \int_0^{L-x'_1} f(z)dz$. Subtracting,

$$\int_{\frac{x_1-x'_2}{2}}^{\frac{x_1-x_2}{2}} f(z)dz \geq \int_{\frac{x'_1-x'_2}{2}}^{\frac{x'_1-x_2}{2}} f(z)dz.$$

Since $\frac{x_1-x'_2}{2} > \frac{x'_1-x'_2}{2}$ and $\frac{x_1-x_2}{2} - \frac{x_1-x'_2}{2} = \frac{x'_1-x_2}{2} - \frac{x'_1-x'_2}{2}$, this contradicts the strict decreasingness of f. Hence equilibrium must be unique; it is unique up to player symmetry since $\{(x_1,x_2),(x_2,x_1),(L-x_1,L-x_2),(L-x_2,L-x_1)\} = \{(x_1,x_2),(x_2,x_1)\}$ by $x_1+x_2 = L$.

Proposition 3.4. *$G = (S,f)$ with $S = [0,L]$ and a strictly decreasing f is a quasi-potential game, with a quasi-potential $P\colon S^2 \to \mathbb{R}$ defined by* (11).

Proof. We show '$\Longleftarrow$' in (7), i.e., that any maximizer of P is an equilibrium. This and the uniqueness of equilibrium (up to player symmetry) imply (7).

Suppose $(x_1,x_2) \in \arg\max_{s\in S^2} P(s)$. Assume $x_2 \leq x_1$ without loss of generality (by $P(x_1,x_2) = P(x_2,x_1)$). Note that if $x_1+x_2 < L$, then for $\epsilon > 0$ such that $x_2+\epsilon < L-x_1-\epsilon$,

$$\begin{aligned}P(x_1+\epsilon, x_2+\epsilon) - P(x_1,x_2) &= \big(\mathbf{L}(x_2+\epsilon) - \mathbf{L}(x_2)\big) + \big(\mathbf{R}(x_1+\epsilon) - \mathbf{R}(x_1)\big) \\ &= \int_{x_2}^{x_2+\epsilon} f(z)dz - \int_{L-x_1-\epsilon}^{L-x_1} f(z)dz > 0.\end{aligned}$$

Likewise, if $x_1+x_2 > L$, then for $\epsilon > 0$ such that $x_2 - \epsilon > L - x_1 + \epsilon$, $P(x_1-\epsilon, x_2-\epsilon) > P(x_1,x_2)$. Since these contradict the maximality of $P(x_1,x_2)$, we must have $x_1+x_2 = L$. Then $\mathbf{L}(x_2) = \mathbf{R}(x_1)$. We distinguish two cases.

1. Case $x_2 = \frac{L}{2}$. In this case that $P(x_1,x_2) \geq P(x'_1,x_2)$ for any $x'_1 \in S$ implies that $\mathbf{R}(x_1) \geq \mathbf{R}(x'_1) + \frac{1}{2}\mathbf{M}(x'_1,x_2)$ for any $x'_1 \geq x_2$ and $\mathbf{L}(x_1) \geq \mathbf{L}(x'_1) + \frac{1}{2}\mathbf{M}(x'_1,x_2)$ for any $x'_1 \leq x_2$. That is, $u_1(x_1,x_2) \geq u_1(x'_1,x_2)$ for any $x'_1 \geq x_2$ and any $x'_1 \leq x_2$, respectively (note that $\mathbf{R}(x_1) = \mathbf{L}(x_1) = \frac{1}{2}(\mathbf{L}(x_1)+\mathbf{R}(x_1))$). Hence $x_1 \in \arg\max_{x'_1\in S} u_1(x'_1,x_2)$. We also have $x_2 \in \arg\max_{x'_2\in S} u_2(x_1,x'_2)$ by player symmetry.

2. Case $x_2 < \frac{L}{2}$. (a) If $x_1' > x_2$ then $P(x_1', x_2) = \mathbf{L}(x_2) + \mathbf{R}(x_1') + \frac{1}{2}\mathbf{M}(x_1', x_2)$ and $u_1(x_1', x_2) = \mathbf{R}(x_1') + \frac{1}{2}\mathbf{M}(x_1', x_2)$, so $P(x_1, x_2) \geq P(x_1', x_2)$ implies $u_1(x_1, x_2) \geq u_1(x_1', x_2)$. (b) If $x_1' < x_2$, recall that there is $x_1'' > x_2$ such that $u_1(x_1'', x_2) > u_1(x_1', x_2)$ (see (12)). Since $u_1(x_1, x_2) \geq u_1(x_1'', x_2)$ by (a), we have $u_1(x_1, x_2) \geq u_1(x_1', x_2)$. (c) If $x_1' = x_2$, then $P(x_1', x_2) \leq P(x_1, x_2)$ reads as $\mathbf{L}(x_2) + \mathbf{R}(x_1') \leq \mathbf{L}(x_2) + \mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)$, so $\mathbf{L}(x_2) + \mathbf{R}(x_1') \leq \mathbf{L}(x_2) + \mathbf{M}(x_2, x_1) + \mathbf{R}(x_1)$. Noting that $u_1(x_1', x_2) = u_2(x_1', x_2)$ by $x_1' = x_2$, and $u_1(x_1, x_2) = u_2(x_1, x_2)$ by $u_1(x_1, x_2) = u_2(x_2, x_1) = u_2(L - x_2, L - x_1) = u_2(x_1, x_2)$, this says that $2u_1(x_1', x_2) \leq 2u_1(x_1, x_2)$. Hence $x_1 \in \arg\max_{x_1' \in S} u_1(x_1', x_2)$. We also have $x_2 \in \arg\max_{x_2' \in S} u_2(x_1, x_2')$ by player symmetry. □

In passing, we note that $G = (S, f)$ with $S = [0, L]$ and a constant f has a unique equilibrium $(\frac{L}{2}, \frac{L}{2})$, as is well known. Clearly, $\bar{P} \colon S^2 \to \mathbb{R}$ defined by

$$\bar{P}(x_1, x_2) := -\left(\left|\frac{L}{2} - x_1\right| + \left|\frac{L}{2} - x_2\right|\right), \tag{15}$$

for example, is a continuous quasi-potential of G.

3.3 Non-existence of a generalized ordinal potential

Having established that G belongs to the class of quasi-potential games, and more strongly best-reply potential games if $S = [0, L]_{\mathbb{Z}}$ or if $S = [0, L]$ with sufficiently decreasing strictly decreasing f, we now show that G is not necessarily a generalized ordinal potential game, to narrow the class to which G belongs.

Proposition 3.5. *$G = (S, f)$ is not necessarily a generalized ordinal potential game.*

Proof. If G has a generalized ordinal potential then it cannot have any non-trivial improvement cycle. Note that this is true irrespective of $S = [0, L]_{\mathbb{Z}}$ or $S = [0, L]$. We provide, for each case, a counterexample, i.e., an example of G having a non-trivial improvement cycle. Let $f(z) = w^z$ with a constant w such that $0 < w \leq 1$. Then, for the game $G = (S, f)$ with $S = [0, 3]_{\mathbb{Z}}$, player 1's payoff matrix is as in Figure 1. If $w > \frac{1}{2}$ then $w^2 > \frac{w}{2}$, so there exists an improvement cycle $(2, 0) \to (1, 0) \to (1, 3) \to (2, 3) \to (2, 0)$. For the game $G = (S, f)$ with $S = [0, 3]$ and the same f, player 1's payoff function restricted to integer points is given (for $w \neq 1$) by a payoff matrix in Figure 2(a). By Proposition 3.3, this game is a best-reply potential game if $\frac{1}{2}f(0) \geq f(\frac{3}{2})$, i.e., if $w \leq (\frac{1}{2})^{\frac{2}{3}}$. The rounded payoff values when $w = \frac{1}{2}$ are as shown in Figure 2(b). As we can see, we have the same improvement cycle $(2, 0) \to (1, 0) \to (1, 3) \to (2, 3) \to (2, 0)$. □

	0	1	2	3
0	$\frac{1+w+w^2+w^3}{2}$	1	$1+\frac{w}{2}$	$1+w$
1	$1+w+w^2$	$\frac{1+2w+w^2}{2}$	$1+w$	$1+w+\frac{w}{2}$
2	$1+w+\frac{w}{2}$	$1+w$	$\frac{1+2w+w^2}{2}$	$1+w+w^2$
3	$1+w$	$1+\frac{w}{2}$	1	$\frac{1+w+w^2+w^3}{2}$

Figure 1: A game having a non-trivial improvement cycle (player 1's payoff when $S = [0,3]_{\mathbb{Z}}$).

	0	1	2	3
0	$\frac{w^3-1}{2\ln w}$	$\frac{w^{\frac{1}{2}}-1}{\ln w}$	$\frac{w-1}{\ln w}$	$\frac{w^{\frac{3}{2}}-1}{\ln w}$
1	$\frac{w^2+w^{\frac{1}{2}}-2}{\ln w}$	$\frac{w^2+w-2}{2\ln w}$	$\frac{w+w^{\frac{1}{2}}-2}{\ln w}$	$\frac{2w-2}{\ln w}$
2	$\frac{2w-2}{\ln w}$	$\frac{w+w^{\frac{1}{2}}-2}{2\ln w}$	$\frac{w^2+w-2}{2\ln w}$	$\frac{w^2+w^{\frac{1}{2}}-2}{\ln w}$
3	$\frac{w^{\frac{3}{2}}-1}{\ln w}$	$\frac{w-1}{\ln w}$	$\frac{w^{\frac{1}{2}}-1}{\ln w}$	$\frac{w^3-1}{2\ln w}$

(a)

	0	1	2	3
0	0.631	0.423	0.721	0.933
1	1.505	0.902	1.144	1.443
2	1.443	1.144	0.902	1.505
3	0.933	0.721	0.423	0.631

(b)

Figure 2: A game having a non-trivial improvement cycle (player 1's payoff when $S = [0,3]$ and $w = \frac{1}{2}$).

4 Concluding remarks

1. Let $G = (S, f)$. We have shown that if $S = [0, L]_{\mathbb{Z}}$ and f is constant or strictly decreasing then G is a best-reply potential game (Proposition 3.1) but may not be a generalized ordinal potential game (Proposition 3.5). Hence it has the FBRP but may not have the FIP. If $S = [0, L]$ and f is strictly decreasing and satisfies $\frac{1}{2}f(0) \geq f(\frac{L}{2})$ then G is a best-reply potential game (Proposition 3.3) but may not be a generalized ordinal potential game (Proposition 3.5). Although the FBRP cannot be claimed for this infinite game, the existence of a pure Nash equilibrium follows from the continuity of the best-reply potential P defined by (11). In general, the same continuous function P is a quasi-potential for G with $S = [0, L]$ and a strictly decreasing f, and such a G has a unique equilibrium up to player symmetry (see fn. 5). Clearly, G with $S = [0, L]$ and a constant f also has a continuous quasi-potential and a unique equilibrium $(\frac{L}{2}, \frac{L}{2})$.

2. Let $G = (S, f)$ with $S = [0, L]$ and a strictly decreasing f. Then we can locate the unique (up to player symmetry) equilibrium (x_1, x_2) of G in the following way.[6] Suppose $x_2 \leq \frac{L}{2}$. Here $x_2 \neq 0$, since otherwise $(x_1, x_2) = (L, 0)$, and $u_1(\epsilon, 0) > \int_{\epsilon}^{L} f(z)dz > \int_{0}^{\frac{L}{2}} f(z)dz =$

[6]The characterization of equilibrium below is also obtained in Anderson et al. (1992). Again, we include our short proof here for the sake of completeness.

$u_1(L,0)$ for a sufficiently small $\epsilon > 0$, a contradiction. Note that for (x_1, x_2) with $x_2 \in]0, \frac{L}{2}[$ (and $x_1 \in]\frac{L}{2}, L[$), we have $\frac{\partial}{\partial x_1} u_1(x_1, x_2) = -f(L - x_1) + \frac{1}{2} f(\frac{x_1 - x_2}{2})$ as in (14), so $\frac{\partial}{\partial x_1} u_1(x_1, x_2) = -f(x_2) + \frac{1}{2} f(\frac{L}{2} - x_2)$ by $x_1 + x_2 = L$. Also $-f(0) + \frac{1}{2} f(\frac{L}{2}) < 0$ by the strict decreasingness of f, and $-f(x_2) + \frac{1}{2} f(\frac{L}{2} - x_2)$ is strictly increasing in x_2 over $]0, \frac{L}{2}[$ due to the strict decreasingness of f. Therefore, if $-f(\frac{L}{2}) + \frac{1}{2} f(0) > 0$, then (x_1, x_2) is found by solving

$$-f(x_2) + \frac{1}{2} f(\frac{L}{2} - x_2) = 0, \tag{16}$$

with $x_1 = L - x_2$. Eq. (16) is the first order condition $\frac{\partial}{\partial x_1} u_1(x_1, x_2) = 0$ that has to be satisfied at the equilibrium (x_1, x_2) such that $x_2 < x_1$. If $-f(\frac{L}{2}) + \frac{1}{2} f(0) \leq 0$, then (16) fails at every $x_2 \in]0, \frac{L}{2}[$, and recalling that $x_2 \neq 0$, we must have $(x_1, x_2) = (\frac{L}{2}, \frac{L}{2})$.

3. We have shown that any $G = (S, f)$ in this paper belongs to some class of potential games, among which the class of quasi-potential games is the most general one. We note that a further generalization of potential is possible: we can replace $\Longleftrightarrow$ in (7) with $\Longleftarrow$. Such a potential function may be called a *weak quasi-potential function*. Note that the continuity of f is not used in the part of the proof of Proposition 3.4, where '$\Longleftarrow$' in (7) is being proved. Thus, $G = (S, f)$ with $S = [0, L]$ and a strictly decreasing not necessarily continuous f can also be said to be a weak quasi-potential game.

4. As a final remark, we note that:

Proposition 4.1. *If the strategy set is circular (as in Salop (1979)), then the two-person location games with payoff functions given by* (1) *or* (2) *is an exact potential game.*

The proof is straightforward. The game is then an identical interest game, which is an exact potential game (Monderer and Shapley, 1996).

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