Research Paper Series

No. 156

Reformulation of the arbitrage-free pricing method under the multi-curve environment

Masaaki Kijima†, Yukio Muromachi‡

October, 2015

†Graduate School of Social Sciences, Tokyo Metropolitan University

‡Graduate School of Social Sciences, Tokyo Metropolitan University

REFORMULATION OF THE ARBITRAGE-FREE PRICING METHOD UNDER THE MULTI-CURVE ENVIRONMENT

MASAAKI KIJIMA AND YUKIO MUROMACHI

ABSTRACT. This paper proposes a unified framework for the pricing of derivatives under the multi-curve setting. It is shown that *any* derivative security can be duplicated by using the underlying assets, collateral account and funding account, appropriately. A risk-neutral measure is defined accordingly under which the derivative price is determined uniquely. This idea is extended to the pricing of OIS and LIBOR discount bonds and interest-rate derivatives under the risk-neutral measure, which explains the existence of multiple yield curves simultaneously in the market. Some specific models are given to demonstrate the usefulness of our approach. Through numerical examples, we find that the discrepancy of derivative prices under the multi-curve setting from the classical ones becomes significant when the spread volatility between the collateral and funding rates exceeds some level.

Keywords: Multi-curve, duplication, OIS, LIBOR, collateral rate, funding rate, risk-neutral measure, forward measure.

1. INTRODUCTION

Since the beginning of the worldwide financial crisis in 2007, LIBOR rates have been deviated from OIS (Overnight Index Swap) rates for the same maturity. Also, a swap rate based on semiannual payments of the six-month LIBOR rate, for example, has been different from the same-maturity swap rate based on quarterly payments of the three-month LIBOR rate. According to Mercurio (2009), while the construction of a no-arbitrage framework that is consistent with the simultaneous existence of such different yield curves can be possible by using the credit and liquidity theories, practitioners seem to agree with an empirical approach, which is based on the construction of many possible curves of rate lengths, called the *multi-curve approach*.

Recall that, in the classic pricing approach, OIS rates and LIBOR rates with different tenors are defined through a unique and fully consistent zero-coupon curve, which is thus used both in the generation of future cash flows of *any* interest-rate derivatives and in the calculation of their present values. On the other hand, in the multi-curve setting, future cash flows are generated through the curves associated with the underlying rates and then discounted by another curve, not necessary by risk-free discount curve.

A pioneering work of the valuation of interest-rate derivatives under the multi-curve setting seems the papers by Boenkost and Schmidt (2005) and Henrard (2007), which are followed and extended by, e.g., Kijima, Tanaka and Wong (2009) and Mercurio (2009). In particular, Kijima, Tanaka and Wong (2009) assume *exogenously* that there exists a risk-neutral measure

---

Date: October 17, 2015.

M. Kijima: Graduate School of Social Sciences, Tokyo Metropolitan University. Address: 1-1 Minami-Ohsawa, Hachiohji, Tokyo 192-0397, Japan. E-mail: kijima@tmu.ac.jp

Y. Muromachi: Graduate School of Social Sciences, Tokyo Metropolitan University. Address: 1-1 Minami-Ohsawa, Hachiohji, Tokyo 192-0397, Japan. E-mail: muromachi-yukio@tmu.ac.jp

The authors are grateful for the research grant funded by the Grant-in-Aid (A) (#26242028) from Japan's Ministry of Education, Culture, Sports, Science and Technology.

under which any price process discounted by the risk-free savings account is a martingale and other yield curves are given consistently in the market.

In this paper, we construct a no-arbitrage framework of financial market that is consistent to the multi-curve setting, because using different yield curves at the same time requires a reformulation of the basic assumptions made in the classic interest-rate models. Of our particular interest are how to define a risk-neutral measure for a *given* financial market and how to price derivative securities in the market. However, as in Mercurio (2009) and many others, we do not take into account the credit and liquidity issues. We rather start from the existence of multiple yield curves and try to build a consistent framework with the actual market.

One of the promising approaches to explain the existence of multiple yield curves is to use the collateralization in derivatives contracts.[1] Namely, we assume that there exist not only the underlying securities but also the collateral account, funding account, and repo account (defined later) in the market. It is shown that *any* derivative security can be duplicated by using these securities appropriately. A risk-neutral measure is defined accordingly and the derivative price is determined uniquely under the measure. This idea can be applied to the pricing of OIS and LIBOR discount bonds and interest-rate derivatives as well, ending up with the simultaneous existence of multiple yield curves in the market.

This idea is not new in the finance literature. For example, Piterbarg (2010) discusses the price of a perfectly collateralized derivative by constructing a self-financing risk-free portfolio. Using a no-arbitrage condition, he derives a partial differential equation (PDE) satisfied by the price function and obtains the price of OIS discount bonds by applying the Feynman-Kac theorem to the PDE. See also Piterbarg (2012) and Han, He and Zhang (2014, 2015) for similar developments.

Application of this framework for the pricing of interest-rate derivatives has been also discussed in many papers. For example, Bianchetti (2013) and Henrard (2014) have discussed the pricing of interest-rate derivatives in the multi-curve setting and derived useful formulas which are widely used in practice for calculating OIS and LIBOR discount curves and for pricing interest-rate derivatives.

The aim of this article is a reformulation of the no-arbitrage framework in financial markets that is consistent to the multi-curve environment. Our discussion starts from the construction of a self-financing duplication portfolio under the physical probability measure. The risk-neutral measure is defined by the change of measure technique due to the Girsanov theorem. Any derivative security including interest-rate derivatives can then be priced under the risk-neutral measure.

This paper is organized as follows. In the next section, we set up the security market model and show that any derivative security can be duplicated by using the underlying securities, collateral account, and funding account in the market appropriately. The risk-neutral measure $\mathbb{Q}$ is defined and derivative securities are priced under $\mathbb{Q}$ accordingly. A remarkable result is that, while the instantaneous rate of return of the derivatives under $\mathbb{Q}$ depends on the definition of collateraization, that of each underlying security is given by its repo rate; hence they are not identical in contrast to the classic single-curve case. This idea is then applied to the pricing of OIS and LIBOR discount curves in Section 3, which explains the simultaneous existence of

[1] In the last decade, collateralization in the derivatives contracts has increased rapidly along with the CSA (Credit Support Annex) to the ISDA (International Swaps and Derivatives Association) master agreement. It is quite difficult now to make a contract without a collateral agreement among the major financial institutions.

multiple yield curves in the market. In Section 4, we consider the pricing of forward rate agreements (FRAs) and interest-rate swaps (IRSs). Section 5 is devoted to some specific models. In particular, we develop a short rate model in which the prices of OIS and LIBOR discount bonds as well as FRA and IRS rates are derived in closed form. Some numerical examples are given to show the deviation of the two-curve setting from the classic single-curve setting. Through numerical examples, we find that the discrepancy of derivative prices under the two-curve setting from the classical ones becomes significant when the spread volatility between the collateral and funding rates exceeds some level. Section 6 concludes this paper. Proofs are provided in Appendix A for the reader's convenience.

Throughout this paper, $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0\le t\le \bar{T}})$ denotes a filtered probability space where $\bar{T}$ is finite and the filtration $\{\mathcal{F}_t\}_{0\le t\le \bar{T}}$ satisfies the usual conditions. The probability measure $\mathbb{P}$ is the physical measure and a martingale (pricing) measure will be denoted by $\mathbb{Q}$. The expectation operator under $\mathbb{Q}$ is denoted by $\mathbb{E}$.

2. Pricing by Duplication

In this section, we consider a financial market in which there are available n risky assets.[2] Let $S_i(t)$ denote the time-t price of risky asset i, $i = 1, 2, \dots, n$. It is assumed that the risky asset price $S_i(t)$ under the physical measure $\mathbb{P}$ follows the stochastic differential equation (SDE)

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = \mu_i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}W_i(t), \qquad i = 1, 2, \dots, n, \tag{2.1}$$

where the instantaneous rates of return $\mu_i(t)$ and the volatilities $\sigma_i(t)$ satisfy the standard conditions and where $W_i(t)$ are correlated standard Brownian motions under $\mathbb{P}$.

On the other hand, there are three kinds of non-risky savings accounts; collateral, funding, and repo accounts. We denote the time-t price per one unit of the collateral account by $B_C(t)$, whereas $B_F^j(t)$ denotes the jth funding account, $j = 1, 2, \dots, m$. While the collateral account is the savings account that is fully secured by collaterals, the funding accounts have no collaterals. Different quality of the funding accounts are assumed to be available in the market. However, in order to keep the description as simple as possible, this section considers the case $m = 1$ only. The next section treats the general case. Denoting by $r_C(t)$ and $r_F(t)$ the instantaneous collateral rate and funding rate at time t, respectively, the associated collateral and funding accounts are defined by

$$\mathrm{d}B_C(t) = r_C(t)B_C(t)\mathrm{d}t, \qquad \mathrm{d}B_F(t) = r_F(t)B_F(t)\mathrm{d}t \tag{2.2}$$

with $B_C(0) = B_F(0) = 1$. Finally, when the funding is secured by asset $S_i(t)$, the short rate $r_R^i(t)$ is applied that represents the credit quality of the asset $S_i(t)$. Here, R stands for 'repo' as in Piterbarg (2010). In the actual markets, we expect that $r_C(t) \le r_R^i(t) \le r_F(t)$ for all i. Note, however, that there is no arbitrage opportunity across those savings accounts, because of the availability of collaterals or not.

Consider a European contingent claim written on $(S_1(t), \dots, S_n(t))$ with payoff function $h(S_1, \dots, S_n)$ and maturity T, $0 \le T \le \bar{T}$. The time-t price of the claim is denoted by $V(t)$. Suppose that the contingent claim $V(t)$ is duplicated through a self-financing strategy of trading $(\theta_1(t), \dots, \theta_n(t), \theta_C(t), \theta_F(t))$ units of the underlying risky assets and the two non-risky savings

[2] For the sake of simplicity, we assume that every asset pays no dividends. It is a straightforward extension to include positive dividend rates.

accounts. That is,

$$V(t)=\sum_{i=1}^{n}\theta_i(t)S_i(t)+\Gamma(t),\qquad 0\le t\le T, \tag{2.3}$$

with $V(T)=h(S_1(T),\ldots,S_n(T))$, where $\Gamma(t)$ denotes the cash amount held in the duplicated portfolio, i.e.,

$$\Gamma(t)=\theta_C(t)B_C(t)+\theta_F(t)B_F(t)-R(t),\qquad R(t)\equiv\sum_{i=1}^{n}\theta_i(t)S_i(t). \tag{2.4}$$

It should be noted from (2.3) and (2.4) that the value $V(t)$ of the contingent claim is divided into the collateral account $C(t)\equiv\theta_C(t)B_C(t)$ and the funding account $F(t)\equiv\theta_F(t)B_F(t)$ according to a predetermined manner.[3] That is, we have

$$V(t)=C(t)+F(t),\qquad 0\le t\le T. \tag{2.5}$$

Also, the repo rate $r_R^i(t)$ is applied for the funding that is secured by asset $S_i(t)$. In other words, the growth of the cash amount $R(t)$ is given by

$$\mathrm{d}R(t)=\sum_{i=1}^{n}r_R^i(t)\left[\theta_i(t)S_i(t)\right]\mathrm{d}t. \tag{2.6}$$

Now, under the self-financing strategy, (2.3) and (2.4) together imply

$$\mathrm{d}V(t)=\sum_{i=1}^{n}\theta_i(t)\mathrm{d}S_i(t)+\theta_C(t)\mathrm{d}B_C(t)+\theta_F(t)\mathrm{d}B_F(t)-\mathrm{d}R(t). \tag{2.7}$$

Substituting (2.1), (2.2) and (2.6) into (2.7), we obtain

$$\begin{aligned}\mathrm{d}V(t)&=\left(\sum_{i=1}^{n}\theta_i(t)(\mu_i(t)-r_R^i(t))S_i(t)+\theta_C(t)r_C(t)B_C(t)+\theta_F(t)r_F(t)B_F(t)\right)\mathrm{d}t\\&\quad+\sum_{i=1}^{n}\theta_i(t)\sigma_i(t)S_i(t)\mathrm{d}W_i(t).\end{aligned} \tag{2.8}$$

Let us define the processes $W_i^*(t)$, $i=1,\ldots,n$, by

$$\mathrm{d}W_i^*(t)=\mathrm{d}W_i(t)+\lambda_i(t)\mathrm{d}t,\qquad \lambda_i(t)\equiv\frac{\mu_i(t)-r_R^i(t)}{\sigma_i(t)}, \tag{2.9}$$

with $W_i^*(0)=0$. Following the tradition in the standard finance literature, we call $\lambda_i(t)$ the *market price of risk* associated with $W_i(t)$.

Substituting (2.9) into (2.8), we then have

$$\mathrm{d}V(t)=(r_C(t)C(t)+r_F(t)F(t))\,\mathrm{d}t+\sum_{i=1}^{n}\theta_i(t)\sigma_i(t)S_i(t)\mathrm{d}W_i^*(t). \tag{2.10}$$

Define $\gamma(t)$ to be the ratio of the collateral $C(t)$ to the total value $V(t)$, i.e., $\gamma(t)=C(t)/V(t)$. Using the ratio $\gamma(t)$, we define the weighted average of the short rates by

$$r_\gamma(t)\equiv r_C(t)\gamma(t)+r_F(t)(1-\gamma(t)). \tag{2.11}$$

[3] In this paper, we assume that the investor who holds the contingent claim can access the funding market at any time and at any amount. Also, the asymmetric collateralization is not considered, i.e., the lending and borrowing rates are the same.

It follows from (2.5) that $r_C(t)C(t) + r_F(t)F(t) = r_\gamma(t)V(t)$, and so we obtain from (2.10) that

$$\mathrm{d}V(t) = r_\gamma(t)V(t)\mathrm{d}t + \sum_{i=1}^{n} \theta_i(t)\sigma_i(t)S_i(t)\mathrm{d}W_i^*(t). \tag{2.12}$$

Associated with $r_\gamma(t)$ defined in (2.11) is the new savings account

$$B_\gamma(t) \equiv \exp\left\{\int_0^t r_\gamma(s)\mathrm{d}s\right\}, \qquad B_\gamma(0) = 1.$$

Consider the denominated price $V_\gamma^*(t) \equiv V(t)/B_\gamma(t)$ with the numéraire $B_\gamma(t)$. It follows from (2.12) that

$$\mathrm{d}V_\gamma^*(t) = \sum_{i=1}^{n} \theta_i(t)\sigma_i(t)S_i^*(t)\mathrm{d}W_i^*(t).$$

Integrating it over $[t, T]$, we obtain

$$V_\gamma^*(T) - V_\gamma^*(t) = \sum_{i=1}^{n} \int_t^T \theta_i(u)\sigma_i(u)S_i^*(u)\mathrm{d}W_i^*(u). \tag{2.13}$$

Let us define a probability measure $\mathbb{Q}$ that makes the processes $(W_1^*(t), \ldots, W_n^*(t))$ standard Brownian motions. The existence of such $\mathbb{Q}$ is guaranteed by the standard Girsanov's theorem under regularity conditions. Taking the conditional expectation of (2.13), we get

$$V(t) = \mathbb{E}_t\left[\exp\left\{-\int_t^T r_\gamma(u)\mathrm{d}u\right\} V(T)\right], \qquad 0 \le t \le T, \tag{2.14}$$

subject to regularity conditions, where $\mathbb{E}_t$ denotes the conditional expectation operator under $\mathbb{Q}$ given $\mathcal{F}_t$.

By substituting (2.9) into the SDE (2.1), we obtain

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = r_R^i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}W_i^*(t), \qquad i = 1, 2, \ldots, n, \tag{2.15}$$

under the martingale measure $\mathbb{Q}$. Hence, under $\mathbb{Q}$, the instantaneous rate of return of risky asset $S_i(t)$ is given by its repo rate $r_R^i(t)$. Again, by adopting the standard terminology in the finance literature, we call $\mathbb{Q}$ the *risk-neutral* probability measure.

Since the denominated price $V_\gamma^*(t) \equiv V(t)/B_\gamma(t)$ is a $\mathbb{Q}$-martingale from (2.14), it follows that

$$\frac{\mathrm{d}V(t)}{V(t)} = r_\gamma(t)\mathrm{d}t + \sigma_\gamma(t)\mathrm{d}W_\gamma^*(t)$$

for some volatility process $\sigma_\gamma(t)$, where $W_\gamma^*(t)$ denotes a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. Hence, in contrast to the underlying assets $S_i(t)$, the instantaneous rate of return of the derivative security is given by $r_\gamma(t)$; cf. Equation (2.15). This is the remarkable difference of the multi-curve setting from the standard single-curve world. In this paper, we call Equation (2.14) the *fundamental pricing formula*.

Summarizing, we have the following result.

Proposition 2.1. *Suppose that there are risky assets $S_i(t)$ and non-risky savings accounts, called collateral and funding accounts, whose short rates are given by $r_C(t)$ and $r_F(t)$, respectively. Then, there exists a pricing measure $\mathbb{Q}$, called the risk-neutral measure, such that* *the time-t price $V(t)$ of a derivative written on the risky assets is given by*

$$V(t) = \mathbb{E}_t\left[\exp\left\{-\int_t^T r_\gamma(u)\mathrm{d}u\right\} V(T)\right],$$

where $r_\gamma(t) = \gamma(t)r_C(t) + (1-\gamma(t))r_F(t)$ for some $\gamma(t)$, and where $\mathbb{E}_t$ denotes the conditional expectation operator under $\mathbb{Q}$. Under $\mathbb{Q}$, the derivative price $V(t)$ follows the SDE

$$\frac{\mathrm{d}V(t)}{V(t)} = r_\gamma(t)\mathrm{d}t + \sigma_\gamma(t)\mathrm{d}W_\gamma^*(t)$$

for some $\sigma_\gamma(t)$, where $W_\gamma^*(t)$ is a standard Brownian motion under $\mathbb{Q}$. On the other hand, the price of risky asset $S_i(t)$ follows the SDE

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = r_R^i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}W_i^*(t),$$

where $r_R^i(t)$ denotes the short rate of funding that is secured by the asset $S_i(t)$ and $W_i^*(t)$ is another standard Brownian motion under $\mathbb{Q}$.

Remark 2.1. In the multi-curve setting, we note that (i) the instantaneous rate of return of the underlying asset $S_i(t)$ is equal to its repo rate $r_R^i(t)$ under the risk-neutral measure $\mathbb{Q}$, (ii) the risk-neutral measure $\mathbb{Q}$ depends on the repo rates, but neither on the collateral rate $r_C(t)$ nor the funding rate $r_F(t)$, (iii) the derivative price $V(t)$ depends on the measure $\mathbb{Q}$ and $\gamma(t)$ (hence, both $r_C(t)$ and $r_F(t)$), and (iv) the numéraire which makes the denominated derivative price a $\mathbb{Q}$-martingale is $B_\gamma(t)$, which depends on the collateral rate $r_C(t)$, the funding rate $r_F(t)$, and the ratio $\gamma(t) = C(t)/V(t)$ of the collateral account to the derivative value.

Remark 2.2. Suppose that $r_R^i(t) = r_C(t)$, i.e., the repo rate of asset $S_i(t)$ is the same as the collateral rate. Then, we define $\theta_C'(t)B_C(t) = \theta_C(t)B_C(t) - R(t)$ in (2.4). In this case, we take the collateral account $B_C(t)$ as the numéraire so that, from (2.3), we have

$$V^*(t) = \sum_{i=1}^n \theta_i(t)S_i^*(t) + \theta_C'(t) + \theta_F(t)B_F^*(t),$$

where $V^*(t) = V(t)/B_C(t)$ and so on. Under the self-financing strategy, it follows that

$$\mathrm{d}V^*(t) = \theta_F(t)(r_F(t) - r_C(t))B_F^*(t)\mathrm{d}t + \sum_{i=1}^n \theta_i(t)\sigma_i(t)S_i^*(t)\mathrm{d}W_i^*(t),$$

where we define the market price of risk in (2.9) by $\lambda_i(t) = (\mu_i(t) - r_C(t))/\sigma_i(t)$ as in the ordinary single-curve setting. But, since $\theta_F(t)B_F(t) = (1-\gamma(t))V(t)$ by the definition of the ratio $\gamma(t)$, we obtain from (2.11) that

$$\mathrm{d}V^*(t) = (r_\gamma(t) - r_C(t))V^*(t)\mathrm{d}t + \sum_{i=1}^n \theta_i(t)\sigma_i(t)S_i^*(t)\mathrm{d}W_i^*(t).$$

Hence, the denominated process $V^*(t)$ with numéraire $B_C(t)$ is not a martingale under the risk-neutral measure $\mathbb{Q}$, unless $\gamma(t) = 1$, i.e., the perfect collateral case.

We now proceed to consider some special cases of the fundamental pricing formula (2.14).

Example 2.1 (Perfect Collateral). Suppose that $\gamma(u) = 1$, i.e., $C(u) = V(u)$, $t \le u \le T$. This case is known as the *perfect collateral*. In this case, from the pricing formula (2.14), we have

$$V(t) = \mathbb{E}_t\left[\exp\left\{-\int_t^T r_C(u)\mathrm{d}u\right\}V(T)\right] = \mathbb{E}_t\left[\frac{B_C(t)}{B_C(T)}V(T)\right], \tag{2.16}$$

which has been obtained by many authors.[4] Because $V^*(t) = V(t)/B_C(t)$ is a martingale under $\mathbb{Q}$ from (2.16), the value process $V(t)$ follows the SDE

$$\frac{\mathrm{d}V(t)}{V(t)} = r_C(t)\mathrm{d}t + \sigma_C(t)\mathrm{d}W_C^*(t) \tag{2.17}$$

for some volatility process $\sigma_C(t)$. On the other hand, the price of risky asset $S_i(t)$ follows the SDE (2.15), i.e.,

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = r_R^i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}W_i^*(t),$$

where $r_R^i(t)$ denotes the repo rate. Hence, only if *every* underlying asset $S_i(t)$ is perfectly healthy so that its repo rate $r_R^i(t)$ is equal to the collateral rate $r_C(t)$, then the instantaneous rate of return of $S_i(t)$ is given by $r_C(t)$ under $\mathbb{Q}$. In this case, we can recover the classic risk-neutral framework in the single-curve world.

Example 2.2 (No Collateral). On the other hand, if $\gamma(u) = C(u) = 0$, $t \le u \le T$, i.e., with no collateral, then the pricing formula (2.14) becomes

$$V(t) = \mathbb{E}_t\left[\exp\left\{-\int_t^T r_F(u)\mathrm{d}u\right\}V(T)\right] = \mathbb{E}_t\left[\frac{B_F(t)}{B_F(T)}V(T)\right], \tag{2.18}$$

and the value process $V(t)$ follows the SDE

$$\frac{\mathrm{d}V(t)}{V(t)} = r_F(t)\mathrm{d}t + \sigma_F(t)\mathrm{d}W_F^*(t) \tag{2.19}$$

for some volatility process $\sigma_F(t)$. Moreover, only if *every* underlying asset $S_i(t)$ is unhealthy so that its repo rate $r_R^i(t)$ is equal to the funding rate $r_F(t)$, then the instantaneous rate of return of $S_i(t)$ is given by $r_F(t)$ under $\mathbb{Q}$. Again, this case is reduced to the classic risk-neutral world under the single-curve setting.

3. Interest-rate Derivatives under Multiple Curves

In this section, we consider the pricing of interest-rate derivatives under the multi-curve setting. It is assumed throughout that the market considered in the previous section is rich enough so as to take the same risk-neutral measure $\mathbb{Q}$ even in this section.

3.1. **OIS Discount Bond.** Suppose that, under the physical measure $\mathbb{P}$, the price dynamics of the collateralized discount bond maturing at time T_i follows the SDE

$$\frac{\mathrm{d}D(t,T_i)}{D(t,T_i)} = \mu_D(t,T_i)\mathrm{d}t + \sigma_D(t,T_i)\mathrm{d}W_i(t), \tag{3.1}$$

where $D(T_i,T_i) = 1$ for all i, and where $W_i(t)$ denote correlated standard Brownian motions under $\mathbb{P}$. The collateralized discount bond $D(t,T)$ is usually called the OIS discount bond and considered to be perfectly secured.

[4] See, e.g., Piterbarg (2010) who applies the Feynman-Kac formula to derive (2.16).

Let $r_C(t)$ be the collateral rate, and suppose that the OIS discount bond $D(t,T)$ is duplicated by the instruments $B_C(t)$ and $D(t,T_i)$ with different (finitely many) maturities T_i. That is, as in the framework of Heath, Jarrow and Morton (1992), the OIS discount bonds $D(t,T_i)$, $T_i > T$, are used as the underlying securities. Because $D(T,T) = 1$, it then follows from (2.16) that

$$D(t,T) = \mathbb{E}_t\left[\exp\left\{-\int_t^T r_C(u)\mathrm{d}u\right\}\right] = \mathbb{E}_t\left[\frac{B_C(t)}{B_C(T)}\right], \qquad t < T, \tag{3.2}$$

which is known as the "OIS discounting" in practice. Note that the denominated price process $D(t,T)/B_C(t)$ is a martingale under $\mathbb{Q}$, and so we have

$$\frac{\mathrm{d}D(t,T)}{D(t,T)} = r_C(t)\mathrm{d}t + \sigma_D(t,T)\mathrm{d}W^*(t),$$

as the *derivative* security; see (2.17)

It is plausible to assume that the repo rate for the OIS discount bonds is equal to the collateral rate $r_C(t)$, because they are perfectly secured. Hence, under the risk-neutral measure $\mathbb{Q}$, we have from (2.15) that

$$\frac{\mathrm{d}D(t,T_i)}{D(t,T_i)} = r_C(t)\mathrm{d}t + \sigma_D(t,T_i)\mathrm{d}W_i^*(t) \tag{3.3}$$

for each T_i, as the *underlying* securities. The market price of risk that changes the SDE (3.1) under $\mathbb{P}$ to the SDE (3.3) under $\mathbb{Q}$ is defined by (2.9) in an obvious manner. Hence, the OIS discount bonds alone are treated as if they were in the single-curve world.

3.2. **LIBOR Discount Bond.** Consider the LIBOR rate with tenor τ_k, $k = 1, 2, \ldots, m$, and denote the time-t price of the associated LIBOR discount bond maturing at time T_j by $L_k(t,T_j)$. Suppose that its price dynamics follows the SDE

$$\frac{\mathrm{d}L_k(t,T_j)}{L_k(t,T_j)} = \mu_L^k(t,T_j)\mathrm{d}t + \sigma_L^k(t,T_j)\mathrm{d}W_j^k(t),$$

where $L_k(T_j,T_j) = 1$ for all j, and where $W_j^k(t)$ denote correlated standard Brownian motions under $\mathbb{P}$.

It is widely believed by practitioners that LIBOR discount bonds are unsecured. Suppose that the funding rate and repo rate for the LIBOR discount bonds are the same and given by $r_F^k(t)$. If the LIBOR discount bond $L_k(t,T)$ can be duplicated by other LIBOR discount bonds with different (finitely many) maturities but with the same tenor τ_k, then we have from (2.18) that

$$L_k(t,T) = \mathbb{E}_t\left[\exp\left\{-\int_t^T r_F^k(u)\mathrm{d}u\right\}\right] = \mathbb{E}_t\left[\frac{B_F^k(t)}{B_F^k(T)}\right], \qquad t < T, \tag{3.4}$$

which is known as the "LIBOR discounting" in practice, where $B_F^k(t)$ is the funding account associated with the funding rate $r_F^k(t)$. Hence, the denominated price process $L_k(t,T)/B_F^k(t)$ is a martingale under $\mathbb{Q}$, and so we have

$$\frac{\mathrm{d}L_k(t,T)}{L_k(t,T)} = r_F^k(t)\mathrm{d}t + \sigma_L^k(t,T)\mathrm{d}W^{k*}(t),$$

as the *derivative* security; see (2.19).

Also, under the risk-neutral measure $\mathbb{Q}$, we have from (2.15) that

$$\frac{\mathrm{d}L_k(t,T_j)}{L_k(t,T_j)} = r_F^k(t)\mathrm{d}t + \sigma_L^k(t,T_j)\mathrm{d}W_j^{k*}(t) \tag{3.5}$$

for each T_j, as the *underlying* securities, because the repo rate of $L_k(t,T_j)$ is equal to the funding rate $r_F^k(t)$. Hence, the LIBOR discount bonds with the same tenor alone are treated as if they were in the single-curve world.

We have thus proved the following, which shows the simultaneous existence of multiple yield curves in the market.

Proposition 3.1. *Let $D(t,T_i)$ and $L_k(t,T_j)$ be the OIS and LIBOR discount bonds defined above. Suppose that the collateral and repo rates of the OIS discount bonds $D(t,T_i)$ are the same and given by $r_C(t)$. Further, suppose that the funding and repo rates of the LIBOR discount bonds $L_k(t,T_j)$ are the same and given by $r_F^k(t)$ for each tenor τ_k. If the market is rich enough, then there is a unique risk-neutral measure $\mathbb{Q}$ under which the denominated price processes $D(t,T_i)/B_C(t)$ and $L_k(t,T_j)/B_F^k(t)$ are martingales simultaneously, where $B_C(t)$ and $B_F^k(t)$ are associated savings accounts defined above.*

Remark 3.1. While the process $L_k(t,T)/B_F^k(t)$ is a martingale under $\mathbb{Q}$ for each k, the process $L_k(t,T)/B(t)$ denominated by the *other* account $B(t)$ is no longer a martingale under the risk-neutral measure $\mathbb{Q}$ in the multi-curve setting.

In the rest of this paper, for the sake of notational simplicity, we treat only the case $m=1$ and simply call $L(t,T)$ the LIBOR discount bond maturing at time T. Extension to the general case is straightforward.

3.3. **Forward-Neutral Method.** We have already seen that the derivative price process follows the SDE

$$\frac{\mathrm{d}V(t)}{V(t)} = r_\gamma(t)\mathrm{d}t + \sigma_\gamma(t,T)\mathrm{d}W_\gamma^*(t) \tag{3.6}$$

for some volatility $\sigma_\gamma(t,T)$ under the risk-neutral measure $\mathbb{Q}$. Similarly, from (3.3) and (3.5), the price processes of OIS and LIBOR discount bonds can be written by

$$\frac{\mathrm{d}D(t,T)}{D(t,T)} = r_C(t)\mathrm{d}t + \sigma_D(t,T)\mathrm{d}W_D^*(t), \qquad t \le T, \tag{3.7}$$

and

$$\frac{\mathrm{d}L(t,T)}{L(t,T)} = r_F(t)\mathrm{d}t + \sigma_L(t,T)\mathrm{d}W_L^*(t), \qquad t \le T, \tag{3.8}$$

for some $\sigma_D(t,T)$ and $\sigma_L(t,T)$, respectively, where $W_D^*(t)$ and $W_L^*(t)$ are standard Brownian motions under $\mathbb{Q}$ with correlations $\mathrm{d}W_\gamma^*(t)\mathrm{d}W_M^*(t) = \rho_{\gamma,M}(t)\mathrm{d}t$ for $M=D,L$.

Let us denote the OIS and LIBOR forward prices by $V_T^D(t) = V(t)/D(t,T)$ and $V_T^L(t) = V(t)/L(t,T)$, respectively. By applying Ito's division rule to (3.6) and (3.7), the OIS forward price is given by

$$\begin{aligned} \frac{\mathrm{d}V_T^D(t)}{V_T^D(t)} &= [r_\gamma(t) - r_C(t) - \sigma_D(t,T)\left(\rho_{\gamma,D}(t)\sigma_\gamma(t,T) - \sigma_D(t,T)\right)]\,\mathrm{d}t \\ &\quad + \sigma_\gamma(t,T)\mathrm{d}W_\gamma^*(t) - \sigma_D(t,T)\mathrm{d}W_D^*(t). \end{aligned} \tag{3.9}$$

Now, define the processes $W_{\gamma,D}^T(t)$ and $W_D^T(t)$ by

$$\mathrm{d}W_{\gamma,D}^T(t) = \mathrm{d}W_\gamma^*(t) - \rho_{\gamma,D}(t)\sigma_D(t,T)\mathrm{d}t, \quad \mathrm{d}W_D^T(t) = \mathrm{d}W_D^*(t) - \sigma_D(t,T)\mathrm{d}t, \tag{3.10}$$

respectively. By substituting (3.10) into (3.9), we obtain

$$\frac{\mathrm{d}V_T^D(t)}{V_T^D(t)} = (r_\gamma(t) - r_C(t))\,\mathrm{d}t + \sigma_\gamma(t,T)\mathrm{d}W_{\gamma,D}^T(t) - \sigma_D(t,T)\mathrm{d}W_D^T(t). \tag{3.11}$$

Let $\mathbb{Q}_{\gamma,D}^T$ be a probability measure that makes the processes $W_{\gamma,D}^T(t)$ and $W_D^T(t)$ standard Brownian motions. The existence of such $\mathbb{Q}_{\gamma,D}^T$ is guaranteed by Girsanov's theorem; however, the OIS-forward price $V_T^D(t)$ cannot be a $\mathbb{Q}_{\gamma,D}^T$-martingale, since the drift term in (3.11) is not eliminated except the case that $\gamma(t) = 1$. This is a remarkable difference from the ordinary forward-neutral method under the single-curve setting. Note also that the measure $\mathbb{Q}_{\gamma,D}^T$ depends on $\gamma(t)$, in contrast to the risk-neutral measure $\mathbb{Q}$.

Next, we define the spread by

$$y_\gamma(t) \equiv r_\gamma(t) - r_C(t) = (1-\gamma(t))(r_F(t) - r_C(t)) \tag{3.12}$$

and the associated savings account by

$$\frac{\mathrm{d}B_{Y,\gamma}(t)}{B_{Y,\gamma}(t)} = y_\gamma(t)\mathrm{d}t, \qquad B_{Y,\gamma}(0) = 1.$$

Note that we have $y_1(t) = 0$ and $B_{Y,\gamma}(t) = 1$ for the perfect collateral case ($\gamma(t) = 1$). It follows from (3.11) that

$$\frac{\mathrm{d}\left(V_T^D(t)/B_{Y,\gamma}(t)\right)}{V_T^D(t)/B_{Y,\gamma}(t)} = \sigma_\gamma(t,T)\mathrm{d}W_{\gamma,D}^T(t) - \sigma_D(t,T)\mathrm{d}W_D^T(t). \tag{3.13}$$

Since the process $V_T^D(t)/B_{Y,\gamma}(t)$ is a $\mathbb{Q}_{\gamma,D}^T$-martingale from (3.13), we can get the following expression:

$$V_T^D(t) = \mathbb{E}_t^{\gamma,D^T}\left[\frac{B_{Y,\gamma}(t)}{B_{Y,\gamma}(T)}V_T^D(T)\right] = \mathbb{E}_t^{\gamma,D^T}\left[\frac{B_{Y,\gamma}(t)}{B_{Y,\gamma}(T)}V(T)\right],$$

where E_t^{γ,D^T} is the conditional expectation operator under $\mathbb{Q}_{\gamma,D}^T$ given the information $\mathcal{F}_t$. Therefore, we obtain

$$V(t) = D(t,T)\mathbb{E}_t^{\gamma,D^T}\left[\exp\left\{-\int_t^T (1-\gamma(s))(r_F(s) - r_C(s))\mathrm{d}s\right\}V(T)\right]. \tag{3.14}$$

Although not exactly the same, we call the measure $\mathbb{Q}_{\gamma,D}^T$ the T-OIS-forward measure because of the resemblance (see (3.15) and (3.16) below) to the ordinary forward measure (see, e.g., Kijima, 2013).

In the case of perfect collateral ($\gamma(t) = 1$), the OIS-forward price $V_T^D(t) = V(t)/D(t,T)$ is a martingale under the T-OIS-forward measure $\mathbb{Q}_{1,D}^T$. It follows from the pricing formula (3.14) that

$$V(t) = D(t,T)\mathbb{E}_t^{1,D^T}[V(T)], \tag{3.15}$$

which is equivalent to the forward-neutral pricing formula in the single-curve setting. On the other hand, in the non-collateral case ($\gamma(u) = 0$), the pricing formula (3.14) becomes

$$V(t) = D(t,T)\mathbb{E}_t^{0,D^T}\left[\exp\left\{-\int_t^T (r_F(s) - r_C(s))\,\mathrm{d}s\right\}V(T)\right], \tag{3.16}$$

which is different from the ordinary pricing formula derived in the previous literature unless $r_F(t) = r_C(t)$.

Similarly, the LIBOR-forward price is given by

$$\begin{aligned}\frac{\mathrm{d}V_T^L(t)}{V_T^L(t)} &= [r_\gamma(t) - r_F(t) - \sigma_L(t,T)\left(\rho_{\gamma,L}(t)\sigma_\gamma(t,T) - \sigma_L(t,T)\right)]\,\mathrm{d}t \\ &\quad + \sigma_\gamma(t,T)\mathrm{d}W_\gamma^*(t) - \sigma_L(t,T)\mathrm{d}W_L^*(t). \qquad (3.17)\end{aligned}$$

Define the processes $W_{\gamma,L}^T(t)$ and $W_L^T(t)$ by

$$\mathrm{d}W_{\gamma,L}^T(t) = \mathrm{d}W_\gamma^*(t) - \rho_{\gamma,L}(t)\sigma_L(t,T)\mathrm{d}t, \quad \mathrm{d}W_L^T(t) = \mathrm{d}W_L^*(t) - \sigma_L(t,T)\mathrm{d}t, \qquad (3.18)$$

respectively. By substituting (3.18) into (3.17), we obtain

$$\frac{\mathrm{d}V_T^L(t)}{V_T^L(t)} = (r_\gamma(t) - r_F(t))\,\mathrm{d}t + \sigma_\gamma(t,T)\mathrm{d}W_{\gamma,L}^T(t) - \sigma_L(t,T)\mathrm{d}W_L^T(t). \qquad (3.19)$$

Let $\mathbb{Q}_{\gamma,L}^T$ be a probability measure that makes the processes $W_{\gamma,L}^T(t)$ and $W_L^T(t)$ standard Brownian motions. We call $\mathbb{Q}_{\gamma,L}^T$ the T-LIBOR-forward measure. Again, $V_T^L(t)$ cannot be a $\mathbb{Q}_{\gamma,L}^T$-martingale, since the drift term in (3.19) is not eliminated except the case that $\gamma(t) = 0$.

Under the T-LIBOR-forward measure $\mathbb{Q}_{\gamma,L}^T$, similar to (3.14), we obtain

$$V(t) = L(t,T)\mathbb{E}_t^{\gamma,L^T}\left[\exp\left\{\int_t^T \gamma(s)(r_F(s) - r_C(s))\mathrm{d}s\right\} V(T)\right]. \qquad (3.20)$$

In the no-collateral case, the pricing formula (3.20) becomes

$$V(t) = L(t,T)\mathbb{E}_t^{0,L^T}\left[V(T)\right], \qquad (3.21)$$

and the LIBOR-forward price $V_T^L(t) = V(t)/L(t,T)$ is a martingale under the T-LIBOR-forward measure $\mathbb{Q}_{0,L}^T$, which is equivalent to the forward-neutral pricing formula in the single-curve setting. However, the forward measure used in (3.21) is different from that in (3.15). The perfect collateral case can be obtained similarly and is given by

$$V(t) = L(t,T)E_t^{1,L^T}\left[\exp\left\{\int_t^T (r_F(s) - r_C(s))\,\mathrm{d}s\right\} V(T)\right]. \qquad (3.22)$$

Again, note the difference of the measures used in (3.16) and (3.22).

4. Forward-rate agreement and interest-rate swap

In this section, we discuss the pricing of forward-rate agreements and interest-rate swaps based on our pricing formulas obtained in the previous section.

Suppose that the OIS and LIBOR discount bonds, denoted by $D(t,T)$ and $L(t,T)$ respectively, are traded in the market, and consider the payment date $0 \le T_0 < T_1 < \cdots < T_N$ with $\delta_i = T_{i+1} - T_i$.

The T_i-forward LIBOR rate at time t is defined by

$$L_i(t) \equiv \frac{L(t,T_i) - L(t,T_{i+1})}{\delta_i L(t,T_{i+1})}, \qquad 0 \le t \le T_i. \qquad (4.1)$$

The LIBOR rate at time T_i is thus equal to $L_i(T_i)$ with $L(T_i,T_i) = 1$.

Similarly, the T_i-forward OIS rate at time t is defined by

$$D_i(t) \equiv \frac{D(t,T_i) - D(t,T_{i+1})}{\delta_i D(t,T_{i+1})}, \qquad 0 \le t \le T_i. \qquad (4.2)$$

The OIS rate at time T_i is given by $D_i(T_i)$ with $D(T_i,T_i) = 1$.

4.1. **Forward-Rate Agreement.** We begin by the definitions of forward-rate agreement (FRA) and FRA rate.

Definition 4.1 (FRA)**.** A forward-rate agreement (FRA) is a contract which allows the holder to lock in at $t < T_i$ the interest rate between $[T_i, T_{i+1}]$ at a fixed value K. At the maturity T_{i+1}, one pays the cash based on the rate K, and another pays the cash based on the LIBOR rate $L_i(T_i)$.

Definition 4.2 (FRA rate)**.** The FRA rate is defined as the rate where the present value of the FRA is equal to zero.

4.1.1. *Perfect collateral case.* In the perfect collateral case ($\gamma(t) = 1$), it follows from (2.16) that the no-arbitrage price of the FRA with maturity T_{i+1} and exercise rate K is given by

$$V_{\mathrm{FRA}}(t; T_i, T_{i+1}, K) = \delta_i \mathbb{E}_t \left[\exp\left\{ -\int_t^{T_{i+1}} r_C(u)\mathrm{d}u \right\} (L_i(T_i) - K) \right].$$

Setting $V_{\mathrm{FRA}}(t; T_i, T_{i+1}, K) = 0$ and then solving it with respect to K yields the FRA rate at time t as

$$\text{FRA}(t; T_i, T_{i+1}) = \frac{\mathbb{E}_t \left[\exp\left\{ -\int_t^{T_{i+1}} r_C(u)\mathrm{d}u \right\} L_i(T_i) \right]}{D(t, T_{i+1})}, \tag{4.3}$$

where we have used the result (3.2) for the denominator.

On the other hand, under the T_{i+1}-OIS-forward measure $\mathbb{Q}_{1,D}^{T_{i+1}}$, we have from (3.15) that

$$V_{\mathrm{FRA}}(t; T_i, T_{i+1}, K) = \delta_i D(t, T_{i+1}) \mathbb{E}_t^{1,D^{T_{i+1}}} [L_i(T_i) - K].$$

The FRA rate is thus obtained as

$$\text{FRA}(t; T_i, T_{i+1}) = \mathbb{E}_t^{1,D^{T_{i+1}}} [L_i(T_i)], \tag{4.4}$$

which has been treated as a definition or an assumption of the FRA rate by many authors such as Bianchetti (2013). Note that $L_i(t)$ is not a $\mathbb{Q}_{1,D}^{T_{i+1}}$-martingale so that the equation $\text{FRA}(t; T_i, T_{i+1}) = L_i(t)$ fails.

Similarly, under the T_{i+1}-LIBOR-forward measure $\mathbb{Q}_{1,L}^{T_{i+1}}$, we have from (3.22) that

$$V_{\mathrm{FRA}}(t; T_i, T_{i+1}, K) = \delta_i L(t, T_{i+1}) \mathbb{E}_t^{1,L^{T_{i+1}}} \left[\exp\left\{ \int_t^{T_{i+1}} y_0(s)\mathrm{d}s \right\} (L_i(T_i) - K) \right],$$

where $y_0(t) = r_F(t) - r_C(t)$ denotes the spread defined in (3.12) with $\gamma(t) = 0$. The FRA rate is thus obtained as

$$\text{FRA}(t; T_i, T_{i+1}) = \frac{\mathbb{E}_t^{1,L^{T_{i+1}}} \left[\exp\left\{ \int_t^{T_{i+1}} y_0(s)\mathrm{d}s \right\} L_i(T_i) \right]}{\mathbb{E}_t^{1,L^{T_{i+1}}} \left[\exp\left\{ \int_t^{T_{i+1}} y_0(s)\mathrm{d}s \right\} \right]}. \tag{4.5}$$

Note the difference of the measures used in (4.4) and (4.5).

4.1.2. *No-collateral case.* In the no-collateral case ($\gamma(t) = 0$), it follows from (2.18) that the no-arbitrage price of the FRA with maturity T_{i+1} and exercise rate K is given by

$$V_{\mathrm{FRA}}(t; T_i, T_{i+1}, K) = \delta_i \mathbb{E}_t \left[\exp\left\{ -\int_t^{T_{i+1}} r_F(u)\mathrm{d}u \right\} (L_i(T_i) - K) \right].$$

Solving $V_{\mathrm{FRA}}(t;T_i,T_{i+1},K)=0$ with respect to K yields the FRA rate at time t as

$$\mathrm{FRA}(t;T_i,T_{i+1})=\frac{\mathbb{E}_t\left[\exp\left\{-\int_t^{T_{i+1}} r_F(u)\mathrm{d}u\right\}L_i(T_i)\right]}{L(t,T_{i+1})}, \tag{4.6}$$

where we have used the result (3.4).

On the other hand, under the T_{i+1}-LIBOR-forward measure $\mathbb{Q}_{0,L}^{T_{i+1}}$, we have from (3.21) that

$$V_{\mathrm{FRA}}(t;T_i,T_{i+1},K)=\delta_i L(t,T_{i+1})\mathbb{E}_t^{0,L^{T_{i+1}}}\left[L_i(T_i)-K\right]$$

and the FRA rate is obtained as

$$\mathrm{FRA}(t;T_i,T_{i+1})=\mathbb{E}_t^{0,L^{T_{i+1}}}\left[L_i(T_i)\right]=L_i(t). \tag{4.7}$$

Recall that $L_i(t)$ is a martingale under $\mathbb{Q}_{0,L}^{T_{i+1}}$.

Similarly, under the T_{i+1}-OIS-forward measure $\mathbb{Q}_{0,D}^{T_{i+1}}$, we have from (3.16) that

$$V_{\mathrm{FRA}}(t;T_i,T_{i+1},K)=\delta_i D(t,T_{i+1})\mathbb{E}_t^{0,D^{T_{i+1}}}\left[\exp\left\{-\int_t^{T_{i+1}} y_0(s)\mathrm{d}s\right\}(L_i(T_i)-K)\right],$$

and the FRA rate is obtained as

$$\mathrm{FRA}(t;T_i,T_{i+1})=\frac{\mathbb{E}_t^{0,D^{T_{i+1}}}\left[\exp\left\{-\int_t^{T_{i+1}} y_0(s)\mathrm{d}s\right\}L_i(T_i)\right]}{\mathbb{E}_t^{0,D^{T_{i+1}}}\left[\exp\left\{-\int_t^{T_{i+1}} y_0(s)\mathrm{d}s\right\}\right]}. \tag{4.8}$$

Note the difference of the measures used in (4.7) and (4.8).

4.2. **Interest-Rate Swap.** Consider a plain-vanilla, interest-rate swap (IRS) which starts at time $T_0 \geq 0$. For simplicity, suppose that the notional amount is unity and cash flows are exchanged at date T_i, $i=1,\cdots,N$.

Definition 4.3 (Swap Contract)**.** In the LIBOR swap, party A pays to party B the interest $\delta_{i-1}K$ with fixed rate K at dates T_i, whereas party B pays to party A the interest $\delta_{i-1}L_{i-1}(T_{i-1})$ with LIBOR rate $L_{i-1}(T_{i-1})$. The OIS swap is the same when the interest is based on the OIS rate $D_{i-1}(T_{i-1})$.

Definition 4.4 (Swap Rate)**.** The swap rate is defined as the rate where the present value of the swap contract is equal to zero.

4.2.1. *Perfect collateral case.* In the perfect collateral case ($\gamma(t)=1$), it follows from (2.16) that the no-arbitrage price of the OIS swap contract with maturity T_N and fixed rate K is given by

$$\begin{aligned}V_{\mathrm{SWP}}^{O}(t,T_N,K) &= \sum_{i=1}^{N}\delta_{i-1}\mathbb{E}_t\left[\exp\left\{-\int_t^{T_i} r_C(u)\mathrm{d}u\right\}D_{i-1}(T_{i-1})\right] \\ &\quad - K\sum_{i=1}^{N}\delta_{i-1}\mathbb{E}_t\left[\exp\left\{-\int_t^{T_i} r_C(u)\mathrm{d}u\right\}\right] \\ &= \sum_{i=1}^{N}\delta_{i-1}D(t,T_i)\mathbb{E}_t^{1,D^{T_i}}\left[D_{i-1}(T_{i-1})\right]-K\sum_{i=1}^{N}\delta_{i-1}D(t,T_i), \end{aligned} \tag{4.9}$$

where we have used the results (3.2) and (3.15). But, the T_i-OIS forward rate $D_{i-1}(t)$ defined in (4.2) is a $\mathbb{Q}^{T_i}_{1,D}$-martingale, so that we have

$$\delta_{i-1}D(t,T_i)\mathbb{E}_t^{1,D^{T_i}}\left[D_{i-1}(T_{i-1})\right] = \delta_{i-1}D(t,T_i)D_{i-1}(t) = D(t,T_{i-1}) - D(t,T_i).$$

It follows from (4.9) that

$$V^O_{\mathrm{SWP}}(t,T_N,K) = D(t,T_0) - D(t,T_N) - K\sum_{i=1}^N \delta_{i-1}D(t,T_i),$$

and the OIS swap rate, $S^O(t,T_N)$, at time t is obtained as

$$S^O(t,T_N) = \frac{D(t,T_0) - D(t,T_N)}{\sum_{i=1}^N \delta_{i-1}D(t,T_i)}, \tag{4.10}$$

which is the well-known *telescope* formula for the OIS swap rate.

On the other hand, from (2.16) again, the no-arbitrage price of the LIBOR swap contract with maturity T_N and fixed rate K is given by

$$\begin{aligned} V^L_{\mathrm{SWP}}(t,T_N,K) &= \sum_{i=1}^N \delta_{i-1}\mathbb{E}_t\left[\exp\left\{-\int_t^{T_i} r_C(u)\mathrm{d}u\right\} L_{i-1}(T_{i-1})\right] \\ &\quad - K\sum_{i=1}^N \delta_{i-1}\mathbb{E}_t\left[\exp\left\{-\int_t^{T_i} r_C(u)\mathrm{d}u\right\}\right]. \end{aligned}$$

Setting $V^L_{\mathrm{SWP}}(t,T_N,K) = 0$ and then solving it with respect to K yields the LIBOR swap rate at time t as

$$S^L(t,T_N) = \frac{\sum_{i=1}^N \delta_{i-1}\mathrm{FRA}(t;T_{i-1},T_i)D(t,T_i)}{\sum_{i=1}^N \delta_{i-1}D(t,T_i)}, \tag{4.11}$$

where we have used the results (3.2) and (4.3). The formula (4.11) has been obtained by many authors including Bianchetti (2013). Note that the swap rate given in (4.11) cannot be reduced to the telescope formula other than the case that $\mathrm{FRA}(t;T_{i-1},T_i) = D_{i-1}(t)$.

4.2.2. *No-collateral case.* In the no-collateral case ($\gamma(t) = 0$), we consider LIBOR swap contracts only. It then follows from (2.18) that

$$\begin{aligned} V^L_{\mathrm{SWP}}(t,T_N,K) &= \sum_{i=1}^N \delta_{i-1}\mathbb{E}_t\left[\exp\left\{-\int_t^{T_i} r_F(u)\mathrm{d}u\right\} L_{i-1}(T_{i-1})\right] \\ &\quad - K\sum_{i=1}^N \delta_{i-1}\mathbb{E}_t\left[\exp\left\{-\int_t^{T_i} r_F(u)\mathrm{d}u\right\}\right], \end{aligned}$$

and thus the LIBOR swap rate is given by

$$S^L(t,T_N) = \frac{\sum_{i=1}^N \delta_{i-1}\mathrm{FRA}(t;T_{i-1},T_i)L(t,T_i)}{\sum_{i=1}^N \delta_{i-1}L(t,T_i)},$$

where we have used the results (3.4) and (4.6). However, in this case, we have from (4.7) and (4.1) that

$$\delta_{i-1}L(t,T_i)\mathrm{FRA}(t;T_{i-1},T_i) = \delta_{i-1}L(t,T_i)\frac{L(t,T_{i-1}) - L(t,T_i)}{\delta_{i-1}L(t,T_i)} = L(t,T_{i-1}) - L(t,T_i).$$

It follows that

$$S^L(t,T_N) = \frac{L(t,T_0) - L(t,T_N)}{\sum_{i=1}^{N} \delta_{i-1} L(t,T_i)}, \tag{4.12}$$

which is again the classic telescope formula under the single-curve setting.

4.2.3. *Telescope property.* We have seen that, in the multi-curve setting, the LIBOR swap-rate (4.11) does not display the so-called telescope formula in the perfect collateral case. In this subsection, we show that the telescope property can be recovered by using some adjusted discount curves. See Ogawa (2015) for a different derivation of the same result.

Let us define the swap spread $B_n(t)$ and annuity $A_n(t)$ by

$$B_n(t) = S^L(t,T_n) - S^O(t,T_n), \qquad A_n(t) = \sum_{i=1}^{n} \delta_{i-1} D(t,T_i), \tag{4.13}$$

respectively, where $0 \le t \le T_0$ and $B_0(t) = 0$. Furthermore, we define the *adjusted* discount curve by

$$D^*(t,T_n) \equiv D(t,T_n) - B_n(t)A_n(t), \qquad 0 \le t \le T_0. \tag{4.14}$$

It is readily seen that

$$\begin{aligned} D^*(t,T_0) - D^*(t,T_n) &= D(t,T_0) - D(t,T_n) + B_n(t)A_n(t) \\ &= S^O(t,T_n)A_n(t) + (S^L(t,T_n) - S^O(t,T_n))A_n(t) \\ &= S^L(t,T_n)A_n(t). \end{aligned}$$

It follows that

$$S^L(t,T_n) = \frac{D^*(t,T_0) - D^*(t,T_n)}{\sum_{i=1}^{n} \delta_{i-1} D(t,T_i)}, \qquad 0 \le t \le T_0. \tag{4.15}$$

Hence, the LIBOR swap rate $S^L(t,T_n)$ displays the telescope property in terms of the adjusted discount curves.

We note that (4.15) suggests the identity

$$\delta_{n-1}\mathrm{FRA}(t;T_{n-1},T_n)D(t,T_n) = D^*(t,T_{n-1}) - D^*(t,T_n). \tag{4.16}$$

Hence, in the perfect collateral case, the FRA rate is given by

$$\mathrm{FRA}(t;T_{n-1},T_n) = \frac{D^*(t,T_{n-1}) - D^*(t,T_n)}{\delta_{n-1} D(t,T_n)}, \tag{4.17}$$

which should be compared with (4.11) and (4.12). The quantities involved in (4.17) are all observed in the market; hence, the formula (4.17) is extremely useful in practice.

To see that (4.16) indeed holds true, we first note that

$$S^O(t,T_{n+1})A_{n+1}(t) - S^O(t,T_n)A_n(t) = D(t,T_n) - D(t,T_{n+1}),$$

where we have used (4.10). Also, for the LIBOR swap, we have from (4.11) that

$$S^L(t,T_{n+1})A_{n+1}(t) - S^L(t,T_n)A_n(t) = \delta_n D(t,T_{n+1})\mathrm{FRA}(t;T_n,T_{n+1}).$$

It follows that

$$\begin{aligned}
& D^*(t,T_n) - D^*(t,T_{n+1}) \\
= \;& D(t,T_n) - D(t,T_{n+1}) - B_n(t)A_n(t) + B_{n+1}(t)A_{n+1}(t) \\
= \;& D(t,T_n) - D(t,T_{n+1}) - (S^O(t,T_{n+1})A_{n+1}(t) - S^O(t,T_n)A_n(t)) \\
& \quad + (S^L(t,T_{n+1})A_{n+1}(t) - S^L(t,T_n)A_n(t)) \\
= \;& D(t,T_n) - D(t,T_{n+1}) - D(t,T_n) + D(t,T_{n+1}) + \delta_n D(t,T_{n+1})\mathrm{FRA}(t;T_n,T_{n+1}),
\end{aligned}$$

which proves (4.16).

5. Some Specific Models

This section provides some specific models under the multi-curve setting. In order to keep our presentation as simple as possible, we consider the *perfect* collateral case only. More general cases including the no-collateral case follow similarly. In this section, we start with a simple extension of the Black–Scholes model (1973) and then move to spot rate models. Forward rate models such as HJM and BGM models are treated in a separate paper.

5.1. **Equity Options.** In the framework of Section 2, consider a European derivative written on a risky asset $S(t)$ whose repo rate is given by $r_R(t)$. Then, from (2.14), the time-t price of the derivative is obtained as

$$V(t) = \mathbb{E}_t\left[\exp\left\{-\int_t^T r_C(u)\mathrm{d}u\right\} h(S(T))\right], \qquad 0 \le t \le T,$$

where the underlying risky asset follows the SDE

$$\frac{\mathrm{d}S(t)}{S(t)} = r_R(t)\mathrm{d}t + \sigma(t)\mathrm{d}W^*(t)$$

under the risk-neutral measure $\mathbb{Q}$. Here, $h(S)$ denotes the payoff function of the derivative and $W^*(t)$ is a standard Brownian motion under $\mathbb{Q}$.

Suppose that the interest rates are positive constants with $r_C \le r_R$. Then, for example, a call option price at time 0 with strike K and maturity T is given by

$$V(0) = \mathrm{e}^{-r_C T}\mathbb{E}_t\left[(S(T)-K)_+\right],$$

where $(X)_+ = \max(X,0)$, $S(T) = S\mathrm{e}^{\nu T + \sigma W^*(T)}$ with $S = S(0)$, and $\nu = r_R - \sigma^2/2$. It follows that

$$V(0) = \mathrm{e}^{(r_R - r_C)T}\mathrm{BS}(S,K,T;r_R,\sigma), \tag{5.1}$$

where $\mathrm{BS}(S,K,T;r,\sigma)$ stands for the Black–Scholes call option price with risk-free spot rate r and volatility σ.

We note that the call option price (5.1) can be very different from the classic Black–Scholes price $\mathrm{BS}(S,K,T;r_C,\sigma)$ when $\Delta_r = r_R - r_C$ is large. Figure 1 shows the discrepancy of the two prices

$$\Delta_V = \mathrm{e}^{(r_R - r_C)T}\mathrm{BS}(S,K,T;r_R,\sigma) - \mathrm{BS}(S,K,T;r_C,\sigma)$$

with respect to Δ_r, where we set $S = 100$ and $r_C = 1\%$. The difference Δ_V increases as Δ_r (and also the maturity T) increases. Also, the discrepancy becomes more significant for the ITM options than the OTM options. However, the impact of volatility σ on option prices seems negligible.

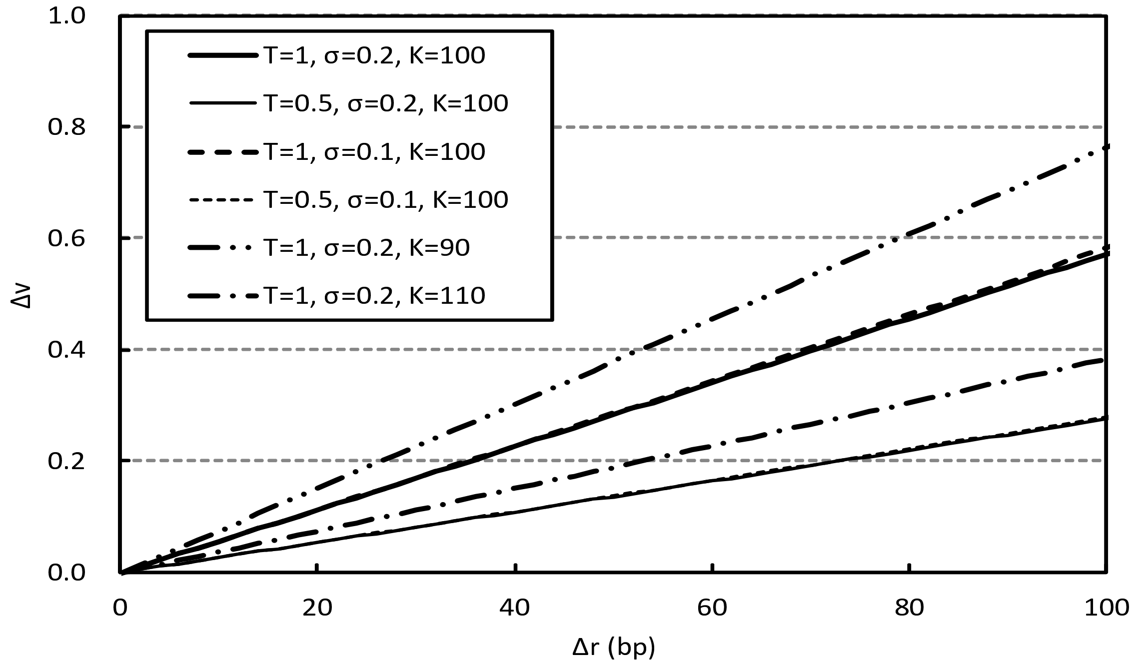


Figure 1: Difference Δ_V of call option price from the classic Black–Scholes price with respect to $r_R - r_C$: The parameters are chosen as $S = 100$ and $r_C = 1\%$.

5.2. **Spot Rate Models.** In the multi-curve setting, several spot rate models have been proposed in the finance literature, Among them, Kijima, Tanaka and Wong (2009) consider a spot rate model in which the collateral rate $r_C(t)$ is modeled by the quadratic Gaussian (QG) model of Pelsser (1997) and the spread $s(t) = r_F(t) - r_C(t)$ follows a Vasicek model (1977). On the other hand, Morino and Runggaldier (2014) consider an (independent) affine factor model $\Phi_i(t)$ in which the collateral rate and the spread are given by $r_C(t) = \Phi_2(t) - \Phi_1(t)$ and $s(t) = \kappa\Phi_1(t) + \Phi_3(t)$, respectively. While Kijima, Tanaka and Wong (2009) assume that $r_C(t)$ and $s(t)$ are mutually independent, Morino and Runggaldier (2014) introduce a possible (negative) correlation between them through the parameter $\kappa > 0$. This section introduces another spot rate model in the framework of Section 2.

Suppose that the two latent factors $x(t)$ and $y(t)$ follow mean-reverting processes

$$\mathrm{d}x(t) = -c_x x(t)\mathrm{d}t + \sigma_x \mathrm{d}W_1^*(t), \qquad \mathrm{d}y(t) = c_y(m_y - y(t))\mathrm{d}t + \sigma_y \mathrm{d}W_2^*(t), \tag{5.2}$$

respectively, where $W_1^*(t)$ and $W_2^*(t)$ are independent standard Brownian motions under $\mathbb{Q}$. We define the spot rates $r_F(t)$ and $r_C(t)$ as follows. Let $s(t)$ be the spread between $r_C(t)$ and $r_F(t)$, and suppose that

$$r_C(t) = y(t) + \kappa x(t), \qquad s(t) = (\ell + x(t))^2. \tag{5.3}$$

Hence, while the OIS spot rate $r_C(t)$ can be negative with positive probability, the spread $s(t)$ is always kept to be non-negative and $r_F(t) = r_C(t) + s(t) \geq r_C(t)$ as desired.[5] Note that the

[5]While negative interest rates are often observed in the recent interest-rate market, the spread should be non-negative by the definition.

parameter κ represents the dependency between $r_C(t)$ and $s(t)$. In fact, from (5.2) and (5.3), we have

$$\mathrm{d}r_C(t)\mathrm{d}s(t) = (\mathrm{d}y(t) + \kappa\mathrm{d}x(t))\mathrm{d}s(t) = 2\kappa\sigma_x^2(\ell + x(t))\mathrm{d}t. \tag{5.4}$$

Hence, if $\kappa < 0$, the OIS rate and the spread are negatively correlated as long as $\ell + x(t) > 0$. Also, denoting the volatility of variable $z(t)$ by $\sigma[z]$, we have

$$\sigma[r_C] = \sqrt{\sigma_y^2 + \kappa^2\sigma_x^2}, \quad \sigma[s] = 2(\ell + x(t))\sigma_x, \quad \sigma[r_F] = \sqrt{\sigma_y^2 + (\kappa + 2(\ell + x(t)))^2\sigma_x^2}, \tag{5.5}$$

respectively.

5.2.1. *Discount Bond Prices.* First, from (3.2), (5.3) and the independence of $W_1^*(t)$ and $W_2^*(t)$, the time-t price of OIS discount bond maturing at time T is given by

$$D(t,T) = \mathbb{E}_t\left[\exp\left\{-\int_t^T y(u)\mathrm{d}u\right\}\right]\mathbb{E}_t\left[\exp\left\{-\rho\int_t^T x(u)\mathrm{d}u\right\}\right].$$

Since both $x(t)$ and $y(t)$ follow the Vasicek model (1977), we obtain

$$D(t,T) = H_1^x(T-t,\rho)H_1^y(T-t)\mathrm{e}^{-\rho H_2^x(T-t)x(t) - H_2^y(T-t)y(t)}, \tag{5.6}$$

where

$$\begin{aligned}
H_1^x(t,\rho) &= \exp\left\{-\frac{\rho^2\sigma_x^2}{4c_x}\left(H_2^x(t)\right)^2 - \frac{\rho^2\sigma_x^2}{2c_x^2}\left(H_2^x(t) - t\right)\right\}, \\
H_1^y(t) &= \exp\left\{-\frac{\sigma_y^2}{4c_y}\left(H_2^y(t)\right)^2 + \left(m_y - \frac{\sigma_y^2}{2c_y^2}\right)\left(H_2^y(t) - t\right)\right\}, \\
H_2^x(t) &= \frac{1 - \mathrm{e}^{-c_x t}}{c_x}, \qquad H_2^y(t) = \frac{1 - \mathrm{e}^{-c_y t}}{c_y}.
\end{aligned}$$

Next, from (5.3), we have

$$r_F(t) = y(t) + (x(t) + \alpha_x)^2 - d, \qquad t \geq 0,$$

where

$$\alpha_x = \ell + \frac{\kappa}{2}, \qquad d = \alpha_x^2 - \ell^2 = \kappa\ell + \frac{\kappa^2}{4}.$$

Since $W_1^*(t)$ and $W_2^*(t)$ are independent, we obtain

$$L(t,T) = \mathrm{e}^{d(T-t)}\mathbb{E}_t\left[\exp\left\{-\int_t^T y(u)\mathrm{d}u\right\}\right]\mathbb{E}_t\left[\exp\left\{-\int_t^T (x(u) + \alpha_x)^2\,\mathrm{d}u\right\}\right]. \tag{5.7}$$

The first expectation in the right hand side of (5.7) is given by

$$\mathbb{E}_t\left[\exp\left\{-\int_t^T y(u)\mathrm{d}u\right\}\right] = H_1^y(T-t)\mathrm{e}^{-H_2^y(T-t)y(t)}, \tag{5.8}$$

where $H_1^y(t)$ and $H_2^y(t)$ are defined above. On the other hand, the second expectation in the right hand side of (5.7) is the quadratic Gaussian (QG) model of Pelsser (1997). The closed form solution of the expectation is obtained in Kijima, Tanaka and Wong (2009) as

$$\begin{aligned}
&\mathbb{E}_t\left[\exp\left\{-\int_t^T (x(u) + \alpha_x)^2\,\mathrm{d}u\right\}\right] \\
&\quad = \exp\left\{A(t,T) - B(t,T)x(t) - C(t,T)x^2(t)\right\},
\end{aligned} \tag{5.9}$$

where $\gamma = \sqrt{c_x^2 + 2\sigma_x^2}$, $\Gamma_a = \gamma - c_x$, $\Gamma_b = \gamma + c_x$, and

$$\begin{aligned}
A(t,T) &= -\sigma_x^2\left(\frac{A_4(t,T)}{\gamma^5 A_5(t,T)} + A_6(t,T)\right) - \alpha_x^2(T-t),\\
B(t,T) &= \frac{2B_1(t,T)}{\gamma^2 A_5(t,T)},\\
C(t,T) &= \frac{\mathrm{e}^{2\gamma(T-t)} - 1}{\Gamma_b \mathrm{e}^{2\gamma(T-t)} + \Gamma_a},\\
A_{1a}(t,T) &= -\mathrm{e}^{\gamma(T-t)} + 4 - \mathrm{e}^{-\gamma(T-t)}(3 + 2\gamma(T-t)),\\
A_{1b}(t,T) &= \mathrm{e}^{-\gamma(T-t)} - 4 + \mathrm{e}^{\gamma(T-t)}(3 - 2\gamma(T-t)),\\
A_4(t,T) &= \alpha_x^2\gamma^2(\Gamma_a A_{1a}(t,T) + \Gamma_b A_{1b}(t,T)),\\
A_5(t,T) &= \Gamma_a \mathrm{e}^{-\gamma(T-t)} + \Gamma_b \mathrm{e}^{\gamma(T-t)},\\
A_6(t,T) &= -\frac{(T-t)(\Gamma_a^{-1} - \Gamma_b^{-1})}{2} + \frac{1}{2\gamma}(\Gamma_a^{-1} + \Gamma_b^{-1})\log\frac{A_5(t,T)}{2\gamma},\\
B_1(t,T) &= -\alpha_x\gamma(\mathrm{e}^{-\gamma T} - \mathrm{e}^{-\gamma t})(\Gamma_a \mathrm{e}^{\gamma t} + \Gamma_b \mathrm{e}^{\gamma T}).
\end{aligned}$$

Substituting (5.8) and (5.9) into (5.7), the LIBOR discount bond price $L(t,T)$ is obtained.

5.2.2. *Interest-Rate Derivatives.* A significant advantage of this spot-rate model is that we can obtain the prices of various interest-rate derivatives in closed form. For example, the FRA rate is given by the next theorem. The proof is given in Appendix A.1.

Theorem 5.1. *Under the spot rate model (5.3), the FRA rate is given by*

$$\mathit{FRA}(t;T_i,T_{i+1}) = \frac{1}{\delta_i}\left(\frac{\mathrm{e}^{K(t,T_i,T_{i+1})}}{H_1^y(\delta_i)\sqrt{1 - 2C(T_i,T_{i+1})\sigma_X^2(t,T_i)}} - 1\right),$$

where

$$\begin{aligned}
K(t,T_i,T_{i+1}) &= -\kappa\delta_i\left(\ell + \frac{\kappa}{4}\right) - A(T_i,T_{i+1}) + \frac{(H_2^y(\delta_i))^2}{2}\sigma_y^2(t,T_i) - \frac{B^2(T_i,T_{i+1})}{4C(T_i,T_{i+1})}\\
&\quad + H_2^y(\delta_i)\mu_y(t,T_i) + \frac{C(T_i,T_{i+1})}{1 - 2C(T_i,T_{i+1})\sigma_X^2(t,T_i)}\mu_X^2(t,T_i,T_{i+1}).
\end{aligned}$$

Here, we define

$$\begin{aligned}
\mu_y(t,T) &= m_y + (y(t) - m_y)\mathrm{e}^{-c_y(T-t)} - \frac{\sigma_y^2}{c_y^2}\left(1 - \mathrm{e}^{-c_y(T-t)} - \mathrm{e}^{-c_y\delta_i}\frac{1 - \mathrm{e}^{-2c_y(T-t)}}{2}\right),\\
\sigma_y^2(t,T) &= \frac{\sigma_y^2}{2c_y}\left(1 - \mathrm{e}^{-2c_y(T-t)}\right),\\
\mu_X(t,T_i,T_{i+1}) &= x(t)\mathrm{e}^{-c_x(T_i-t)} - \frac{\kappa\sigma_x^2}{c_x^2}\left(1 - \mathrm{e}^{-c_x(T_i-t)} - \mathrm{e}^{-c_x\delta_i}\frac{1 - \mathrm{e}^{-2c_x(T_i-t)}}{2}\right)\\
&\qquad + \frac{B(T_i,T_{i+1})}{2C(T_i,T_{i+1})},\\
\sigma_X^2(t,T) &= \frac{\sigma_x^2}{2c_x}\left(1 - \mathrm{e}^{-2c_x(T-t)}\right),
\end{aligned}$$

and the functions $H_1^y(t)$, $H_2^y(t)$, $A(t,T)$, $B(t,T)$ and $C(t,T)$ are given above.

Recall that the LIBOR swap rate is given by (4.11) using the FRA rates in Theorem 5.1 and the OIS swap rate is given by (4.10).

Next, from (3.15), the price of a LIBOR caplet is given by

$$\mathrm{Cplt}(T_{i+1}, K) = D(0, T_{i+1})\mathbb{E}_t^{1,D^{T_{i+1}}}\left[(L_i(T_i) - K)_+\right].$$

The next theorem provides the analytical solution of the caplet price. The proof is given in Appendix A.2.

Theorem 5.2. *The price of a LIBOR caplet is given by*

$$\mathrm{Cplt}(T_{i+1}, K) = \int_{-\infty}^{\infty} \mathrm{Cplt}(T_{i+1}, K|x(T_i) = x)\frac{1}{\sqrt{2\pi\sigma_x^2(T_i)}}\mathrm{e}^{-\frac{(x-\mu_x(T_i))^2}{2\sigma_x^2(T_i)}}\mathrm{d}x,$$

where

$$\begin{aligned}
&\mathrm{Cplt}(T_{i+1}, K|x(T_i) = x)\\
&= \frac{D(0, T_{i+1})}{\delta_i}\left[S(x)\mathrm{e}^{-\sigma_Z^2(T_i)\left(2\mu_Z(T_i)+\sigma_Z^2(T_i)\right)}\Phi\left(\frac{\log\frac{S(x)}{1+K\delta_i} + \mu_Z(T_i)}{\sigma_Z(T_i)} + \sigma_Z(T_i)\right)\right.\\
&\quad \left. -(1 + K\delta_i)\Phi\left(\frac{\log\frac{S(x)}{1+K\delta_i} + \mu_Z(T_i)}{\sigma_Z(T_i)}\right)\right].
\end{aligned}$$

Here, we define $\mu_x(T_i) = \mu_X(0, T_i, T_{i+1}) - \frac{B(T_i,T_{i+1})}{2C(T_i,T_{i+1})}$, $\sigma_x^2(T_i) = \sigma_X^2(0, T_i)$, $\mu_Z(T_i) = H_2^y(\delta_i)\mu_y(0, T_i)$, $\sigma_Z^2(T_i) = [H_2^y(\delta_i)\sigma_y(0, T_i)]^2$*, and*

$$S(x) = \frac{\mathrm{e}^{-d\delta_i - A(T_i,T_{i+1}) + B(T_i,T_{i+1})x + C(T_i,T_{i+1})x^2}}{H_1^y(\delta_i)}.$$

The other functions such as $\mu_X(t, T_i, T_{i+1})$ *are defined in Theorem 5.1.*

The caplet price given in Theorem 5.2 involves one-dimensional numerical integration which can be easily evaluated by using, e.g., the Gaussian quadrature.

5.2.3. *Numerical Example.* In the following numerical examples, we set

t	δ_i	$x(0)$	c_x	$y(0)$	c_y	m_y	ℓ^2
0	0.5	0	0.1	0.01	0.1	0.03	0.001

as the base-case parameters, and consider the four cases; Case (1) to Case (4).

To this end, we fix $\sigma[r_C] = 0.01$ and $\sigma_x = 0.01$ for Case (1), $\sigma_x = 0.02$ for Case (2), $\sigma_x = 0.04$ for Case (3), and $\sigma_x = 0.08$ for Case (4). The other parameters are determined from (5.4) and (5.5) with $x(0) = 0$, i.e.,

$$\rho = \frac{\kappa\sigma_x}{\sigma[r_C]}, \quad \sigma[r_C] = \sqrt{\sigma_y^2 + \kappa^2\sigma_x^2}, \quad \sigma[s] = 2\ell\sigma_x, \quad \sigma[r_F] = \sqrt{\sigma_y^2 + (\kappa + 2\ell)^2\sigma_x^2}.$$

In each case, we consider the positively correlated case ($\rho = 0.5$) and negatively correlated case ($\rho = -0.5$). Hence, for example, for Case (1) with $\rho = 0.5$, we first determine the parameter κ from the relation $\rho = \kappa\sigma_x/\sigma[r_C]$, and then σ_y from $\sigma[r_C] = \sqrt{\sigma_y^2 + \kappa^2\sigma_x^2} = 0.01$. The volatilities $\sigma[s]$ and $\sigma[r_F]$ are calculated by these parameter values. Summarizing, we have the following set of volatilities (in terms of $\%$).

	Case 1: $\sigma_x = 0.01$		Case 2: $\sigma_x = 0.02$		Case 3: $\sigma_x = 0.04$		Case 4: $\sigma_x = 0.08$	
ρ	-0.5	0.5	-0.5	0.5	-0.5	0.5	-0.5	0.5
$\sigma[r_C]$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\sigma[r_F]$	0.970	1.033	0.943	1.069	0.901	1.148	0.866	1.327
$\sigma[s]$	0.063	0.063	0.126	0.126	0.253	0.253	0.506	0.506

Note that for the negatively correlated case ($\rho = -0.5$), the volatility $\sigma[r_F]$ of the funding rate decreases as σ_x increases; however, for a range ($\sigma_x > 0.08$), $\sigma[r_F]$ turns to be increasing in σ_x. If $\sigma_x = 0.16$, the three volatilities become similar (about 1%).

Figure 2 depicts the swap spread defined by (4.13), i.e.,

$$B_n \equiv S^L(0,T_n) - S^O(0,T_n), \qquad T_n = \sum_{i=1}^n \delta_i,$$

with respect to T_n. In the upper panel, we consider the negatively correlated case $(\rho = -0.5)$, whereas the lower panel considers the positively correlated case $(\rho = 0.5)$. The swap spread B_n becomes wider as the spread volatility $\sigma[s]$ gets large for the both cases. The increase of the spread amplitude seems much faster than the increase of the volatility $\sigma[s]$, and the correlation ρ has only little impact on the swap spread. Recall that, when the correlation is negative, the volatility $\sigma[r_F]$ of the funding rate decreases as the spread volatility $\sigma[s]$ increases in the range of $\sigma_x \le 0.08$. Hence, the spread volatility $\sigma[s]$ between the collateral and funding rates seems to play a significant role for the swap spread.

Recall that, in the multi-curve setting, the FRA rate can be different from the forward LIBOR rate $L_i(t)$, because $L_i(t)$ is not a $\mathbb{Q}_{1,D}^{T_{i+1}}$-martingale; see (4.4).[6] As a result, the modern pricing formula for LIBOR swap rates does not display the telescope property; see (4.11). Motivated by this stylized fact, let us define the discrepancy of the modern swap pricing from the classic pricing by

$$\Delta_S(T_n) = S_m^L(0,T_n) - S_c^L(0,T_n), \qquad T_n = \sum_{i=1}^n \delta_i.$$

Here, m stands for 'modern' and $S_m^L(0,T_N)$ is calculated according to (4.11), whereas c stands for 'classic' and $S_c^L(0,T_N)$ is given by (4.12). Figure 3 depicts the discrepancy $\Delta_S(T_n)$ with respect to T_n. In the upper panel, we consider the negatively correlated case $(\rho = -0.5)$, whereas the lower panel considers the positively correlated case $(\rho = 0.5)$. Again, the swap discrepancy $\Delta_S(T_n)$ becomes wider as the volatility $\sigma[s]$ gets large for the both cases. Surprisingly, the spread $\Delta_S(T_n)$ is negligible when the maturity is short (say, shorter than 1 year) and the spread volatility $\sigma[s]$ is small. However, the spread suddenly becomes significant when $\sigma[s]$ exceeds some level. This suggests the importance of considering the regime-switching model for the volatility $\sigma[s]$, i.e., high volatility regime such as the credit crisis period and low volatility regime for the calm period.

[6]When the spread volatility $\sigma[s]$ is small, the swap spread becomes nearly constant, i.e., $B_n \approx \ell^2$ for all n; see Figure 2. In this case, it follows from (4.14) and (4.17) that $\mathrm{FRA}(0;T,T+\delta_n) \approx D_n(0) + \ell^2$ for all n.

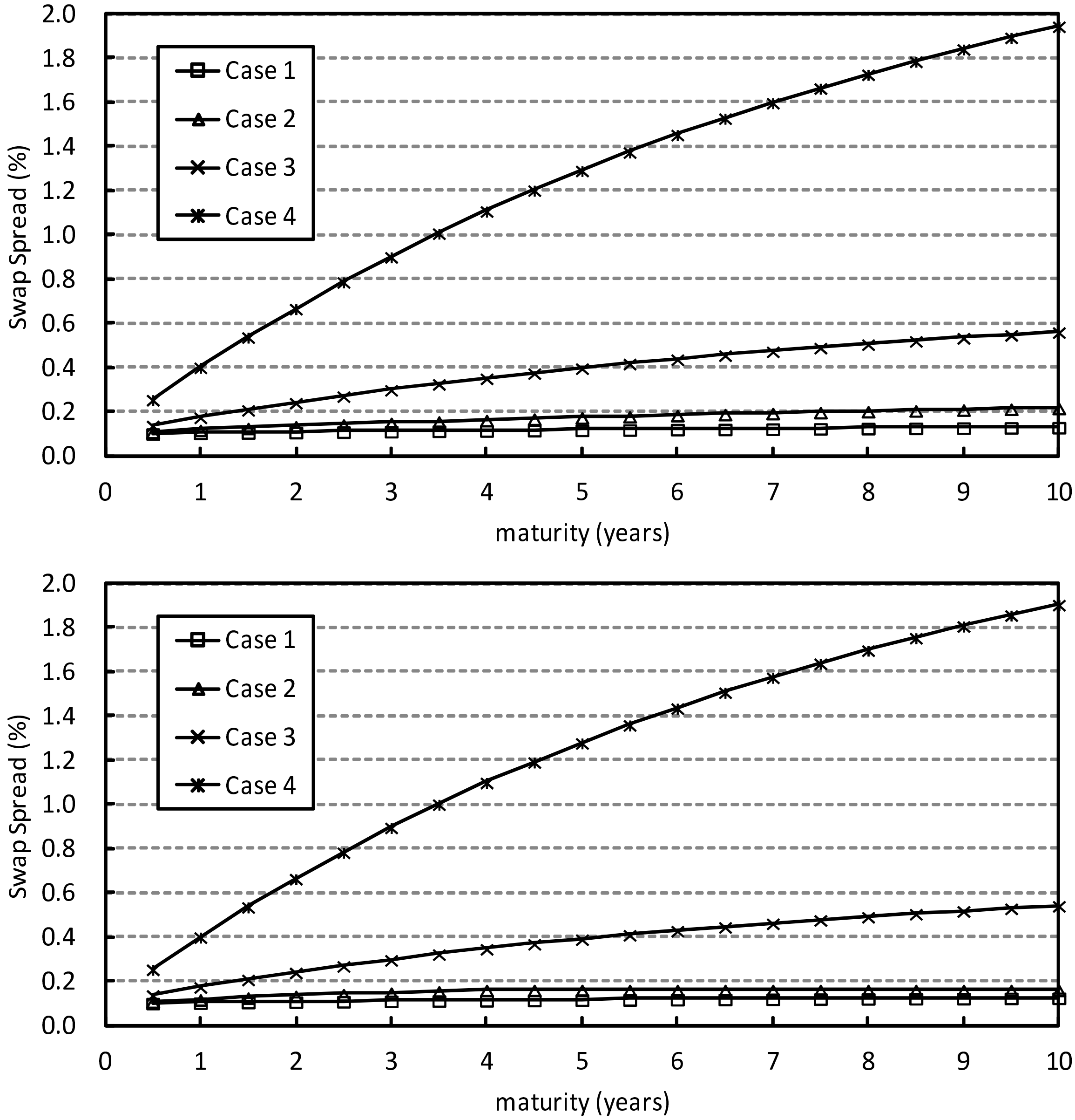
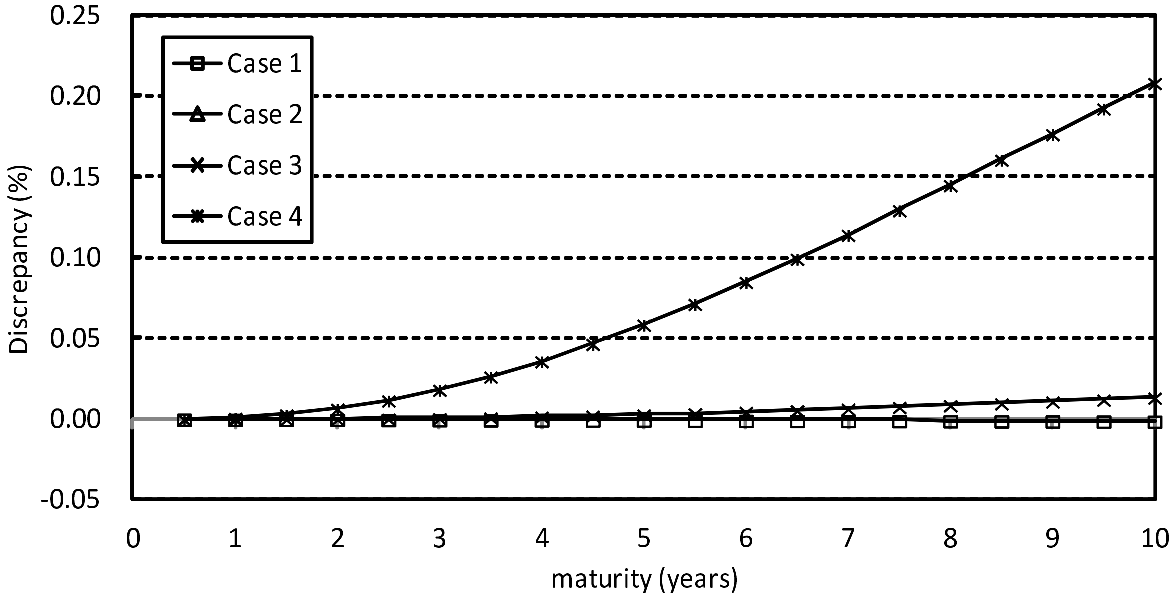


Figure 2: Swap spread $B_n(T_n)$ between the LIBOR and the OIS swap rates (upper panel for the negatively correlated case with $\rho = -0.5$, lower panel for the positively correlated case with $\rho = 0.5$)

6. CONCLUDING REMARKS

In this paper, we construct a no-arbitrage framework of financial market that is consistent to the multi-curve setting. It is shown that *any* derivative security can be duplicated by using the underlying assets and collateral and funding accounts appropriately. A risk-neutral measure is defined accordingly and the derivative price is determined uniquely under the measure. This idea is extended to the pricing of OIS and LIBOR discount bonds, which shows the simultaneous existence of multiple yield curves in the market.



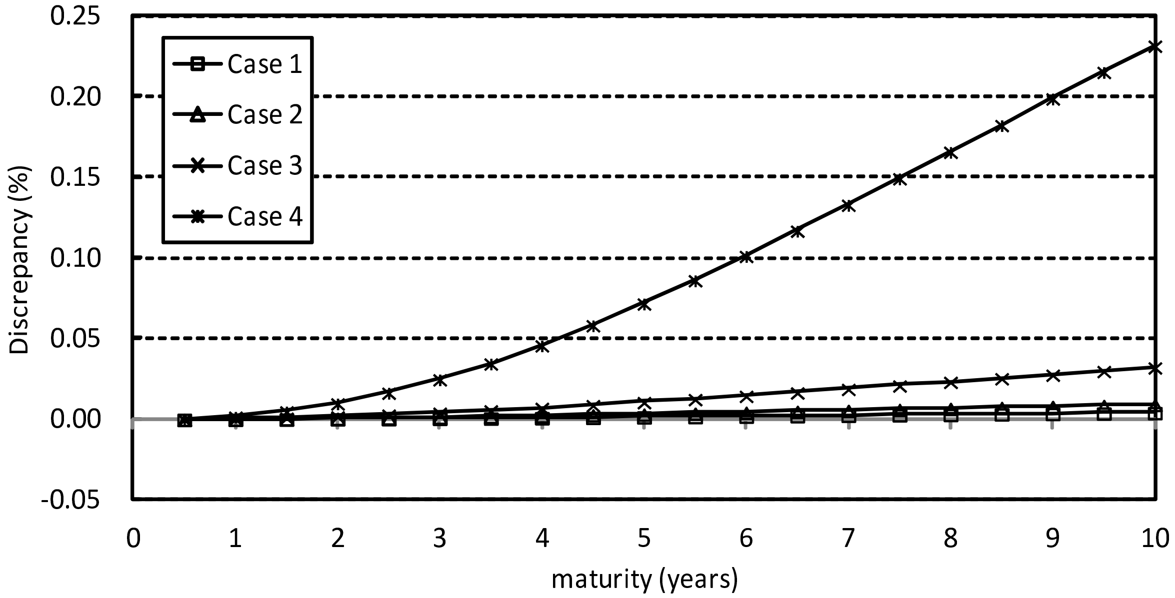


Figure 3: Discrepancy $\Delta_S(T_n)$ of the modern swap rate from the classic swap rate (upper panel for the negatively correlated case with $\rho = -0.5$, lower panel for the positively correlated case with $\rho = 0.5$)

Some specific spot rate model is provided to derive the formulas of interest-rate derivatives such as forward rate agreement and LIBOR swap rates in closed form under the multi-curve setting. Numerical examples are given to demonstrate the discrepancy of the derivative prices under the multi-curve setting from the classic ones. Our important finding is that the discrepancy suddenly becomes significant when the spread volatility $\sigma[s]$ between the collateral and funding rates exceeds some level. This suggests the importance of considering the regime-switching model for the volatility $\sigma[s]$, i.e. high volatility regime such as the credit crisis period and low volatility regime for the calm period. This theme is left for future research.

APPENDIX A. PROOFS

From (5.6), we have

$$\mathrm{d}\log D(t,T) = (\ \dots\ )\mathrm{d}t - \kappa\sigma_x H_2^x(T-t)\mathrm{d}W_1^*(t) - \sigma_y H_2^y(T-t)\mathrm{d}W_2^*(t).$$

It follows from (3.7) that the OIS discount bond price follows the SDE

$$\frac{\mathrm{d}D(t,T)}{D(t,T)} = r_C(t)\mathrm{d}t + \sigma_D(t,T)\mathrm{d}W_D^*(t),$$

where

$$\sigma_D(t,T) = \sqrt{\kappa^2\sigma_x^2(H_2^x(T-t))^2 + \sigma_y^2(H_2^y(T-t))^2}$$

and

$$\mathrm{d}W_D^*(t) = -\frac{\kappa\sigma_x H_2^x(T-t)\mathrm{d}W_1^*(t) + \sigma_y H_2^y(T-t)\mathrm{d}W_2^*(t)}{\sigma_D(t,T)}. \tag{A.1}$$

Similarly, from (5.7), we have

$$\mathrm{d}\log L(t,T) = (\ \dots\ )\mathrm{d}t - \sigma_x[B(t,T) + 2C(t,T)x(t)]\mathrm{d}W_1^*(t) - \sigma_y H_2^y(T-t)\mathrm{d}W_2^*(t).$$

Hence, from (3.8), the LIBOR discount bond price follows the SDE

$$\frac{\mathrm{d}L(t,T)}{L(t,T)} = r_F(t)\mathrm{d}t + \sigma_L(t,T)\mathrm{d}W_L^*(t),$$

where

$$\sigma_L(t,T) = \sqrt{\sigma_x^2[B(t,T) + 2C(t,T)x(t)]^2 + \sigma_y^2(H_2^y(T-t))^2}$$

and

$$\mathrm{d}W_L^*(t) = -\frac{\sigma_x[B(t,T) + 2C(t,T)x(t)]\mathrm{d}W_1^*(t) + \sigma_y H_2^y(T-t)\mathrm{d}W_2^*(t)}{\sigma_L(t,T)}. \tag{A.2}$$

It follows from (A.1) and (A.2) that

$$\mathrm{d}W_D^*(t)\mathrm{d}W_L^*(t) = \frac{\kappa\sigma_x^2 H_2^x(T-t)[B(t,T) + 2C(t,T)x(t)] + \sigma_y^2(H_2^y(T-t))^2}{\sigma_D(t,T)\sigma_L(t,T)}\mathrm{d}t. \tag{A.3}$$

Recall that $W_D^*(t)$ and $W_L^*(t)$ are standard Brownian motions under the risk-neutral measure $\mathbb{Q}$ with the correlation given in (A.3).

A.1. **Pricing of FRA Rate.** From (4.4) and (4.1), the FRA rate is given by

$$\mathrm{FRA}(t;T_i,T_{i+1}) = \mathbb{E}_t^{1,D^{T_{i+1}}}[L_i(T_i)] = \frac{1}{\delta_i}\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\frac{1}{L(T_i,T_{i+1})} - 1\right] \tag{A.4}$$

under the T_{i+1}-OIS-forward measure $\mathbb{Q}_{1,D}^{T_{i+1}}$, where the LIBOR discount bond price $L(T_i,T_{i+1})$ is given by (5.7). Let us denote the correlations by $\rho_{1,D}(t)\mathrm{d}t = \mathrm{d}W_1^*(t)\mathrm{d}W_D^*(t)$ and $\rho_{2,D}(t)\mathrm{d}t = \mathrm{d}W_2^*(t)\mathrm{d}W_D^*(t)$. It is readily seen that

$$\rho_{1,D}(t) = -\frac{\kappa\sigma_x H_2^x(T-t)}{\sigma_D(t,T)}, \qquad \rho_{2,D}(t) = -\frac{\sigma_y H_2^y(T-t)}{\sigma_D(t,T)}.$$

Now, define the processes $W_{1,D}^{T_{i+1}}(t)$ and $W_{2,D}^{T_{i+1}}(t)$ as

$$\begin{cases} \mathrm{d}W_{1,D}^{T_{i+1}}(t) &=& \mathrm{d}W_1^*(t) - \rho_{1,D}(t)\sigma_D(t,T_{i+1})\mathrm{d}t, \\ \mathrm{d}W_{2,D}^{T_{i+1}}(t) &=& \mathrm{d}W_2^*(t) - \rho_{2,D}(t)\sigma_D(t,T_{i+1})\mathrm{d}t, \end{cases} \tag{A.5}$$

respectively. It is well known that $(W_{1,D}^{T_{i+1}}(t), W_{2,D}^{T_{i+1}}(t))$ are independent standard Brownian motions under $\mathbb{Q}_{1,D}^{T_{i+1}}$.[7] Substituting (A.5) into (5.2), we obtain

$$\text{(A.6)} \qquad \left\{ \begin{array}{rcl} \mathrm{d}x(t) &=& [-c_x x(t) - \kappa\sigma_x^2 H_2^x(T_{i+1}-t)]\mathrm{d}t + \sigma_x \mathrm{d}W_{1,D}^{T_{i+1}}(t), \\ \mathrm{d}y(t) &=& [c_y(m_y - y(t)) - \sigma_y^2 H_2^y(T_{i+1}-t)]\mathrm{d}t + \sigma_y \mathrm{d}W_{2,D}^{T_{i+1}}(t), \end{array} \right.$$

under the T_{i+1}-OIS-forward measure $\mathbb{Q}_{1,D}^{T_{i+1}}$. It follows from (5.7) that

$$\begin{aligned} &\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\frac{1}{L(T_i,T_{i+1})}\right] \\ = \quad &\frac{\mathrm{e}^{-d\delta_i - A(T_i,T_{i+1})}}{H_1^y(\delta_i)}\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\mathrm{e}^{H_2^y(\delta_i)y(T_i)}\right]\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\mathrm{e}^{B(T_i,T_{i+1})x(T_i)+C(T_i,T_{i+1})x^2(T_i)}\right], \end{aligned}$$

because $x(t)$ and $y(t)$ are independent.

Note that the process $y(t)$ given in (A.6) follows a mean-reverting process which can be solved as

$$\begin{aligned} y(s) \quad = \quad & y(t)\mathrm{e}^{-c_y(s-t)} + \int_t^s \left(c_y m_y - \sigma_y^2 H_2^y(T_{i+1}-u)\right)\mathrm{e}^{-c_y(s-u)}\mathrm{d}u \\ & + \sigma_y \int_t^s \mathrm{e}^{-c_y(s-u)}\mathrm{d}W_{2,D}^{T_{i+1}}(u). \end{aligned}$$

Hence, $y(T_i)$ is normally distributed with mean

$$\mu_y(t,T_i) \equiv m_y + (y(t)-m_y)\mathrm{e}^{-c_y(T_i-t)} - \frac{\sigma_y^2}{c_y^2}\left(1 - \mathrm{e}^{-c_y(T_i-t)} - \mathrm{e}^{-c_y\delta_i}\frac{1-\mathrm{e}^{-2c_y(T_i-t)}}{2}\right)$$

and variance

$$\sigma_y^2(t,T_i) \equiv \frac{\sigma_y^2}{2c_y}\left(1 - \mathrm{e}^{-2c_y(T_i-t)}\right).$$

It follows that

$$\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\mathrm{e}^{H_2^y(\delta_i)y(T_i)}\right] = \exp\left\{H_2^y(\delta_i)\mu_y(t,T_i) + \frac{(H_2^y(\delta_i))^2}{2}\sigma_y^2(t,T_i)\right\}.$$

Similarly, from (A.6), we obtain

$$x(s) = x(t)\mathrm{e}^{-c_x(s-t)} - \rho\sigma_x^2\int_t^s H_2^x(T_{i+1}-u)\mathrm{e}^{-c_x(s-u)}\mathrm{d}u + \sigma_x\int_t^s \mathrm{e}^{-c_x(s-u)}\mathrm{d}W_{1,D}^{T_{i+1}}(u).$$

Hence, $x(T_i)$ is normally distributed with mean

$$\mu_x(t,T_i) \equiv x(t)\mathrm{e}^{-c_x(T_i-t)} - \frac{\kappa\sigma_x^2}{c_x^2}\left(1 - \mathrm{e}^{-c_x(T_i-t)} - \mathrm{e}^{-c_x\delta_i}\frac{1-\mathrm{e}^{-2c_x(T_i-t)}}{2}\right)$$

and variance

$$\sigma_x^2(t,T_i) \equiv \frac{\sigma_x^2}{2c_x}\left(1 - \mathrm{e}^{-2c_x(T_i-t)}\right).$$

Defining

$$X(T_i) = x(T_i) + \frac{B(T_i,T_{i+1})}{2C(T_i,T_{i+1})},$$

[7] A concrete derivation is available from the authors upon request.

we have

$$\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\mathrm{e}^{B(T_i,T_{i+1})x(T_i)+C(T_i,T_{i+1})x^2(T_i)}\right]=\mathrm{e}^{-\frac{B^2(T_i,T_{i+1})}{4C(T_i,T_{i+1})}}\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\mathrm{e}^{C(T_i,T_{i+1})X^2(T_i)}\right].$$

Note that $X(T_i)$ is normally distributed with mean

$$\mu_X(t,T_i,T_{i+1})\equiv x(t)\mathrm{e}^{-c_x(T_i-t)}-\frac{\kappa\sigma_x^2}{c_x^2}\left(1-\mathrm{e}^{-c_x(T_i-t)}-\mathrm{e}^{-c_x\delta_i}\frac{1-\mathrm{e}^{-2c_x(T_i-t)}}{2}\right)+\frac{B(T_i,T_{i+1})}{2C(T_i,T_{i+1})}$$

and variance $\sigma_X^2(t,T_i)=\sigma_x^2(t,T_i)$. Hence, $X^2(T_i)/\sigma_X^2(x,T_i)$ follows a non-central χ^2-distribution with 1 degree of freedom and non-centrality $\mu_X^2(t,T_i)/\sigma_X^2(t,T_i)$. It follows that[8]

$$\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\mathrm{e}^{C(T_i,T_{i+1})X^2(T_i)}\right]=\frac{1}{\sqrt{1-2C(T_i,T_{i+1})\sigma_X^2(t,T_i)}}\mathrm{e}^{\frac{C(T_i,T_{i+1})\mu_X^2(t,T_i,T_{i+1})}{1-2C(T_i,T_{i+1})\sigma_X^2(t,T_i)}}.$$

Putting these results into (A.4), we obtain the FRA rate given in Theorem 5.1.

A.2. **Pricing of Caplet.** From (4.1) and (3.15), we have

$$\mathrm{Cplt}(T_{i+1},K)=\frac{D(0,T_{i+1})}{\delta_i}\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\left(\frac{1}{L(T_i,T_{i+1})}-(1+K\delta_i)\right)_+\right].$$

Given $x(T_i)=x$, it follows that

$$\begin{aligned}&\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\left(\frac{1}{L(T_i,T_{i+1})}-(1+K\delta_i)\right)_+\middle|\,x(T_i)=x\right]\\ =\ &\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\left(\frac{\mathrm{e}^{-d\delta_i-A(T_i,T_{i+1})+B(T_i,T_{i+1})x+C(T_i,T_{i+1})x^2}}{H_1^y(\delta_i)}\mathrm{e}^{H_2^y(\delta_i)y(T_i)}-(1+K\delta_i)\right)_+\right].\end{aligned}$$

Now, defining $Z(T_i)=H_2^y(\delta_i)y(T_i)$, we have

$$\begin{aligned}&\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\left(\frac{1}{L(T_i,T_{i+1})}-(1+K\delta_i)\right)_+\middle|\,x(T_i)=x\right]\\ =\ &\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\left(S(x)\mathrm{e}^{Z(T_i)}-(1+K\delta_i)\right)_+\right]\\ =\ &S(x)\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\mathrm{e}^{Z(T_i)}1_{\{S(x)\mathrm{e}^{Z(T_i)}>1+K\delta_i\}}\right]-(1+K\delta_i)\mathbb{E}_t^{1,D^{T_{i+1}}}\left[1_{\{S(x)\mathrm{e}^{Z(T_i)}>1+K\delta_i\}}\right],\end{aligned}$$

where

$$S(x)=\frac{\mathrm{e}^{-d\delta_i-A(T_i,T_{i+1})+B(T_i,T_{i+1})x+C(T_i,T_{i+1})x^2}}{H_1^y(\delta_i)}.$$

Note that $Z(T_i)$ is normally distributed with mean $\mu_Z(T_i)\equiv H_2^y(\delta_i)\mu_y(0,T_i)$ and variance $\sigma_Z^2(T_i)\equiv[H_2^y(\delta_i)\sigma_y(0,T_i)]^2$. It follows that

$$\mathbb{E}_t^{1,D^{T_{i+1}}}\left[1_{\{S(x)\mathrm{e}^{Z(T_i)}>1+K\delta_i\}}\right]=\Phi\left(\frac{\log\frac{S(x)}{1+K\delta_i}+\mu_Z(T_i)}{\sigma_Z(T_i)}\right),$$

[8]If Y follows a non-central χ^2-distribution with ν degree of freedom and non-centrality δ, its moment generating function is given by

$$\mathbb{E}\left[\mathrm{e}^{tY}\right]=(1-2t)^{-\frac{\nu}{2}}\exp\left\{\frac{\delta t}{1-2t}\right\},\qquad t<\frac{1}{2}.$$

where $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution. Also, recall that

$$\mathrm{e}^x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} = \mathrm{e}^{-\sigma^2(2\mu+\sigma^2)} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu-\sigma^2)^2}{2\sigma^2}\right\}.$$

Defining $Z'(T_i) = Z(T_i) + \sigma_Z^2(T_i)$ and using the above relation, we obtain

$$\begin{aligned}
\mathbb{E}_t^{1,D^{T_{i+1}}}\left[\mathrm{e}^{Z(T_i)} 1_{\{S(x)\mathrm{e}^{Z(T_i)}>1+K\delta_i\}}\right] &= \mathrm{e}^{-\sigma_Z^2(T_i)\left(2\mu_Z(T_i)+\sigma_Z^2(T_i)\right)} \mathbb{E}_t^{1,D^{T_{i+1}}}\left[1_{\{S(x)\mathrm{e}^{Z'(T_i)}>1+K\delta_i\}}\right] \\
&= \mathrm{e}^{-\sigma_Z^2(T_i)\left(2\mu_Z(T_i)+\sigma_Z^2(T_i)\right)} \mathbb{Q}_{1,D}^{T_{i+1}}\left\{Z'(T_i) > \log\frac{1+K\delta_i}{S(x)}\right\} \\
&= \mathrm{e}^{-\sigma_Z^2(T_i)\left(2\mu_Z(T_i)+\sigma_Z^2(T_i)\right)} \Phi\left(\frac{\log\frac{S(x)}{1+K\delta_i} + \mu_Z(T_i) + \sigma_Z^2(T_i)}{\sigma_Z(T_i)}\right).
\end{aligned}$$

Putting these results together, we obtain the caplet price given in Theorem 5.2.

REFERENCES

[1] Bianchetti, M. (2013), "Modern pricing of interest rate derivatives including funding and collateral," *Interest Rate Modeling after the Financial Crisis*, Risk Books, 113–152.

[2] Black, F. and M. Scholes (1973), "The pricing of options and corporate liabilities," *Journal of Political Economy*, **81**, 637–654.

[3] Boenkost, W. and W.M. Schmidt (2005), "Cross currency swap valuation," HfB-Business School of Finance & Management, working paper.

[4] Han, M, Y. He and H. Zhang (2015), "A note on evaluation of derivatives with collaterals," working paper, SSRN-id2575633.pdf.

[5] Han, M., Y. He and H. Zhang (2014), "A note on discounting and funding value adjustments for derivatives," *Journal of Financial Engineering*, **1**, 1450008.1—-1450008.34.

[6] Heath D., R. Jarrow, and A. Morton (1992), "Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation," *Econometrica*, **60**, 77–105.

[7] Henrard, M. (2007), "The irony in the derivatives discounting," *Wilmott Magazine*, **30**, 92–98.

[8] Henrard, M. (2014), *Interest Rate Modelling in the Multi-curve Framework*, Palgrave Macmillan.

[9] Kijima, M. (2013), *Stochastic Processes with Applications to Finance*, 2nd Edition, Chapman & Hall, London.

[10] Kijima, M., K. Tanaka and T. Wong (2009), "A multi-quality model of interest rates," *Quantitative Finance*, **9**, 133–145.

[11] Mercurio, F. (2009) "Interest rates and the credit crunch: New formulas and market models," working paper.

[12] Morino, L. and W. J. Runggaldier (2014), "On multicurve models for the term structure," *Nonlinear Economic Dynamics and Financial Modeling* (eds. R. Dieci et al.), .275–290.

[13] Ogawa, K. (2015), "Pricing basic instruments with telescopic property in a two-curve environment," working paper.

[14] Pelsser, A. (1997), "A tractable yield-curve model that guarantees positive interest rates," *Review of Derivatives Research*, **1**, 269–284.

[15] Piterbarg, V. (2010), "Funding beyond discounting," *RISK*, **February**, 97–102.

[16] Piterbarg, V. (2012), "Cooking with collateral," *RISK*, **August**, 58–63.

[17] Vasicek, O.A. (1977), "An equilibrium characterization of the term structure," *Journal of Financial Economics*, **5**, 177–188.