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Takuya Iimura*

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*Graduate School of Social Sciences, Tokyo Metropolitan University

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Abstract

We show that a two-person finite game is solved by the iterated elimination of never best responses (IENBR) if and only if it is best response acyclic and strongly solvable in the sense of Nash (1951). Thus the rationalizable strategies (Bernheim (1984), Pearce (1984)) are equivalent to the Nash equilibrium strategies in two-person finite games if and only if the two conditions are met. We prove this for both mixed strategy games and pure strategy games.

JEL Classification: C72 (Noncooperative game)

1 Introduction

Nash (1951) defined the *solution* of a noncooperative game as the set of mixed strategy equilibria satisfying the *interchangeability* condition. This condition stipulates that if there are multiple equilibria then any combination of the strategies therein is also an equilibrium. Thus, for a *solvable* game that has the solution, the *equilibrium strategies* are determined, and the set of equilibria is a Cartesian product of the sets of equilibrium strategies. The *strong solution* is a solution that has an additional property that any strategy profile reached by a unilateral deviation from an equilibrium and having the same payoff as the equilibrium payoff for the deviant is also an equilibrium. Since deviations to a pure strategy in the support of mixed strategy do not change the deviant's payoff, any *strongly solvable* game, which is a game having the strong solution, has a pure strategy equilibrium (Nash, 1951, page 290).

The notion of solvable game in the sense of Nash was later generalized by Friedman (1983), and then by Kats and Thisse (1992), to mean any finite or infinite game whose set

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†School of Business Administration, Tokyo Metropolitan University, Tokyo 192-0397, Japan, E-mail: t.iimura@tmu.ac.jp (T. Iimura).

of equilibria, pure or mixed, satisfies the interchangeability condition (the existence of an appropriate equilibrium is needed then). In this paper, we focus on two-person finite games and their mixed extensions, and call a game a *mixed strategy game* when we consider the game in mixed strategy; a *pure strategy game* when we do it in pure strategy. In these settings, we show a relationships between the solvability in the sense of Nash and the solvability by the *iterated elimination of never best responses (IENBR)*. We show that a two-person finite game is solved by the IENBR if and only if it is best response acyclic and strongly solvable in the sense of Nash. We prove this for both mixed strategy games and pure strategy games.

A game is solved by IENBR, or IENBR-solvable, if the outcome of IENBR coincides with the set of equilibria. Since the outcome of IENBR is known as the set of profiles of *rationalizable* strategies (Bernheim (1984) and Pearce (1984); to be more precise *point rationalizable* strategies in the sense of Bernheim (1984)), our results show the necessary and sufficient conditions for the rationalizable strategies to be equivalent to the equilibrium strategies in two-person finite games. To see the necessity of these two conditions, consider the following best response graphs of two 2×2 pure strategy games.

$$\begin{tikzcd} 1,0 \arrow[r] & 0,1 \arrow[d] \\ 0,1 \arrow[u] & 1,0 \arrow[l] \end{tikzcd} \qquad\qquad \begin{tikzcd} 1,1 & 0,0 \arrow[l] \arrow[d] \\ 0,0 \arrow[u] \arrow[r] & 1,1 \end{tikzcd}$$

In both games, every strategy is a best response to some strategy and cannot be eliminated by the IENBR. Thus these are the outcomes of IENBR. Now, in the first (left) game, there exists a best response cycle and no equilibrium exists. The second (right) game is best response acyclic and has two equilibria, but they do not constitute a Cartesian product. These two are not IENBR-solvable, as expected. However, if we assume that a game is best response acyclic and strongly solvable in the sense of Nash, both are impossible as the outcomes of IENBR. The IENBR of a best response acyclic and strongly solvable two-person finite game will lead to an outcome that coincides with the set of equilibria. The core of this paper will be a formal proof of this sufficiency.

The rest of the paper is organized as follows. The definitions and known facts are provided in Section 2. In Section 3, we report our results. Some concluding remarks are given in Section 4.

2 Preliminaries

2.1 Games

Let $G^p = (S_1^p, S_2^p; u_1, u_2)$ be a two-person finite normal form game and $G = (S_1, S_2; u_1, u_2)$ its mixed extension, where S_i^p is a finite set of pure strategies, S_i is the set of mixed strategies, and u_i is the payoff function, for each player $i = 1, 2$. Letting $m_i = |S_i^p|$ for $i = 1, 2$, we identify S_i^p with the standard basis of Euclidean m_i-space and S_i with its convex hull. That is, S_i is spanned by S_i^p, and S_i^p is the set of vertices of the $(m_i - 1)$-dimensional simplex S_i, $i = 1, 2$. A face of S_i is a set spanned by a nonempty subset of S_i^p. The payoff function u_i is a bilinear function defined on the set of mixed strategy profiles $S = S_1 \times S_2$ (Cartesian product) by

$$u_i(s_1, s_2) = \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \theta_1^{k_1} \theta_2^{k_2} u_i^p(t_1^{k_1}, t_2^{k_2}),$$

with a payoff function u_i^p on the set of pure strategy profiles $S^p = S_1^p \times S_2^p$, and $\theta_j^{k_j} \in \mathbb{R}$, $t_j^{k_j} \in S_j^p$ such that

$$s_j = \sum_{k_j=1}^{m_j} \theta_j^{k_j} t_j^{k_j},\ 0 \leq \theta_j^{k_j}\ \forall k_j = 1, \ldots, m_j,\ \sum_{k_j=1}^{m_j} \theta_j^{k_j} = 1,\ j = 1, 2.$$

We call G a *mixed strategy game* (or a *game in mixed strategy*) and G^p a *pure strategy game* (or a *game in pure strategy*). For convenience, we use the restriction of u_i to S^p in the same notation for G^p instead of u_i^p.

2.2 Best responses

In a mixed strategy game G, the player one's strategy $s_1 \in S_1$ is a *best response* to the player two's strategy $s_2 \in S_2$ if $u_1(s_1, s_2) \geq u_1(s_1', s_2)$ for all $s_1' \in S_1$. The player two's best response to the player one's strategy is analogously defined. A Nash equilibrium, or an *equilibrium* for short, is a strategy profile $s \in S$ such that s_1 is a best response to s_2 and s_2 is a best response to s_1. Let $i, j = 1, 2$, $i \neq j$, and denote by $\beta_i(s_j)$ the set of best responses of player i to the player j's strategy s_j. Since u_i is linear in s_i, for any $s_j \in S_j$, $\beta_i(s_j)$ is a face of S_i, which is compact and convex. For any s_i in the relative interior of $\beta_i(s_j)$ ($s_i \in \mathrm{ri}(\beta_i(s_j))$ in notation), the set of vertices of $\beta_i(s_j)$ is said to be the support of s_i, which is a set of pure strategies that are used in s_i with positive probability. Deviations from s_i to any strategy in its support do not change the payoff of the deviant. Since u_i is continuous on S, the best response correspondence $s_j \mapsto\to \beta_i(s_j)$ is upper hemicontinuous by Berge's maximum theorem. Define $\beta \colon S \to\to S$ by $\beta(s) = \beta_1(s_2) \times \beta_2(s_1)$. Since β_i are compact-and convex-valued upper hemicontinuous correspondences on a compact and convex set S, so is β. Then there exists an $s \in S$ such that $s \in \beta(s)$ by Kakutani's fixed point theorem, which is an equilibrium of G. The best response and the best response correspondence in the pure strategy game G^p are similarly defined, by restricting the strategy sets S_i to S_i^p, $i = 1, 2$. However, the existence of an equilibrium of G^p is not always guaranteed.

2.3 Best response acyclic games

A *best response path* in G is a finite or infinite sequence of strategy profiles $s^0, s^1, s^2, \dots$ in S such that if s_1^k and s_2^k are not best responses to each other then for some i such that s_i^k is not a best response to s_j^k, we have that s_i^{k+1} is a best response to s_j^k and $s_j^{k+1} = s_j^k$, where $i, j = 1, 2$ and $i \neq j$. Hence, for s^k and s^{k+1} such that $s_i^{k+1} \neq s_i^k$,

$$s_i^{k+1} \in \beta_i(s_j^k) \quad \text{and} \quad u_i(s^{k+1}) > u_i(s^k).$$

A best response path *terminates* at s^k if s_1^k and s_2^k are best responses to each other. A *best response cycle* is an infinite best response path that repeats a finite best response path infinitely. We call G *best response acyclic* if G contains no best response cycle. The best response cycle and the best response acyclicity in a pure strategy game G^p are similarly defined, by restricting the strategy sets S_i to S_i^p, $i = 1, 2$. In both mixed and pure strategy games, an equilibrium exists in its domain of strategies if the game is best response acyclic, because any best response path terminates at some s^k, which is an equilibrium. Note that if G is best response acyclic, so is G^p.

2.4 Solvable games

Let $E(G)$ be the set of equilibria of a mixed strategy game G. $E(G)$ is called the *solution* of G (Nash, 1951) if

$$s \in E(G) \text{ and } s' \in E(G) \implies s'' \in E(G) \ \forall s'' \in \{s_1, s_1'\} \times \{s_2, s_2'\}.$$

That is, $E(G)$ is the solution of G if $E(G) = \pi_1(E(G)) \times \pi_2(E(G))$, where $\pi_i(\cdot)$ is the operator taking the projection to S_i, $i = 1, 2$. Let $E_i(G) = \pi_i(E(G))$, $i = 1, 2$. $E_i(G)$ is called the set of *equilibrium strategies* of i, $i = 1, 2$. $E_i(G)$ are closed and convex (Nash, 1951, Theorem 4). The solution $E(G)$ is a *strong solution* if for any $i, j = 1, 2$ such that $i \neq j$ and for any $s \in E(G)$ and $s' \in S$ such that $s_i' \neq s_i$ and $s_j' = s_j$

$$u_i(s') = u_i(s) \implies s' \in E(G).$$

A game having the (strong) solution is called *(strongly) solvable.* Suppose that G is strongly solvable and $(s_1, s_2) \in E(G)$. Then since deviations to a pure strategy in the support of mixed strategy do not change the payoff of the deviant, $s_1' \in \beta_1(s_2) \cap S_1^p$ satisfies $(s_1', s_2) \in E(G)$, and $s_2' \in \beta_2(s_1') \cap S_2^p$ satisfies $(s_1', s_2') \in E(G)$. Hence G has a pure strategy equilibrium. For a strongly solvable game, the set of equilibrium strategies is the convex hull of the set of equilibrium pure strategies. The (strong) solvability of G^p in the sense of Nash is similarly defined, by restricting the strategy sets S_i to S_i^p, $i = 1, 2$. Note that if G is strongly solvable, so is G^p, and the set of equilibrium strategies $E_i(G)$ is the convex hull of the set of pure equilibrium strategies $E_i(G^p)$, $i = 1, 2$.

2.5 IENBR and the rationalizable strategies

In a mixed strategy game G, the player one's strategy $s_1 \in S_1$ is a *never best response* if for any $s_2 \in S_2$ there exists an $s_1' \in S_1$ such that $u_1(s_1, s_2) < u_1(s_1', s_2)$. The player two's never best response is analogously defined. The *iterated elimination of never best responses* (*IENBR*) is a procedure that successively eliminates the never best responses of the players. Formally, letting $\lambda \colon 2^S \to 2^S$ be the elimination of never best responses defined by $\lambda(B) = (\bigcup_{s \in B} \beta_1(s_2)) \times (\bigcup_{s \in B} \beta_2(s_1))$ for $B \subseteq S$, IENBR reduces S to $P(G) \subseteq S$ such that $P(G) = \bigcap_{k=0}^{\infty} \lambda^k(S)$, where $\lambda^k = \lambda \circ \lambda^{k-1}$ for $k \geq 2$ (Bernheim, 1984). If S_i are compact and u_i are continuous, then $P(G)$ is nonempty and coincides with the maximal set $B \subseteq S$ satisfying $B = \lambda(B)$ (Bernheim, 1984, Proposition 3.1). Clearly $P(G) = \pi_1(P(G)) \times \pi_2(P(G))$. Let $P_i(G) = \pi_i(P(G))$, $i = 1, 2$. We call $P(G)$ (and its components) the *outcome* of IENBR. A strategy $s_i \in P_i(G)$ is said to be *rationalizable* (Bernheim (1984) and Pearce (1984); to be precise *point rationalzable* in Bernheim (1984)). For any $s_i \in P_i(G)$, there exists some $s_j \in P_j(G)$ such that $s_i \in \beta_i(s_j)$, where $i, j = 1, 2$ and $i \neq j$. Note that for any $s_j \in P_j(G)$, we have $\beta_i(s_j) \subseteq P_i(G)$. Since $\beta_i(s_j)$ is a face of S_i, the outcome of IENBR is a finite union of faces of S_i, $i = 1, 2$. The never best response and the IENBR are similarly defined for pure strategy games.

2.6 IENBR-solvable games

Let $G = (S_1, S_2; u_1, u_2)$ be a mixed strategy game. G is *solved by IENBR* (or *IENBR-solvable*) if the outcome $P(G) \subseteq S$ of the IENBR of G coincides with the set of equilibria of G, i.e., if $P(G) = E(G)$. The IENBR-solvability of a pure strategy game is similarly defined: Letting $P(G^p) \subseteq S^p$ be the outcome of the IENBR of G^p, $E(G^p) \subseteq S^p$ the set of equilibria of G^p, G^p is said to be IENBR-solvable if $P(G^p) = E(G^p)$.

3 The results

We first consider the mixed strategy games. Let $G = (S_1, S_2; u_1, u_2)$ be a two-person finite game in mixed strategy.

Lemma 3.1. *If G is solved by IENBR then it is best response acyclic.*

Proof. If there exists a best response cycle in G, which is two-person, then any strategy of player one in the cycle is a best response to some strategy of player two in the cycle, which in turn is a best response to some strategy of player one in the cycle. Thus every strategy in the cycle is not a never best response, and the IENBR cannot remove the cycle. Since there exists a non-equilibrium profile in the cycle (indeed every profile is a non-equilibrium in the cycle), this says that G cannot be solved by IENBR if it has a best response cycle. Hence, if G is solved by IENBR, it has to be best response acyclic. □

Lemma 3.2. *If G is solved by IENBR then G is strongly solvable in the sense of Nash.*

Proof. Since the outcome of IENBR is a Cartesian product, the set of equilibria is a Cartesian product $E = E_1 \times E_2$ if G is solved by IENBR. This shows the solvability of G in the sense of Nash. To see that G is strongly solvable in the sense of Nash, let $s \in E$ and $s' \in S$ be such that $s'_i \neq s_i$, $s'_j = s_j$, and $u_i(s') = u_i(s)$, where $i, j = 1, 2$ and $i \neq j$. Then since s'_i is also a best response to s_j, it cannot be removed by IENBR. Hence, if G is solved by IENBR, it has to be that $s'_i \in E_i$, and $s' \in E$, i.e., G has to be strongly solvable in the sense of Nash. □

Theorem 3.1. *G is solved by IENBR if and only if it is best response acyclic and strongly solvable in the sense of Nash.*

Proof. The necessity part is by Lemmas 3.1 and 3.2. To see the sufficiency part, let $P_i \subseteq S_i$ be the outcome of IENBR for player i and $E_i \subseteq S_i$ the set of equilibrium strategies of player i, which is determined by the solvability in the sense of Nash, $i = 1, 2$. We have $E_i \subseteq P_i$ since $s_i \in E_i$ is not a never best response. We want to show that $P_i \setminus E_i = \emptyset$ for $i = 1, 2$. Let $D_i = P_i \setminus E_i$, $i = 1, 2$. First, observe that if $s_1 \in D_1$, then s_1 is a best response only to some $s_2 \in D_2$, since if it is a best response to some $s'_2 \in E_2$, then $u_1(s_1, s'_2) = u_1(s'_1, s'_2)$ for some $s'_1 \in E_1$, contradicting the strong solvability in the sense of Nash. Thus $D_1 \neq \emptyset$ implies $D_2 \neq \emptyset$. Similarly $D_2 \neq \emptyset$ implies $D_1 \neq \emptyset$. Suppose $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$ by way of contradiction. For each $i = 1, 2$, P_i is a finite union of faces of S_i that survived IENBR. Let $\mathcal{F}_i$ be the set of faces of S_i that survived IENBR. Since $E_i \in \mathcal{F}_i$, let $\mathcal{F}'_i = \mathcal{F}_i \setminus \{E_i\}$. Then $P_i = \bigcup \mathcal{F}_i$ and $D_i = (\bigcup \mathcal{F}'_i) \setminus E_i$. From the observation above, for every $F_i \in \mathcal{F}'_i$, there exists a $F_j \in \mathcal{F}'_j$ such that $F_i = \beta_i(s_j)$ for some $s_j \in \mathrm{ri}(F_j)$ (note that $\mathrm{ri}(F_j) \cap E_j = \emptyset$). Let us denote this relation by $F_j \lhd_i F_i$. We have shown that for every $F_1 \in \mathcal{F}'_1$ there exists $F_2 \in \mathcal{F}'_2$ such that $F_2 \lhd_1 F_1$, and for every $F_2 \in \mathcal{F}'_2$ there exists $F_1 \in \mathcal{F}'_1$ such that $F_1 \lhd_2 F_2$. Then, since $\mathcal{F}'_i$ are finite, there must exist a cycle $F_1^0 \lhd_2 F_2^1 \lhd_1 F_1^2 \lhd_2 \cdots \lhd_2 F_2^{t-1} \lhd_1 F_1^t$, where t is even and $F_1^t = F_1^0$. For each $k = 0, \ldots, t-1$, choose $s_j^k \in \mathrm{ri}(F_j^k)$ such that $\beta_i(s_j^k) = F_i^{k+1}$. (We let $k+1$ be zero if $k = t-1$.) Then for s^k and s^{k+1} such that $s_i^{k+1} \neq s_i^k$,

$$s_i^{k+1} \in \beta_i(s_j^k) \quad \text{and} \quad u_i(s^{k+1}) \geq u_i(s^k).$$

We have three cases.

i) Case $t = 2$: In this case $s_2^1 \in \beta_2(s_1^0)$ and $s_1^0 \in \beta_1(s_2^1)$, i.e., $(s_1^0, s_2^1) \in D_1 \times D_2$ is an equilibrium, contradicting the solvability of G in the sense of Nash.

ii) Case $t > 2$ and $u_i(s^{k+1}) = u_i(s^k)$ for some i such that $s_i^{k+1} \in \beta_i(s_j^k)$ for some k: In this case $s^k \in D_1 \times D_2$ is an equilibrium, contradicting the solvability of G in the sense of Nash.

iii) Case $t > 2$ and $u_i(s^{k+1}) > u_i(s^k)$ for i such that $s_i^{k+1} \in \beta_i(s_j^k)$ for every k: In this case there exists a best response cycle $s_1^0 \to s_2^1 \to s_1^2 \to \cdots \to s_2^{t-1} \to s_1^t$ such that $s_1^t = s_1^0$, contradicting the best response acyclicity of G. □

Before proceeding to the case of pure strategy games, note that if a finite game is strongly solvable in mixed strategy then it is strongly solvable in pure strategy; but *not* vice versa as the two-person game in Figure 1 shows. As a pure strategy game, this game has a unique strict equilibrium (marked by an asterisk), and as such, this is strongly solvable in pure strategy. As a mixed strategy game, however, this game has equilibria $((1,0,0),(1,0,0))$ and $((0,\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2},\frac{1}{2}))$, so this is not even solvable in the sense of Nash. This suggests that the solvability in mixed strategy and in pure strategy are generally different.

$1,1^*$	$0,0$	$0,0$
$0,0$	$0,1$	$1,0$
$0,0$	$1,0$	$0,1$

Figure 1: A game strongly solvable in pure strategy but not in mixed strategy

Let $G^p = (S_1^p, S_2^p; u_1, u_2)$ be a finite two-person pure strategy game. We have results similar to mixed strategy games as follows.

Lemma 3.3. *If G^p is solved by IENBR then it is best response acyclic.*

Proof. Identical to the proof of Lemma 3.1. □

Lemma 3.4. *If G^p is solved by IENBR then G^p is strongly solvable in the sense of Nash.*

Proof. Identical to the proof of Lemma 3.2. □

Theorem 3.2. *G^p is solved by IENBR if and only if it is best response acyclic and strongly solvable in the sense of Nash.*

Proof. Almost identical to the proof of Theorem 3.1. Note that the set of equilibria E^p is nonempty by the best response acyclicity and $E^p = \pi_1(E^p) \times \pi_2(E^p)$ by the solvability in the sense of Nash. Let $E_i^p = \pi_i(E^p)$, where $E_i^p \subseteq S_i^p$, $i = 1, 2$. Let $P_i^p \subseteq S_i^p$ be the outcome of IENBR, $i = 1, 2$. Letting $D_i^p = P_i^p \setminus E_i^p$, $i = 1, 2$, we have $D_1^p \neq \emptyset \iff D_2^p \neq \emptyset$ by the strong solvability in the sense of Nash. Assuming $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$ by way of contradiction, we can choose a cyclic sequence of profiles $s^0, s^1, s^2, \ldots, s^{t-1}, s^t$ in $D^p = D_1^p \times D_2^p$ satisfying for s^k and s^{k+1} such that $s_i^{k+1} \neq s_i^k$,

$$s_i^{k+1} \in \beta_i(s_j^k) \quad \text{and} \quad u_i(s^{k+1}) \geq u_i(s^k).$$

We then have three contradictory cases identical to those in the proof of Theorem 3.1. □

4 Concluding remarks

An example of a two-person game that is best response acyclic and strongly solvable in the sense of Nash is a game with weak payoff externalities by Ania (2008). Let $G = (S_1, S_2; u_1, u_2)$ be a mixed strategy game having the *weak payoff externalities* (*WPE*) such that for any $s, s' \in S$ such that $s'_i \neq s_i$ and $s'_j = s_j$, where $i, j = 1, 2$ and $i \neq j$,

$$|u_i(s') - u_i(s)| > |u_j(s') - u_j(s)|.$$

For instance,

$1, 1$	$0, 3$
$3, 0$	$2, 2^*$

has WPE both in pure and mixed strategy. (Letting $s_i = (\theta_i, 1 - \theta_i)$, $0 \leq \theta_i \leq 1$, $i = 1, 2$, we have $u_1(s) = -2\theta_1 + \theta_2 + 2$ and $u_2(s) = \theta_1 - 2\theta_2 + 2$.) To see that G is best response acyclic, note that the sum of payoffs $u_1 + u_2$ is a *generalized ordinal potential function* (Monderer and Shapley, 1996) of G satisfying for any $s, s' \in S$ such that $s'_i \neq s_i$ and $s'_j = s_j$

$$u_i(s') > u_i(s) \implies (u_1 + u_2)(s') > (u_1 + u_2)(s).$$

G is thus a generalized ordinal potential game, which is contained in the class of best response acyclic games. To see that G is strongly solvable, let $s \in E(G)$. Then since $u_i(s') \neq u_i(s)$ for any $s' \in S$ such that $s'_i \neq s_i$ and $s'_j = s_j$, the strong solvability condition $u_i(s') = u_i(s) \implies s' \in E(G)$ is vacuously satisfied.

We have shown the necessary and sufficient conditions of IENBR-solvability in two-person finite games. The two conditions are independent. To see that acyclicity does not imply the strong solvability, it suffices to recall the second example in the introduction, which is of a coordination type. To see that strong solvability does not imply the acyclicity, consider the two-person 3×3 game in Figure 2. This game is strongly solvable (both in pure and mixed) but not best response acyclic (hence not solved by IENBR).

$2, 2^*$	$1, 0$	$1, 0$
$0, 1$	$2, -2$	$-2, 2$
$0, 1$	$-2, 2$	$2, -2$

Figure 2: A strongly solvable game that is not best response acyclic

Concerning the extension of our results to more-than-two-person games, we note that Lemmas 3.2 and 3.4, namely, the necessity of the strong solvability in the sense of Nash, are easily extended to n-person case. The best response acyclicity is not necessary in n-person game if $n > 2$, as the three-person $2 \times 2 \times 2$ game in Figure 3 shows. Also, as the three-person $2 \times 2 \times 2$ game in Figure 4 shows, the best response acyclicity and the strong solvability in the sense of Nash are not sufficient for the IENBR-solvability of more-than-two-person games.

$3, 3, 3^*$	$2, 2, 3$		$3, 3, 2$	$3, 2, 2$
$2, 3, 3$	$2, 2, 3$		$4, 1, 2$	$2, 2, 2$

Figure 3: A three-person game that is solved by IENBR but not acyclic

$1, 0, 1$	$0, 1, 1$		$0, 1, 0$	$0, 0, 0$
$0, 0, 1$	$1, 1, 0$		$1, 0, 0$	$1, 1, 1^*$

Figure 4: A three-person game that is acyclic and strongly solvable but not solved by IENBR

As a final remark, we note on the computational aspect of our games. Let $G = (S_1, S_2; u_1, u_2)$, a mixed strategy game. If $s_i \in S_i$ is a never best response and F_i is a face of S_i such that $s_i \in \mathrm{ri}(F_i)$, then any $s'_i \in \mathrm{ri}(F_i)$ is a never best response. Thus the elimination of never best responses λ eliminates the relative interiors of the faces in one call, and since the number of faces is finite, we have indeed $P(G) = \bigcap_{k=0}^{T} \lambda^k(S)$ for some finite T. Still, the number of faces of S_i is $2^{m_i} - 1$ ($m_i = |S_i^p|$), so T can be exponential in the number of pure strategies. Also deciding whether or not a mixed strategy is a never best response may be hard. Recall that if G is strongly solvable then the set of equilibrium strategies $E_i(G)$ is the convex hull of $E_i(G^p)$. Thus, if G is strongly solvable and best response acyclic, we can solve it by the IENBR in pure strategy, because G^p is also strongly solvable and best response acyclic: the convex hull of the outcome is $E(G)$. It is known that the outcome of IENBR in pure strategy is independent of the order of elimination (Apt (2005)). Given this order independence, if both players have m pure strategies, then deciding whether or not a pure strategy is a never best response consumes at most $m(m-1)$ times of payoff evaluations, so $m^2(m-1)$ per player, and $2m^2(m-1)$ per game, which is of the order $O(m^3)$. Thus the strongly solvable best response acyclic two-person finite game is solved by the IENBR, in pure strategy with complete set of mixed strategy equilibria, in polynomial time.

References

Ania, A.B. (2008), Evolutionary stability and Nash equilibrium in finite populations, with an application to price competition, *Journal of Economic Behavior and Organization*, 65:472–88.

Apt, K. R. (2005), Order independence and rationalizability, Proceedings of the 10th conference on Theoretical aspects of rationality and knowledge, 22–38.

Bernheim, B. D. (1984), Rationalizable strategic behavior, *Econometrica*, 52:1007–1028.

Friedman, J. W. (1983), On characterizing equilibrium points in two person strictly competitive games, *International Journal of Game Theory*, 12:245–247.

Kats, A. and Thisse, J.-F. (1992), Unilaterally competitive games, *International Journal of Game Theory*, 21:291–9.

Monderer, D. and Shapley, L. S. (1996), Potential games, *Games and Economic Behavior*, 14:124–143.

Nash, J. (1951), Non-cooperative games, *Annals of Mathematics*, 54:286–295.

Pearce, D. G. (1984), Rationalizable strategic behavior and the problem of perfection *Econometrica*, 52:1029–1050.