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Unilaterally competitive games with more than two players

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Abstract

We show some interesting properties of unilaterally competitive games [Kats and Thisse, *Int. J. Game. Theory*, 1992], when there are more than two players. We show, in particular, that the games possess: (i) a Nash equilibrium, (ii) maximin solvability, (iii) strong solvability in the sense of [Nash, *Ann. Math.*, 1951], and (iv) weak acyclicity, all in pure strategies of finite or infinite games. The property (ii) is a consequence of (i) and the result of [De Wolf, CORE Discussion Paper, 1999], for which we will provide a different proof. The property (iv) is shown in two ways for finite games, one directly and the other by showing a generalization of a weak acyclicity sufficiency theorem for finite games in [Fabrikant et al. *Theor. Comput. Syst.*, 2013].

JEL Classification: C72 (Noncooperative game)

1 Introduction

Unilaterally competitive (UC) games due to Kats and Thisse (1992) are n-person generally nonzero-sum games whose set of Nash equilibria, if nonempty, is known to possess nice properties such as payoff equivalence and interchangeability (Kats and Thisse (1992, Theorems 1 and 2)). Here the nonemptyness of the set of Nash equilibria makes sense since there are UC games having no Nash equilibrium, such as two-person Matching Pennies played in pure strategies. Payoff equivalence and interchangeability are properties of a set of mixed Nash equilibria of two-person zero-sum games, which are maximin solvable, i.e., whose set of equilibria coincides with the set of maximin strategy profiles. Likewise, as De Wolf (1999) had shown, UC games are maximin solvable if there exists a Nash equilibrium (De Wolf, 1999, Theorem 3.4).

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In this paper, we show some interesting properties of UC games, *when there are more than two players*. We show, in particular, that UC games with more than two players possess: (i) a Nash equilibrium, (ii) maximin solvability, (iii) strong solvability in the sense of Nash (1951), and (iv) weak acyclicity, all in pure strategies of finite or infinite games. The property (ii) is a consequence of (i) and the result of De Wolf (1999) (see Theorem 3 of current paper), for which we will provide a different proof. The property (iv) is shown in two ways for finite games, one directly and the other by showing a generalization of a weak acyclicity sufficiency theorem in Fabrikant et al. (2013) (see Theorem 4 of current paper).

The game in Figure 1, which is presented in De Wolf (1999, Figure 7) as a non-degenerate three-person UC game, may serve to illustrate these results. This game is non-degenerate in that there is no dominated strategy. There is an improvement cycle $(u, r, A) \to (u, r, B) \to (u, l, B) \to (d, l, B) \to (d, l, A) \to (d, r, A) \to (u, r, A)$, so this is not an acyclic game. However, the game is weakly acyclic, i.e., from any strategy profile there exists an improvement path to a pure Nash equilibrium, in this case (u, l, A), which is a unique pure Nash equilibrium of the game and also a unique maximin strategy profile of the game. This game also illustrates Theorem 1 of Fabrikant et al. (2013) (Theorem 4 of current paper) that if every subgame of a game (including the game itself) has a unique pure Nash equilibrium then the game is weakly acyclic. As we will see later, that every subgame has a unique pure Nash equilibrium is a special case of that every subgame is strongly solvable in the sense of Nash, in pure strategies, which is the case for more-than-two-person UC games in pure strategies. Perhaps not so apparent at first sight for this game is that a subgame with player three's strategy fixed to A (call subgame A) is an ordinal potential game (Monderer and Shapley (1996)), with player three's payoff function being a diminishing type of ordinal potential function. So is a subgame B, and indeed every player restricted subgame is an ordinal potential game. The equilibrium (d, l, B) of subgame B is not an equilibrium of entire game, however, because player three improves by deviating. As we will see later, if a deviation from an equilibrium of a subgame like (d, l, B) of subgame B is profitable for the restricted player then it secures a higher payoff for him at an equilibrium of a new subgame, in this case (u, l, A) of subgame A. Since player three's payoff is minimized at each subgame equilibrium, an equilibrium of a subgame having the maximum minimum payoff for player three is an equilibrium of the entire game. We will prove these observations formally in the sequel.

We are mostly concerned with pure Nash equilibria of finite or infinite normal form games. This does not preclude mixed Nash equilibria of finite games, however, for it can be thought of as pure Nash equilibria of infinite games. Mixed Nash equilibria of finite games

	l	r
u	$2,2,2$	$7,1,3$
d	$1,3,7$	$0,4,6$

A

	l	r
u	$3,7,1$	$6,0,4$
d	$4,6,0$	$5,5,5$

B

Figure 1: A three-person UC game. Player three chooses A or B.

will be discussed in the final section.

The rest of the paper is organized as follows. Section 2 gives some preliminaries, Section 3 gives our main results, and Section 4 concludes with some comments.

2 Preliminaries

2.1 Games, subgames, and player subgames

Throughout, we denote by G an n-person normal form game $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$, where $N = \{1, \ldots, n\}$ is a set of players, S_i is a set of strategies of player $i \in N$, and u_i is a real-valued payoff function of player $i \in N$ defined on the set of strategy profiles $S = \prod_{i \in N} S_i$. Here S is either finite or infinite. If infinite we assume that S_i are compact and u_i are continuous, endowing the product topology to S. Let $S_{-i} = \prod_{j \neq i} S_j$. As usual, a strategy profile $s = (s_1, \ldots, s_n) \in S$ is also denoted as $s = (s_i, s_{-i}) \in S_i \times S_{-i}$. A strategy profile $s^* = (s_i^*, s_{-i}^*) \in S$ is a *pure Nash equilibrium* if

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*) \; \forall s_i \in S_i \; \forall i \in N.$$

We denote by $E(G)$ the set of pure Nash equilibria of G.

A *subgame* of a game G is a game G' obtained from G by replacing S_i with their nonempty and closed subsets $S_i' \subseteq S_i$ and restricting the domain of u_i to $S' = \prod_{i \in N} S_i'$. Note that if $s^* \in E(G)$ and $s^* \in S'$ then $s^* \in E(G')$. We denote by $G(s_i)$ a subgame of G such that player i's strategy set is replaced with $\{s_i\}$. Thus, in $G(s_i)$, player i is inactive. We call such a player restricted subgame a *player subgame*. The set of strategy profiles of $G(s_i)$ is $S(s_i)$, where $S(s_i) = \{s_i\} \times S_{-i}$. The set of equilibria of $G(s_i)$ is $E(G(s_i))$, where $E(G(s_i)) \subseteq S(s_i)$. We also denote by $G(s_i, s_j)$ a player subgame of G such that players i and j's strategy sets are replaced with $\{s_i\}$ and $\{s_j\}$, respectively. The set of strategy profiles (resp. equilibria) of $G(s_i, s_j)$ is $S(s_i, s_j)$ (resp. $E(G(s_i, s_j))$), where $S(s_i, s_j) = \{s_i\} \times \{s_j\} \times S_{-ij}$ (resp. $E(G(s_i, s_j)) \subseteq S(s_i, s_j)$).

2.2 Unilaterally competitive games

A game G is *unilaterally competitive (UC)* (Kats and Thisse (1992)) if, for any $i \in N$,

$$u_i(s'_i, s_{-i}) > u_i(s) \iff u_j(s'_i, s_{-i}) < u_j(s) \ \forall j \in N \setminus \{i\} \ \forall s'_i \in S_i \ \forall s = (s_i, s_{-i}) \in S.$$

The following two theorems are proved in Kats and Thisse (1992).[1]

Theorem 1 (Payoff equivalence: Kats and Thisse (1992, Theorem 1))**.** *If G is UC and $s^*, s^{**} \in E(G)$ then for each $i \in N$, $u_i(s') = u_i(s'')$ for all $s', s'' \in \prod_{i \in N}\{s^*_i, s^{**}_i\}$.*

Theorem 2 (Interchangeability: Kats and Thisse (1992, Theorem 2))**.** *If G is UC and $s^*, s^{**} \in E(G)$ then $s \in E(G)$ for all $s \in \prod_{i \in N}\{s^*_i, s^{**}_i\}$. In other words, $E(G)$ is a Cartesian product, if nonempty.*

Nash (1951) proposed, after having established the existence of a mixed Nash equilibrium, the *solution* of noncooperative finite games as the set of mixed Nash equilibria satisfying the above interchangeability in mixed strategies, and called a game *solvable* if it has the solution. Kats and Thisse (1992) generalized this concept to general noncooperative games, and called UC games solvable in the sense of Nash (Kats and Thisse, 1992, page 291). It is important to note, however, that the existence of an equilibrium should be guaranteed in order for a game to be called solvable, so we will call a game solvable in the sense of Nash if it has a nonempty set of equilibria forming a Cartesian product.

Nash (1951) also defined the *strong solution* and called a game *strongly solvable* if it has the strong solution; we will introduce this stronger concept in Section 3 below.

Remark 1. If G is UC then its subgames are UC, in particular, player subgames are UC. The payoff equivalence also holds for the inactive players of player subgames, i.e., $u_i(s^*) = u_i(s^{**})$ for any $s^*, s^{**} \in E(G(s_i))$. To see this, note that $u_j(s^{**}_j, s^*_{-j}) = u_j(s^*)$ implies $u_i(s^{**}_j, s^*_{-j}) = u_i(s^*)$ by UC, and apply this unilateral deviation from s^*_j to s^{**}_j for each $j \in N \setminus \{i\}$ until we reach to s^{**}.

2.3 Maximin strategies and the maximin solvability of games

A *maximin strategy* of a player i in an n-person game is a strategy $\underline{s}_i \in S_i$ that secures the *maximin payoff* $\underline{v}_i$ such that

$$\underline{v}_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

[1] In Kats and Thisse (1992) the payoff equivalence is proved for the class of *weakly unilaterally competitive* (WUC) games, which is a superclass of UC games, and the interchangeability is also shown for two-person WUC games. The results of the current paper do not extend to WUC games.

That is, $\underline{s}_i$ is a strategy such that

$$u_i(\underline{s}_i, s_{-i}) \geq \underline{v}_i \quad \forall s_{-i} \in S_{-i}.$$

See De Wolf (1999) for more on the maximin strategies and maximin payoffs for n-person games. We say that a game is *maximin solvable* if the set of equilibria coincides with the set of profiles of maximin strategies of players. De Wolf (1999) shows the following result.

Theorem 3 (Maximin solvability: De Wolf (1999, Theorem 3.4))**.** *Let G be a UC game. If there exists an equilibrium, then $s = (s_1, \ldots, s_n)$ is an equilibrium of G if and only if s_i is a maximin strategy of player i for any $i \in N$.*

2.4 Improvement path, acyclic games, and weakly acyclic games

Let us call a sequence $(s^k)_{k=0}^t$ of strategy profiles in S a *sequence of unilateral deviations in S* if, for each $k = 1, \ldots, t$, $s_i^k \neq s_i^{k-1}$ and $s_{-i}^k = s_{-i}^{k-1}$ for some $i \in N$ (i depends on k). An *improvement path in S* is a sequence of unilateral deviations in S such that, for each $k = 1, \ldots, t$,

$$s_i^k \neq s_i^{k-1},\ s_{-i}^k = s_{-i}^{k-1},\ \text{and } u_i(s^k) > u_i(s^{k-1}) \text{ for some } i \in N \ (i \text{ depends on } k).$$

Here, s_i^k such that $s_i^k \neq s_i^{k-1}$ is a *better-response* of player i to $s^{k-1} = (s_i^{k-1}, s_{-i}^{k-1})$. If this is a *best-response* of i to s_{-i}^{k-1}, i.e., if s_i^k additionally satisfies that $u_i(s_i^k, s_{-i}^{k-1}) \geq u_i(s_i, s_{-i}^{k-1})\ \forall s_i \in S_i$, then we call it a *best-improvement path in S*. We also call these paths in S paths in the game G. If $s^t = s^0$ then the path is called a *cycle*. We say that a game G is *acyclic* if it contains no improvement cycle; *best-response acyclic* if there is no best-improvement cycle. A game G is said to be *weakly acyclic* if for any $s \in S$ there exists an improvement path from s to some $s^* \in E(G)$. G is *best-response weakly acyclic*, or *weakly acyclic under best-response*, if for any $s \in S$ there exists a best-improvement path from s to some $s^* \in E(G)$.

Note that acyclicity and weak acyclicity here are defined not only for finite games but also for infinite games, i.e., for games with infinite S as well as finite S. The following theorem is proved by Fabrikant et al. (2013) for finite games.

Theorem 4 (Fabrikant et al. (2013, Theorem 1))**.** *Every finite game G that has a unique pure Nash equilibrium in every subgame G' of G is weakly acyclic, even under best-response.*

2.5 Ordinal potential games

A function $P\colon S \to \mathbb{R}$ is an *ordinal potential function* of G (Monderer and Shapley (1996)) if

$$u_i(s_i', s_{-i}) > u_i(s) \iff P(s_i', s_{-i}) > P(s)\ \forall i \in N\ \forall s_i' \in S_i\ \forall s = (s_i, s_{-i}) \in S.$$

If $-P$ is an ordinal potential function of G, we call P a *diminishing* ordinal potential function of G. A game G that has a (diminishing) ordinal potential function is called an *ordinal potential game* (Monderer and Shapley (1996)). In the following, we will make use of a diminishing type of ordinal potential function. Any minimizer of a diminishing ordinal potential function P of G, if any, is a pure Nash equilibrium of G, i.e., $\arg\min_{s\in S} P(s) \subseteq E(G)$. It is easy to see that if G is an ordinal potential game then G is acyclic, i.e., G does not have an improvement cycle. *A fortiori*, G is then best-response acyclic. Note that if G is an ordinal potential game then its subgames are also ordinal potential games, where the (diminishing) ordinal potential function is given by the restriction of the original one.

3 Main results

3.1 Existence of a pure Nash equilibrium when $n > 2$

Recall that we assume compact S_i and continuous u_i for infinite games. These guarantee the existence of $\min_{s_{-i}} u_i(s_i, s_{-i})$ and $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, which is trivial for finite games. We first prove the following lemma. This lemma holds for $n \geq 2$.

Lemma 1. *Let G be an n-person UC game with $n \geq 2$. Then every player subgame of G is an ordinal potential game having a diminishing ordinal potential function. Moreover, the set of its pure Nash equilibria coincides with the set of potential minimizers.*

Proof. It suffices to show for a player subgame in which only one player is inactive. Fix an arbitrary $i \in N$ and $s_i \in S_i$, and consider the player subgame $G(s_i)$, whose set of strategy profiles is $S(s_i) = \{s_i\} \times S_{-i}$. Letting $u_j(s_i|\cdot)$ be the restriction of u_j to $S(s_i)$ for all $j \in N$, UC property says that

$$u_j(s_i|s'_j, s_{-ij}) > u_j(s_i|s_{-i}) \iff u_i(s_i|s'_j, s_{-ij}) < u_i(s_i|s_{-i}) \; \forall j \in N\backslash\{i\} \; \forall s'_j \; \forall s_{-i} = (s_j, s_{-ij}),$$

i.e., $u_i(s_i|\cdot)$ is a diminishing ordinal potential function of $G(s_i)$. Hence $G(s_i)$ is an ordinal potential game having a diminishing ordinal potential function.

Note that every minimizer of u_i over $S(s_i)$ is an element of $E(G(s_i))$ by potential argument, i.e., $\arg\min_{s\in S(s_i)} u_i(s) \subseteq E(G(s_i))$. To see the converse, let $s \in E(G(s_i))$. If $s' \in \arg\min_{s''\in S(s_i)} u_i(s'')$, then $s' \in E(G(s_i))$, and $u_i(s) = u_i(s')$ by Theorem 1 (see also Remark 1). Thus $s \in \arg\min_{s''\in S(s_i)} u_i(s'')$, and we have $E(G(s_i)) \subseteq \arg\min_{s\in S(s_i)} u_i(s)$. Hence $E(G(s_i)) = \arg\min_{s\in S(s_i)} u_i(s)$. □

The following lemma asserts, in essence, that player subgames are "well-ordered".

Lemma 2. *Let G be an n-person UC game with $n > 2$. If $s \in E(G(s_i))$, $u_i(s'_i, s_{-i}) > u_i(s)$, and $s' \in E(G(s'_i))$, then $u_i(s') > u_i(s)$.*

Proof. Assume that $s \in E(G(s_i))$, $u_i(s'_i, s_{-i}) > u_i(s)$, $s' \in E(G(s'_i))$, and suppose by way of contradiction that $u_i(s') \leq u_i(s)$. We let $\hat{G}$ be a finite subgame of G such that the set of strategy profiles is $\hat{S} = \prod_{h \in N}\{s_h, s'_h\}$, which is UC, and show that there exist two contradictory sequences $(x^k)_{k=0}^p$ and $(y^k)_{k=0}^q$ of unilateral deviations in $\hat{G}$. Note that $s \in E(\hat{G}(s_i))$ and $s' \in E(\hat{G}(s'_i))$. Also $s' \neq (s'_i, s_{-i})$ (since $u_i(s') \leq u_i(s) < u_i(s'_i, s_{-i})$), so there exists some $j \in N \setminus \{i\}$ such that $s'_j \neq s_j$. Fix such j.

(i) $(x^k)_{k=0}^p$: Let $x^0 = s$, $z = (s'_i, s_{-i})$, and $x^2 = (s'_i, s'_j, s_{-ij})$. We have $u_i(x^0) < u_i(z)$ by assumption. There are two cases. (a) $u_j(z) \geq u_j(x^2)$: In this case let $x^1 = z$. Then $u_i(x^0) < u_i(x^1)$ with $x_i^0 \neq x_i^1$, and $u_j(x^0) > u_j(x^1) \geq u_j(x^2)$ by UC property. (b) $u_j(z) < u_j(x^2)$: In this case let $x^1 = (s'_j, s_{-j})$. Then $u_j(x^0) \geq u_j(x^1)$ by $x^0 = s \in E(\hat{G}(s_i))$. Also $u_i(x^1) < u_i(x^2)$, since if $u_i(x^2) \leq u_i(x^1)$ with $x_i^2 \neq x_i^1$, then by $u_j(x^1) \leq u_j(x^0)$ with $x_j^1 \neq x_j^0$, $u_i(x^0) < u_i(z)$ with $x_i^0 \neq z_i$, and $u_j(z) < u_j(x^2)$ with $z_j \neq x_j^2$, we have $u_h(x^2) \geq u_h(x^1) \geq u_h(x^0) > u_h(z) > u_h(x^2)$ by UC property for $h \neq i, j$, a contradiction. Hence $u_i(x^1) < u_i(x^2)$, and $u_j(x^0) \geq u_j(x^1) > u_j(x^2)$ by UC property. Now, we have $u_j(x^0) > u_j(x^2)$ in both cases. If $x^2 \notin E(\hat{G}(s'_i, s'_j))$ then append a best-improvement path in $\hat{G}(s'_i, s'_j)$ from x^2 to some $x^p \in E(\hat{G}(s'_i, s'_j))$, along which u_j is diminishing; otherwise let $p = 2$. Since $s' \in E(\hat{G}(s'_i))$, we have $s' \in E(\hat{G}(s'_i, s'_j))$, and we have $u_j(x^p) = u_j(s')$ by Theorem 1. Hence

$$u_j(s) = u_j(x^0) > u_j(x^p) = u_j(s'). \tag{1}$$

(ii) $(y^k)_{k=0}^q$: The construction is similar to (i). Let $y^0 = s'$, $z = (s_i, s'_{-i})$, and $y^2 = (s_i, s_j, s'_{-ij})$. We have $u_i(y^0) \leq u_i(z)$, i.e. $u_i(s') \leq u_i(s_i, s'_{-i})$ (since $u_i(s') \leq u_i(s) \leq u_i(s_i, s'_{-i})$; this second inequality is by $s \in E(\hat{G}(s_i))$ and Lemma 1). Again there are two cases. (a) $u_j(z) \geq u_j(y^2)$: In this case let $y^1 = z$. Then $u_i(y^0) \leq u_i(y^1)$ with $y_i^0 \neq y_i^1$, and $u_j(y^0) \geq u_j(y^1) \geq u_j(y^2)$ by UC property. (b) $u_j(z) < u_j(y^2)$: In this case let $y^1 = (s_j, s'_{-j})$. Then $u_j(y^0) \geq u_j(y^1)$ by $y^0 = s' \in E(\hat{G}(s'_i))$. Also $u_i(y^1) < u_i(y^2)$, since if $u_i(y^2) \leq u_i(y^1)$ with $y_i^2 \neq y_i^1$, then by $u_j(y^1) \leq u_j(y^0)$ with $y_j^1 \neq y_j^0$, $u_i(y^0) < u_i(z)$ with $y_i^0 \neq z_i$, and $u_j(z) < u_j(y^2)$ with $z_j \neq y_j^2$, we have $u_h(y^2) \geq u_h(y^1) \geq u_h(y^0) > u_h(z) > u_h(y^2)$ by UC property for $h \neq i, j$, a contradiction. Hence $u_i(y^1) < u_i(y^2)$, and $u_j(y^0) \geq u_j(y^1) > u_j(y^2)$ by UC property. Now, we have $u_j(y^0) \geq u_j(y^2)$ in both cases. If $y^2 \notin E(\hat{G}(s_i, s_j))$ then append a best-improvement path in $\hat{G}(s_i, s_j)$ from y^2 to some $y^q \in E(\hat{G}(s_i, s_j))$, along which u_j is diminishing; otherwise let $q = 2$. Then since $s \in E(\hat{G}(s_i, s_j))$, we have $u_j(y^q) = u_j(s)$ by Theorem 1. Hence

$$u_j(s') = u_j(y^0) \geq u_j(y^q) = u_j(s). \tag{2}$$

Since (2) contradicts (1), we must have that $u_i(s') > u_i(s)$. □

See Figure 2 for the illustration of the choice of x^1 and y^1. The (dotted) arrows represent the directions of (weak form of) improvement. The sequences $(x^k)_{k=0}^p$ and $(y^k)_{k=0}^q$ are chosen so that $u_j(\cdot)$ is decreasing.

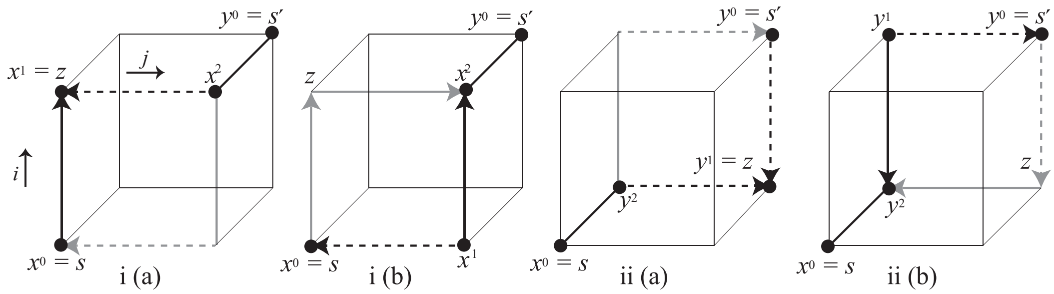

Figure 2: Choice of x^1 and y^1. i(a) $u_j(z) \geq u_j(x^2)$, i(b) $u_j(z) < u_j(x^2)$; ii(a) $u_j(y^2) \leq u_j(z)$, ii(b) $u_j(y^2) > u_j(z)$. The sequences are chosen so that $u_j(\cdot)$ is decreasing.

Now we are ready to prove the following existence theorem.

Theorem 5. *Every n-person UC game with $n > 2$ has a pure Nash equilibrium.*

Proof. By Lemma 1, $E(G(s_i))$ is nonempty for any $i \in N$ and $s_i \in S_i$. In addition, any $s, s' \in E(G(s_i))$ are payoff equivalent by Theorem 1. Let $v_i(s_i) = u_i(s)$ with $s \in E(G(s_i))$. Then, by Lemma 2, we have for any s_i and $s = (s_i, s_{-i}) \in E(G(s_i))$ that

$$u_i(s_i', s_{-i}) > u_i(s) \implies v_i(s_i') > v_i(s_i)$$

i.e., by taking the contrapositive,

$$v_i(s_i') \leq v_i(s_i) \implies u_i(s_i', s_{-i}) \leq u_i(s).$$

Let $s_i^* \in S_i$ be such that $v_i(s_i^*) = \max_{s_i \in S_i} v_i(s_i)$, i.e., $v_i(s_i^*) = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$, and $s^* = (s_i^*, s_{-i}^*) \in E(G(s_i^*))$. Then since $v_i(s_i') \leq v_i(s_i^*)$ for all $s_i' \in S_i$, we have $u_i(s_i', s_{-i}^*) \leq u_i(s^*)$ for all $s_i' \in S_i$. Also $u_j(s_j', s_{-j}^*) \leq u_j(s^*)$ for all $s_j' \in S_j$ and all $j \in N \setminus \{i\}$ by $s^* \in E(G(s_i^*))$. Hence s^* is a pure Nash equilibrium of G. □

3.2 Maximin solvability

Recall that a game G is maximin solvable if the set of equilibria and the set of maximin strategy profiles coincide. While the latter is always nonempty, the former is not, so the maximin solvability requires the set of equilibria be nonempty. By Theorems 3 and 5, we have established the following result.

Theorem 6. *Every n-person UC game with $n > 2$ is maximin solvable in pure strategies.*

We here give a proof of Theorem 3 differently to De Wolf (1999).

Proof of Theorem 3. Let G be an n-person UC game with $n \geq 2$. By Lemma 1, we observe that player i chooses s_i to maximize $u_i(s_i, s_{-i})$ given s_{-i} and player $-i$ ("other than i") chooses s_{-i} to minimize $u_i(s_i, s_{-i})$ given s_i, and this is true for any $i \in N$. Since the minimization of u_i is equivalent to the maximization of $-u_i$, this game of players i and $-i$ is equivalent to a two-person zero-sum game, for any $i \in N$. Now, if there exists an equilibrium in a two-person zero-sum game then a pair of strategies is an equilibrium if and only it is a pair of maximin strategies (see e.g. Osborne and Rubinstein (1994, Section 2.5)). If s is an equilibrium of G, then (s_i, s_{-i}) is an equilibrium of two-person game of i and $-i$, for any $i \in N$. Thus, in particular, if there exists an equilibrium in G, then $s = (s_1, \ldots, s_n)$ is an equilibrium of G if and only if s_i is a maximin strategy of player i, for any $i \in N$. □

3.3 Strong solvability in the sense of Nash

Recall that a game G is solvable in the sense of Nash if it has the solution, namely, a nonempty set of interchangeable equilibria. By Theorems 2 and 5, we can now say that every n-person UC game with $n > 2$ is solvable in pure strategies. Actually, more can be said.

A solvable game G is said to be *strongly solvable* (Nash, 1951) if it has the *strong solution* $E(G)$, which is the solution with additional property such that

$$s \in E(G),\ (s'_i, s_{-i}) \in S,\ u_i(s'_i, s_{-i}) = u_i(s) \implies (s'_i, s_{-i}) \in E(G).$$

Thus, if s is a unique equilibrium of a game G, G is strongly solvable if and only if s is a strict unique equilibrium of G. We now prove the following result.

Theorem 7. *Every n-person UC game with $n > 2$ is strongly solvable in pure strategies.*

Proof. Assume that $s \in E(G)$, $(s'_i, s_{-i}) \in S$, and $u_i(s'_i, s_{-i}) = u_i(s)$. If $(s'_i, s_{-i}) \notin E(G)$ then there exists $j \in N \setminus \{i\}$ such that $u_j(s'_j, s'_i, s_{-ji}) > u_j(s_j, s'_i, s_{-ji})$ for some $s'_j \in S_j$. Let

$$x^0 = (s_j, s_i, s_{-ji}),\ x^1 = (s_j, s'_i, s_{-ji}),\ x^2 = (s'_j, s'_i, s_{-ji}),\ x^3 = (s'_j, s_i, s_{-ji}).$$

Then $u_i(x^0) = u_i(x^1)$ with $x^0_i \neq x^1_i$, $u_j(x^1) < u_j(x^2)$ with $x^1_j \neq x^2_j$, and $u_j(x^3) \leq u_j(x^0)$ with $x^3_j \neq x^0_j$ since $x^0 \in E(G)$. Note that $x^2_i \neq x^3_i$. If $u_i(x^2) \geq u_i(x^3)$ then by UC property

$$u_j(x^0) = u_j(x^1) < u_j(x^2) \leq u_j(x^3) \leq u_j(x^0),$$

a contradiction. If $u_i(x^2) < u_i(x^3)$ then, for $h \neq i, j$, by UC property

$$u_h(x^0) = u_h(x^1) > u_h(x^2) > u_h(x^3) \geq u_h(x^0),$$

another contradiction. Hence we must have that $(s'_i, s_{-i}) \in E(G)$. □

3.4 Weak acyclicity

Assume first that G is a *finite* UC game with more than two players. Lemmas 1 and 2 readily suggest the existence of a best-improvement path from any profile s to some $s^* \in E(G)$, as shown in the proof of the following theorem.

Theorem 8. *Every finite n-person UC game with $n > 2$ is weakly acyclic, even under best-response.*

Proof. Pick an arbitrary $s \in S$. We are done if $s \in E(G)$. Otherwise fix an $i \in N$, and follow a best-improvement path in $G(s_i)$ to some $(s_i, s'_{-i}) \in E(G(s_i))$, which is a best-improvement path in G. If s_i is not a best-response to s'_{-i} then switch to a best-response s'_i, and follow a best-improvement path in $G(s'_i)$ to some $(s'_i, s''_{-i}) \in E(G(s'_i))$, which is a best-improvement path in G. Iteration of this procedure terminates at a pure Nash equilibrium of G in finite steps by Lemma 2 and finiteness of S_i. □

Although it may not be so common to consider weak acyclicity in infinite games, we have the following result for infinite games.

Theorem 9. *Every n-person UC game with $n > 2$ is weakly acyclic.*

Proof. Pick an arbitrary $s \in S$. We are done if $s \in E(G)$. Otherwise pick an arbitrary $s^* \in E(G)$, and let $\hat{G}$ be a finite subgame of G such that the set of strategy profiles is $\hat{S} = \prod_{h=1}^{n} \{s_h, s_h^*\}$, which is UC. Here $u_i(s) < u_i(s_i^*, s_{-i})$ for some i. To see this, suppose to the contrary. Then $u_h(s) \geq u_h(s_h^*, s_{-h})$ for all h, i.e., s is an equilibrium of $\hat{G}$. We then have $s, s^* \in E(\hat{G})$, so $s \in \hat{S}$ are all equilibria by Theorem 2, and payoff equivalent by Theorem 1. However, since $s \notin E(G)$, there must exist $j \in N$ and $s'_j \in S_j$ such that $u_j(s) < u_j(s'_j, s_{-j})$, i.e., $u_j(s_j^*, s_{-j}) < u_j(s'_j, s_{-j})$ since $u_j(s) = u_j(s_j^*, s_{-j})$. Notice that $s^* \in E(G(s_j^*))$ and $u_j(s^*) = u_j(s_j^*, s_{-j})$ imply $(s_j^*, s_{-j}) \in E(G(s_j^*))$ by Lemma 1, and $u_j(s_j^*, s_{-j}) < u_j(s'_j, s_{-j})$ implies that $u_j(s^*) < u_j(s^{**})$ with $s^{**} \in E(G(s'_j))$ by Lemma 2, contradicting that s_i^* of $s^* \in E(G)$ is a maximin strategy of i (Theorem 6). Hence $u_i(s) < u_i(s_i^*, s_{-i})$ for some i. Let $s^0 = s$, $s^1 = (s_i^*, s_{-i})$, and follow the best-improvement path $s^1, s^2, \ldots, s^t$ in $\hat{G}(s_i^*)$ to some $s^t \in E(\hat{G}(s_i^*))$, which is an improvement path in G. Note that $s^t \in E(\hat{G}(s_i^*))$ is also an equilibrium of G, because by $s^* \in E(\hat{G}(s_i^*))$, $u_i(s^*) = u_i(s^t)$ by Theorem 1, and $s^t \in E(G)$

by Lemma 1. We thus have an improvement path $s^0, s^1, \ldots, s^t$ in G from s to a pure Nash equilibrium $s^t \in E(G)$. $\square$

For finite G, there is an alternative proof of Theorem 8. Recall Theorem 4 (Fabrikant et al. (2013, Theorem 1)). For the games of Theorem 8 the uniqueness of equilibrium does not hold in general. However, we can generalize Theorem 4 in the following way.

Theorem 10. *Every finite game G that has a strong solution in every subgame G' of G is weakly acyclic, even under best-response.*

This theorem does not require G be UC. Notice that that every subgame has a unique pure Nash equilibrium implies that every subgame is strongly solvable. To see this, let s be the unique pure Nash equilibrium of a subgame G' whose set of strategy profiles is S', and consider the subgame of G' whose set of strategy profiles is $S'_i \times \{s_{-i}\}$. Since s must also be a unique pure Nash equilibrium of this subgame, we have $u_i(s'_i, s_{-i}) < u_i(s)$ for all $s'_i \in S'_i \setminus \{s_i\}$, and this is true for any $i \in N$, i.e., s is a *strict* unique pure Nash equilibrium of G', which shows that G' is strongly solvable.

In the following, we provide a proof of Theorem 10 along the line of proof in Fabrikant et al. (2013). We use Lemma 1 of Fabrikant et al. (2013) as follows. Here $BR_G(s)$ is the set of strategy profiles reached by best-responses in a game G starting from a profile $s \in S$.

Lemma 3 (Fabrikant et al.'s Lemma 1)**.** *If s is a strategy profile in G, and G' is the subgame of G spanned by $BR_G(s)$, then any best-improvement path $s, s^1, \ldots, s^k$ in G' that starts at s is a best-improvement path in G.*

Proof. See Fabrikant et al. (2013). $\square$

Proof of Theorem 10. Every single profile subgame of G is trivially best-response weakly acyclic. Suppose, by way of induction, that for some subgame G' of G every strict subgame of G' is best-response weakly acyclic. We show that G' is then best-response weakly acyclic. Let $s \in G'$, and G'' the subgame of G' spanned by $BR_{G'}(s)$. Letting S' and S'' be the sets of strategy profiles of G' and G'', respectively, we consider the cases (i) $E(G') \cap S'' \neq \emptyset$ and (ii) $E(G') \cap S'' = \emptyset$. In the former case we assume that $s \notin E(G')$ since otherwise trivial.

Case (i): $E(G') \cap S'' \neq \emptyset$. Pick an arbitrary $s' \in E(G') \cap S''$. Then, since $s' \in S''$ and S'' is spanned by $BR_{G'}(s)$, there is for any $j \in N$ such that $s'_j \neq s_j$ a best-improvement path in G' from s to some $\hat{s} \in S''$ such that $\hat{s}_j = s'_j$. Fix one such j and consider $G'(\hat{s}_j)$, a strict subgame of G' where j is restricted to playing $\hat{s}_j = s'_j$. The inductive hypothesis guarantees a best-improvement path in $G'(\hat{s}_j)$ from $\hat{s}$ to some $s'' \in E(G'(\hat{s}_j))$, where $s''_j = \hat{s}_j = s'_j$. The path is also a best-improvement path in G'. Notice that $s' \in E(G')$ is also in $E(G'(\hat{s}_j))$.

Since $E(G'(\hat{s}_j))$ is a solution, we must have $\prod_{i\in N}\{s'_i, s''_i\} \subseteq E(G'(\hat{s}_j))$, where $s'_j = s''_j$. Since the solution is strong, we must have $(s''_1, s'_{-1}) \in E(G')$ since $u_1(s''_1, s'_{-1}) = u_1(s')$, $(s''_1, s''_2, s'_{-12}) \in E(G')$ since $u_2(s''_1, s''_2, s'_{-12}) = u_2(s''_1, s'_{-1})$, and likewise, $s'' \in E(G')$. Hence we have a best-improvement path in G' from s to $s'' \in E(G')$ via $\hat{s}$.

Case (ii): $E(G') \cap S'' = \emptyset$. Then $E(G') \cap E(G'') = \emptyset$. Also G'' is a strict subgame of G'. Because $s'' \in E(G'')$ implies $s'' \notin E(G')$, any $s'' \in E(G'')$ must have an outgoing best-improvement edge to some profile $\hat{s}(s'')$ in G'. But the inductive hypothesis ensures that $s'' \in BR_{G''}(s)$ for some $s'' \in E(G'')$. By Lemma 3, $s'' \in BR_{G'}(s)$, which then ensures that $\hat{s}(s'')$ must also be in $BR_{G'}(s)$, and hence in G''. Thus, s'' is not an equilibrium of G'', which is a contradiction. This case (ii) is impossible. □

Note that, by Theorem 7, if $n > 2$ then all the subgames of an n-person UC game are strongly solvable since they are n-person UC with $n > 2$. Thus the games of Theorem 8 satisfy the condition of Theorem 10, and we can prove Theorem 8 also by using Theorem 10.

4 Concluding comments

Since subgames of a UC game are also UC, our results also hold for the subgames of n-person UC games with $n > 2$. The weak acyclicity are strengthened in *player* subgames to acyclicity because they are ordinal potential games.

Take the game of Fig. 1 again. As we have seen, this three-person UC game has a cycle in the entire game. Note that, when applied to this game, Lemma 2 is saying the impossibility of a cycle in the entire game that involves equilibria of player subgames. This is a sharp contrast to two-person UC games, which permits a cycle involving equilibria of player subgames (e.g. Matching Pennies).

Our results rest on Theorems 1 and 2 that are proved in Kats and Thisse (1992), for which we did not give the proofs. We also used and cited the results of De Wolf (1999) and Fabrikant et al. (2013), namely, Theorems 3 and 4 of current paper, respectively, for which we gave alternative proofs. Except for the last Theorem 4 (and its base Lemma 3), all the theorems mentioned here are valid for both finite and infinite games. Also our Lemmas 1, 2, and Theorems 5, 6, 7, are valid for both finite and infinite games, whereas Theorems 8 and 10 are for finite games, and Theorem 9 for infinite games; these differ in whether the weak acyclicity is in best-response or not.

We have hitherto considered the pure Nash equilibria of finite or infinite games. Let us now briefly look at the mixed Nash equilibria of finite games. Since such an equilibrium, i.e., an equilibrium of the *mixed extension* of finite games, is thought of as a variant of a pure

Nash equilibrium of infinite games, our theorems mentioned above as being valid for infinite games are also valid for mixed Nash equilibria of finite games. We note, in particular, that if G is a finite game that is UC in mixed strategies then Theorem 7 applied to the mixed extension of G implies that G is strongly solvable in Nash's original sense. As remarked in Nash (1951, page 290), G that is strongly solvable (in mixed strategies) has a pure Nash equilibrium. It can be shown that the set of equilibria of such games is the convex hull of the set of pure Nash equilibria. Therefore, such games are sufficient to be considered in pure strategies. However, as De Wolf (1999) has remarked, such games are very restrictive, and tend to be degenerate in the sense that most of pure strategies are dominated.

As a final comment, we note, analogously to the convexity theorem of Kats and Thisse (1992, Theorem 3), that the set of pure Nash equilibria of finite games with linearly ordered finite strategy sets that are UC in pure strategies, if nonempty, is *convex in the sense of order* if the payoff functions are *quasiconcave with respect to the own strategy*; see Iimura and Watanabe (2016, Theorem 3.2). The results of this paper also apply to such ordered finite n-person UC games, provided that $n > 2$.

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