Research Paper Series

No. 152

Pure strategy equilibria in finite symmetric concave games and an application to symmetric discrete Cournot games

Takuya Iimura† and Takahiro Watanabe‡

July, 2015

†School of Business Administariton, Tokyo Metropolitan University

‡School of Business Administariton, Tokyo Metropolitan University

Pure strategy equilibria in finite symmetric concave games and an application to symmetric discrete Cournot games*

Takuya Iimura† Takahiro Watanabe‡

July 31, 2015

Abstract

We consider a finite symmetric game where the set of strategies for each player is a one-dimensional integer interval. We show that a pure strategy equilibrium exists if the payoff function is concave with respect to the own strategy and satisfies a pair of symmetrical conditions near the symmetric strategy profiles. As an application, we consider a symmetric Cournot game in which each firm chooses an integer quantity of product. If the inverse demand function is a nonincreasing concave function and the cost function for each firm is an identical convex function, then the payoff function of the firm satisfies the conditions and this symmetric game has a pure strategy equilibrium.

1 Introduction

Many problems in economics can be modeled as games where the sets of strategies for players are one-dimensional compact intervals of real numbers (hereafter referred to as games with real intervals). The strategies correspond to various economic variables, for example selling prices in Bertrand games, quantities of product in Cournot games, and locations of firms in spatial competition games. Moreover, for simplicity, many applications postulate identical players in addition to one-dimensional strategy sets, which are described by symmetric games. If the payoff function is quasiconcave and continuous, then symmetric games with compact real intervals have a symmetric pure strategy equilibrium. A simple proof is established by Moulin (1986), although the existence of a not necessarily symmetric pure strategy equilibrium is implied by a more classical theorem in Rosen (1965). Results about the existence of pure strategy equilibria for

*This work was supported by JSPS KAKENHI Grant Number 25380233.

†Takuya Iimura, School of Business Administration, Tokyo Metropolitan University, Tokyo 192-0397, Japan. +81-42-677-1111, t.iimura@tmu.ac.jp.

‡Takahiro Watanabe, School of Business Administration, Tokyo Metropolitan University, Tokyo 192-0397, Japan. +81-42-677-1111, contact_nabe10@nabenavi.net.

symmetric games with compact and convex sets of strategies were extended to those with discontinuous payoffs by Dasgupta and Maskin (1986) and Reny (1999).

This class of games is familiar to us and is useful for formulating problems, but strategies for the games in the real world take discrete values. The prices and the quantities of product are always integral, or perhaps rational numbers. Hence, it would be desirable to examine the existence of a pure strategy equilibrium in finite symmetric games where the sets of strategies are one-dimensional finite integer intervals (hereafter referred to as games with integer intervals), but few studies have examined this.

We have shown previously that finite symmetric games with integer intervals have pure strategy equilibria when the payoff functions are integrally concave Iimura and Watanabe (2014). This result gives a condition for the existence of equilibria, but the concept of integral concavity, defined in Favati and Tardella (1990), is rather technical. In addition, it can be difficult to check whether the payoff functions are integrally concave.

In this paper, we extend this result and show that a finite symmetric game with integer intervals has pure strategy equilibria if the payoff function for each player is concave with respect to the own strategy, defined as a decreasing marginal payoff with respect to the own strategy, and satisfies a pair of symmetrical conditions near the symmetric strategy profiles. The pair of conditions is satisfied when the payoff function is integrally concave, and checking them is easier than checking integral concavity.

We also apply the results to symmetric Cournot games where the quantity that each player selects is an integer. We show that there exists an equilibrium when the inverse demand function is a nonincreasing concave function and the cost function is a convex function. There are many studies regarding the existence and stability of equilibria in n-firm Cournot games for one homogeneous good, such as Okuguchi (1964, 1973) and Novshek (1985). In particular, McManus (1964) and Roberts and Sonnenschein (1976) treat Cournot games with n identical firms; that is, symmetric Cournot games. Many studies assume that the quantity of product for each firm belongs to a real interval, but there are few studies about Cournot games with integer quantities.

This paper is organized as follows. In Section 2, we give some definitions. In Section 3, we prove our existence theorem. In Section 4, we apply the existence theorem to a discrete symmetric Cournot game. In Section 5, we give some concluding remarks.

2 Definitions

A game is a three-tuple $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$, where $N = \{1, \ldots, n\}$ $(n \geq 2)$ is a set of players, S_i is a set of strategies for $i \in N$, and u_i is a real-valued payoff function for $i \in N$ defined on the set of strategy profiles $S := S_1 \times \cdots \times S_n$. In this paper, we only consider symmetric games where the set of strategies is a one-dimensional integer interval. The set of strategies for each player $i \in N$ is an identical integer interval $S_i = \{0, \ldots, m\}$

($m \geq 1$), and u_i satisfies

$$u_i(s) = u_{\pi(i)}(\varphi_\pi(s)), \quad \forall \pi \in \Pi,\ \forall s \in S,\ \forall i \in N,$$

where Π is the set of all permutations of N, and φ_π is a permutation of $s \in S$ associated with $\pi \in \Pi$ such that

$$\varphi_\pi(s) = (s_{\pi^{-1}(1)}, \ldots, s_{\pi^{-1}(n)}).$$

A strategy profile $s = (s_1, \ldots, s_n) \in S$ is also denoted by $s = (s_i, s_{-i})$, where s_i is the strategy of player $i \in N$ and s_{-i} is the $(n-1)$-tuple of the strategies of the other players. Letting e^i be the ith unit vector for each $i \in N$, a strategy profile $(s_i + 1, s_{-i})$ is also denoted by $s + e^i$. A strategy profile $s \in S$ is a pure strategy equilibrium if

$$u_i(s) \geq u_i(s'_i, s_{-i}) \quad \forall s'_i \in S_i\ \forall i \in N.$$

A real-valued function $f : S_i \to \mathbb{R}$ is said to be *concave* if it satisfies

$$f(x) - f(x-1) \geq f(x+1) - f(x)$$

for any x such that $1 \leq x \leq m-1$. For any function defined on a real interval $[0, m]$, the restriction of the function to the integer interval $\{0, \ldots, m\}$ is concave if the function is concave in the usual sense for continuous functions.

The payoff function u_i is *concave* with respect to the own strategy if $u_i(\cdot, s_{-i})$ is concave for any s_{-i}. In other words, u_i is concave if, for any s_{-i} and any integer s_i such that $1 \leq s_i \leq m-1$,

$$u_i(s) - u_i(s - e^i) \geq u_i(s + e^i) - u_i(s).$$

Thus, the payoff function u_i is concave with respect to the own strategy if and only if the marginal payoff is decreasing with respect to the own strategy.

A finite game $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is said to be concave if the payoff u_i is concave with respect to the own strategy for every $i \in N$. Note that when $m = 1$—that is, when every player has only two strategies—every finite symmetric game is concave.

As is the case with concavity on a real interval, any local maximum of a concave function defined on an integer interval is a global maximum. Lemma 1 shows this fact formally. In the following, we use the convention $u_i(x) = -\infty$ if $x \notin S$.

Lemma 1. *Suppose that u_i is concave with respect to the own strategy. Then:*

(i) *for any $s \in S$, if $u_i(s) \geq u_i(s - e^i)$, then $u_i(s) \geq u_i(s'_i, s_{-i})$ for any $s'_i \in S_i$ such that $s'_i \leq s_i$, and*

(ii) *for any $s \in S$, if $u_i(s) \geq u_i(s + e^i)$, then $u_i(s) \geq u_i(s'_i, s_{-i})$ for any $s'_i \in S_i$ such that $s'_i \geq s_i$.*

Proof. We will show (i). For $s'_i \in S_i$ such that $s'_i < s_i$, let $t = s_i - s'_i$. Then, because u_i is concave with respect to the own strategy, for any $k \in \{1, \dots, t\}$ we have

$$\begin{aligned} u_i(s-(k-1)e^i) - u_i(s-ke^i) &\geq u_i(s-(k-2)e^i) - u_i(s-(k-1)e^i) \\ &\vdots \\ &\geq u_i(s) - u_i(s-e^i). \end{aligned}$$

Therefore

$$\begin{aligned} u_i(s) - u_i(s'_i, s_{-i}) &= \sum_{k=1}^{t} \left(u_i(s-(k-1)e^i) - u_i(s-ke^i)\right) \\ &\geq t\left(u_i(s) - u_i(s-e^i)\right). \end{aligned}$$

Thus, if $u_i(s) \geq u_i(s-e^i)$, then $u_i(s) \geq u_i(s'_i, s_{-i})$ and (i) is proved. The proof of (ii) is similar. □ □

3 Existence of a pure strategy equilibrium

Let $V_z = \{s \in S \mid s_i \in \{z, z+1\},\ i = 1, \dots, n\}$ for each $z \in \{0, \dots, m-1\}$. Our main result in the following theorem asserts that there exists an equilibrium in V_z for some $z \in \{0, \dots, m-1\}$ if the payoff function in a finite symmetric game is concave with respect to the own strategy and satisfies a pair of conditions.

Theorem 1. *Let $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a finite symmetric game. If u_i is concave with respect to the own strategy and satisfies the conditions*

$$u_i(s-e^i) > u_i(s) \implies u_i(s-e^j) > u_i(s-e^j+e^i) \quad \forall s, s-e^j+e^i \in V_z, \tag{1}$$

$$u_i(s) < u_i(s+e^i) \implies u_i(s+e^j-e^i) < u_i(s+e^j) \quad \forall s, s+e^j-e^i \in V_z, \tag{2}$$

then there exists an equilibrium $s \in S$ in V_x for some $x \in \{0, \dots, m-1\}$.

Proof. Let $\hat{s}^z = (z, \dots, z)$ for each $z \in S_1 = \{0, \dots, m\}$. If there exists a z such that $u_1(\hat{s}^z) \geq u_1(\hat{s}^z - e^1)$ and $u_1(\hat{s}^z) \geq u_1(\hat{s}^z + e^1)$, then $u_1(\hat{s}^z) \geq u_1(s_1, \hat{s}^z_{-1})$ for any $s_1 \in S_1$ by Lemma 2.1. By symmetry, $u_i(\hat{s}^z) \geq u_i(s_i, \hat{s}^z_{-i})$ for any $s_i \in S_i$ and $i \in N$, and so $\hat{s}^z$ is a symmetric equilibrium.

If there is no such z, then by concavity, we have either $u_1(\hat{s}^z - e^1) < u_1(\hat{s}^z) < u_1(\hat{s}^z + e^1)$ or $u_1(\hat{s}^z - e^1) > u_1(\hat{s}^z) > u_1(\hat{s}^z + e^1)$ for each z. Because $u_1(\hat{s}^0) > u_1(\hat{s}^0 - e^1) = -\infty$ and $u_1(\hat{s}^m) > u_1(\hat{s}^m + e^1) = -\infty$, there exists $x \in S_1$, $0 \leq x \leq m-1$, such that $u_1(\hat{s}^x) < u_1(\hat{s}^x + e^1)$ and $u_1(\hat{s}^{x+1} - e^1) > u_1(\hat{s}^{x+1})$. The latter implies that $u_n(\hat{s}^{x+1} - e^n) > u_n(\hat{s}^{x+1})$ by symmetry. Let $s^0, \dots, s^n$ be points in V_x such that $s^0 = \hat{s}^x$ and $s^{k+1} = s^k + e^{k+1}$ for $k = 0, \dots, n-1$. Note that s^k is given by

$$s^k_i = \begin{cases} x+1 & \text{if } i \leq k, \\ x & \text{if } i > k. \end{cases}$$

Then $u_1(s^0) < u_1(s^1)$ and $u_n(s^{n-1}) > u_n(s^n)$. Therefore there exists a $k \in \{1, \ldots, n-1\}$ such that

$$u_k(s^{k-1}) < u_k(s^k), \tag{3}$$

$$u_{k+1}(s^k) \geq u_{k+1}(s^{k+1}). \tag{4}$$

We will show that

$$u_{k+1}(s^k) \geq u_{k+1}(s_{k+1}, s^k_{-(k+1)}) \; \forall s_{k+1} \in S_{k+1}$$

and

$$u_k(s^k) \geq u_k(s_k, s^k_{-k}) \; \forall s_k \in S_k,$$

that is, players k and $k+1$ are playing an equilibrium strategy. Because $s^k_i = s^k_k = x+1$ for each player $i \in \{1, \ldots, k\}$ and $s^k_i = s^k_{k+1} = x$ for each player $i \in \{k+1, \ldots, n\}$, this implies by symmetry that s^k is a pure strategy equilibrium.

Now, $u_{k+1}(s^k) \geq u_{k+1}(s^k + e^{k+1})$ by (4). We also have $u_{k+1}(s^k - e^{k+1}) \leq u_{k+1}(s^k)$. To see this, suppose by way of contradiction that $u_{k+1}(s^k - e^{k+1}) > u_{k+1}(s^k)$. Then (1) implies that $u_{k+1}(s^k - e^k) > u_{k+1}(s^k - e^k + e^{k+1})$. We then have $u_{k+1}(s^k - e^k) = u_k(s^{k-1})$ and $u_{k+1}(s^k + e^{k+1} - e^k) = u_k(s^k)$ by symmetry, and so $u_k(s^{k-1}) > u_k(s^k)$. However, this contradicts (3). Hence $u_{k+1}(s^k) \geq u_{k+1}(s^k + e^{k+1})$ and $u_{k+1}(s^k - e^{k+1}) \leq u_{k+1}(s^k)$. Thus, by Lemma 1, $u_{k+1}(s^k) \geq u_{k+1}(s_{k+1}, s^k_{-(k+1)})$ for all $s_{k+1} \in S_{k+1}$.

Consider (3), that $u_k(s^{k-1}) < u_k(s^k)$, again. This means that $u_k(s^k - e^k) < u_k(s^k)$. We also have $u_k(s^k) \geq u_k(s^k + e^k)$. To see this, suppose by way of contradiction that $u_k(s^k) < u_k(s^k + e^k)$. Then (2) implies that $u_k(s^k + e^{k+1} - e^k) < u_k(s^k + e^{k+1})$. We have $u_k(s^k + e^{k+1} - e^k) = u_{k+1}(s^k)$ and $u_k(s^k + e^{k+1}) = u_{k+1}(s^{k+1})$ by symmetry, and so $u_{k+1}(s^k) < u_{k+1}(s^{k+1})$. However, this contradicts (4). Hence $u_k(s^k - e^k) < u_k(s^k)$ and $u_k(s^k) \geq u_k(s^k + e^k)$. Therefore, by Lemma 1, $u_k(s^k) \geq u_k(s_k, s^k_{-k})$ for all $s_k \in S_k$. Hence $s^k \in V_x$ is a pure strategy equilibrium. □ □

For $n = 2$, namely two-person symmetric games, (1) and (2) can be expressed more compactly. For $i = 1$, both s and $s - e^2 + e^1$ belong to V_z if and only if $s = (z, z+1)$, and both s and $s + e^2 - e^1$ belong to V_z if and only if $s = (z+1, z)$. (1) is effective only if $z \geq 1$ because $s - e^1 \notin S$ for $z = 0$. Similarly, (2) is effective only if $z \leq m-2$, because $s + e^1 \notin S$ for $z = m-1$. If (1) and (2) hold for $i = 1$, then they also hold for any $i \in N$ by symmetry. Thus we obtain the following corollary.

Corollary 1. *Let $(\{1,2\}, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a two-person finite symmetric game. If u_1 is concave with respect to the own strategy and satisfies the conditions*

$$\begin{aligned} &u_1(z-1, z+1) > u_1(z, z+1) \\ &\implies u_1(z, z) > u_1(z+1, z) \qquad \forall z \in \{1, \ldots, m-1\}, \end{aligned} \tag{5}$$

$$\begin{aligned} &u_1(z+1, z) < u_1(z+2, z) \\ &\implies u_1(z, z+1) < u_1(z+1, z+1) \quad \forall z \in \{0, \ldots, m-2\}, \end{aligned} \tag{6}$$

then there exists an equilibrium $s \in S$ in V_x for some $x \in \{0, \dots, m-1\}$.

None of (1), (2), and concavity are redundant for ensuring the existence of an equilibrium. We show this using examples of two-person symmetric games with more than two strategies.[1] In Figure 1, the left symmetric game with three strategies satisfies (6) and concavity, but not (5), while the right symmetric game satisfies (5) and concavity, but not (6). Neither of these has a pure strategy equilibrium.

In Figure 2, the symmetric game with four strategies satisfies (5) and (6), but this game is not concave: the payoff function for player 1 is not concave with respect to the own strategy at $s_2 = 3$. This game also has no pure strategy equilibrium.[2]

$1\backslash 2$	0	1	2
0	$(-1,-1)$	$(-1,1)$	$(1,0)$
1	$(1,-1)$	$(0,0)$	$(0,1)$
2	$(0,1)$	$(1,0)$	$(-1,-1)$

$1\backslash 2$	0	1	2
0	$(-1,-1)$	$(1,0)$	$(0,1)$
1	$(0,1)$	$(0,0)$	$(1,-1)$
2	$(1,0)$	$(-1,1)$	$(-1,-1)$

Figure 1: The left symmetric concave game violates (5) and the right symmetric concave game violates (6).

$1\backslash 2$	0	1	2	3
0	$(1,1)$	$(0,2)$	$(-1,1)$	$(2,0)$
1	$(2,0)$	$(1,1)$	$(0,2)$	$(-1,1)$
2	$(1,-1)$	$(2,0)$	$(1,1)$	$(0,2)$
3	$(0,2)$	$(1,-1)$	$(2,0)$	$(1,1)$

Figure 2: A symmetric game that satisfies (5) and (6), but is not concave.

4 An application: discrete Cournot games

Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a symmetric game such that for each $i \in N$, u_i is defined by

$$u_i(s) = f(s_1 + \cdots + s_n)s_i - c(s_i), \quad s \in S,$$

where f is an *inverse demand function* and c is a *cost function* that is identical for all players (firms). We call G a symmetric discrete Cournot game.

Recall that f is *concave* if, for any z, $1 \le z \le m-1$,

$$f(z) - f(z-1) \ge f(z+1) - f(z).$$

[1]Every symmetric game with two strategies has a pure strategy equilibrium as shown in Cheng et al (2004).

[2]It can be shown that any two-person symmetric game with three strategies satisfying (5) and (6) has a pure strategy equilibrium, even if it is not concave.

We say that c is *convex* if, for any z, $1 \le z \le m-1$,

$$c(z) - c(z-1) \le c(z+1) - c(z).$$

We show the following.

Theorem 2. *A symmetric discrete Cournot game has a pure strategy equilibrium if the inverse demand function is a nonincreasing concave function and the cost function is a convex function.*

Proof. We show that the payoff function u_i defined by $u_i(s) = f(s_1 + \cdots + s_n)s_i - c(s_i)$ is concave with respect to the own strategy, and satisfies conditions (1) and (2) of Theorem 1.

For $s = (s_i, s_{-i}) \in S$, let $a = \sum_{h \ne i} s_h$ and let $x = s_i$. Then $x \ge 0$ implies that

$$\begin{aligned}
u_i(s) - u_i(s - e^i) &= (f(a+x)x - c(x)) - (f(a+x-1)(x-1) - c(x-1)) \\
&= (f(a+x) - f(a+x-1))x \\
&\quad + f(a+x-1) - (c(x) - c(x-1)) \\
&\ge (f(a+x+1) - f(a+x))x \\
&\quad + f(a+x+1) - (c(x+1) - c(x)) \\
&= (f(a+x+1)(x+1) - c(x+1)) - (f(a+x)x - c(x)) \\
&= u_i(s+e^i) - u_i(s),
\end{aligned}$$

where we use the three inequalities

$$\begin{aligned}
f(a+x+1) - f(a+x) &\le f(a+x) - f(a+x-1), \\
f(a+x+1) &\le f(a+x-1), \text{ and} \\
c(x) - c(x-1) &\le c(x+1) - c(x), \qquad (7)
\end{aligned}$$

given by the concavity of f, the nonincreasing property of f, and the convexity of c, respectively. The inequality $u_i(s) - u_i(s-e^i) \ge u_i(s+e^i) - u_i(s)$ for all $s \in S$ says that u_i is concave with respect to the own strategy.

To see that u_i satisfies (1) of Theorem 1, suppose by way of contradiction that $u_i(s-e^i) > u_i(s)$ and $u_i(s-e^j) \le u_i(s-e^j+e^i)$. Then

$$\begin{aligned}
f(a+x)x - c(x) &< f(a+(x-1))(x-1) - c(x-1) \text{ and} \\
f((a-1)+(x+1))(x+1) - c(x+1) &\ge f((a-1)+x)x - c(x).
\end{aligned}$$

Then, by subtracting and rearranging,

$$f(a+x-1) + [c(x+1) - c(x)] < f(a+x) + [c(x) - c(x-1)].$$

This is impossible because f is nonincreasing and because of (7). Hence (1) follows. Condition (2) can be shown similarly. □ □

A simple algorithm for locating an equilibrium is the following. First, search through the symmetric profiles to find a symmetric equilibrium by checking whether

$$u_1(\hat{s}^z - e^1) \leq u_1(\hat{s}^z) \quad \text{and} \quad u_1(\hat{s}^z) \geq u_1(\hat{s}^z - e^1)$$

(see the proof of Theorem 1). Second, if this fails, find two adjacent symmetric profiles $\hat{s}^x$ and $\hat{s}^{x+1}$ such that

$$u_1(\hat{s}^x) < u_1(\hat{s}^x + e^1) \quad \text{and} \quad u_1(\hat{s}^{x+1} - e^1) > u_1(\hat{s}^{x+1}),$$

and then search through $s^1, s^2, \dots s^{n-1}$ as defined in the proof of Theorem 1 to find an asymmetric equilibrium. Our result is also applicable to the restriction of a symmetric continuous Cournot game to the integer lattice. For example, if the inverse demand function and the cost function are given by

$$f(y) = 2000 - 0.5y^{1.6} \quad \text{and} \quad c(x) = 10 + 0.01x^{1.2},$$

then letting $S_i = \{0, \dots, 180\}$, our algorithm locates an equilibrium for the three-firm symmetric discrete Cournot game at $(46, 46, 45)$.

5 Concluding remarks

As mentioned in the introduction, there are numerous studies on the existence and stability of equilibria in n-firm Cournot games for one homogeneous good when the quantity of product for each firm comes from a real interval. In contrast, few studies have examined the case where each firm selects an integer quantity of product. Monderer and Shapley (1996) showed that some Cournot games are exact potential games. Typically, such games have pure strategy equilibria irrespective of whether the set of strategies are infinite. However, a Cournot game can be an exact potential game only if the inverse demand function is linear. Thus, a discrete Cournot game satisfying the conditions of Theorem 2 is not an exact potential game when the inverse demand function is strictly concave.

We can also apply our results to symmetric discrete games whose payoff functions are of the form

$$u_i(s) = f(s_i) + g(\sum_{j \in N} s_j)$$

for two increasing concave functions f and g. This is the case, in particular, for games of private provision of pure public goods.

References

Cheng SF, Reeves DM, Vorobeychik Y, Wellman MP (2004) Notes on equilibria in symmetric games. In: AAMAS-04 Workshop on Game-Theoretic and Decision-Theoretic Agents.

Dasgupta P, Maskin E (1986) The existence of equilibrium in discontinuous economic games, i: Theory. The Review of Economic Studies 53(1):pp. 1–26

Favati P, Tardella F (1990) Convexity in nonlinear integer programming. Ricerca Operativa 53:3–44

Iimura T, Watanabe T (2014) Existence of a pure strategy equilibrium in finite symmetric games where payoff functions are integrally concave. Discrete Applied Mathematics 166:26–33

McManus M (1964) Equilibrium, numbers and size in Cournot oligopoly. Yorkshire Bulletin of Social and Economic Research 16(2):68–75

Monderer D, Shapley LS (1996) Potential games. Games and Economic Behavior 14(1):124–143

Moulin H (1986) Game theory for the social sciences. NYU Press

Novshek W (1985) On the existence of Cournot equilibrium. The Review of Economic Studies 52(1):85–98

Okuguchi K (1964) The stability of the Cournot oligopoly solution: A further generalization. The Review of Economic Studies pp. 143–146

Okuguchi K (1973) Quasi-competitiveness and Cournot oligopoly. The Review of Economic Studies pp. 145–148

Reny PJ (1999) On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica 67(5):pp. 1029–1056

Roberts J, Sonnenschein H (1976) On the existence of Cournot equilbrium without concave profit functions. Journal of Economic Theory 13(1):112–117

Rosen JB (1965) Existence and uniqueness of equilibrium points for concave n-person games. Econometrica 33(3):pp. 520–534