Research Paper Series

No. 119

Risk evaluation of a portfolio including forward-looking stress events with probabilities

Yukio Muromachi†

January, 2013

† Graduate School of Social Sciences, Tokyo Metropolitan University

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Yukio Muromachi

Graduate School of Social Sciences, Tokyo Metropolitan University
1-1 Minami-Ohsawa, Hachiohji, Tokyo 192-0397, Japan

January 11, 2013

Abstract

After the recent worldwide financial crisis, it is pointed out that the stress tests, especially the tests based on the forward-looking stress scenarios, are very important for the financial risk management. However, since the stress tests do not have clear statistical meanings, we can know the characteristics of the risk included in the portfolio from the results, but cannot obtain fruitful probabilistic information such as VaR (Value at Risk). In this article, we propose a risk evaluation model for a portfolio including stress events with probabilities implied from the market data. Our model is based on the implied copula model proposed by Hull and White (2006) for pricing CDOs (Collateralized Debt Obligations). We apply the implied copula model to the risk evaluation of a portfolio by using a framework for constructing a risk evaluation model proposed by Kijima and Muromachi (2000). The "sub-filtration approach" for modeling credit risk is used in our model, and the conditional independence of default times is assumed. One key point is the stochastic modeling of default probabilities, and under some assumptions, the distributions of the default probabilities can be calibrated from the liquid CDO tranche prices. According to Hull and White (2006), even in the prosperous period there existed small probability mass in the extremely high default probability regions, which reflected the latent fear of the market participants against catastrophic default events. We construct a simple one-period model, and calculate the loss distributions of CDO tranches and a bond portfolio numerically. The results show that the small probability mass in the extremely high default probability region has strong influences on the risk measures such as VaR, and additionally show that most of the losses derives from the crash of the CDO prices, not from the actual default losses. We think that our method is one of the possible ways to connect the existing statistical models with the stress tests.

Key words: risk evaluation model, implied copula, market-implied stress scenarios.

1 Introduction

In the last several years a lot of financial institutions have suffered a great amount of losses all over the world, and some of the great companies have disappeared. In the progress of this worldwide crisis, the financial technology of the securitization and credit derivatives played very important roles on the magnification and spread of this financial disaster. See, for example, Brigo et al.[3], Crouhy et al.[5], Hull[7] and so on. The crisis clarifies some problems on the financial markets and institutions, especially on the risk management in the financial institutions and financial systems. It is regretful that no existing risk evaluation models, such as RiskMetrics$^{\mathrm{TM}}$[15], CreditMetrics$^{\mathrm{TM}}$[14] and CREDITRISK^{+}[4], can predict the occurrence of this crisis to serve notice in advance. However, it is quite natural because they are purely statistical models based on the historical data, so that it is impossible to predict the crisis if the market has not experienced any crises yet such as the securitisation market which is near the source of this crisis. After the crisis, many people say that the forward-looking risk evaluations, especially the forward-looking stress tests, are very important. However, the stress tests have some fatal problems. For example, it is difficult to make plausible stress scenarios, and moreover, even if we can set plausible stress scenarios, the scenario-based stress tests have no clear statistical meanings. Therefore, we cannot discuss the results obtained from the stress tests with those from the statistical risk evaluation models.

On the other hand, it is widely said that the market prices of the financial products, especially, the prices of the derivatives include the forward-looking information or insights. For example, in the options market, the implied volatilities calculated from the market prices are apparently different from the historical volatilities, which reflect the insights of the major market participants in future. In the CDO (Collateralized Debt Obligations) markets, the market prices of some liquid CDO tranches are too high to be explained by the observed data and the standard pricing techniques even in the prosperous period. Such high prices can be interpreted that there exists a small probability of the catastrophic loss. These examples imply that we can derive some forward-looking information from derivatives prices, and can use them for the risk evaluation.

In this article we propose a new risk evaluation model which belongs to the statistical models, and simultaneously, some forward-looking information derived from the market prices of derivatives is also used in the statistical evaluation. That is, the latent fear of the major market participants is reflected in the evaluation of the model. In order to construct such a new risk model, we use Hull and White's implied copula framework[10] and Kijima and Muromachi's framework[16].

The implied copula model was proposed by Hull and White[10] in order to explain the market prices of the CDO tranches, especially the tranches of the synthetic CDO whose reference assets are CDSs (Credit Default Swaps). The standard model for pricing synthetic CDO is the one-factor Gaussian copula model. In the model, the joint distribution of the default times of the reference entities is described by using the Gaussian copula and the marginal distributions, and the variance-covariance matrix of the Gaussian copula is constructed by a simplest multivariate normal variance mixture model, that is, one-factor latent normal variable model. However, the market prices of all CDO tranches cannot be explained simultaneously by the one-factor Gaussian copula model, and it is widely known that the implied correlations (the correlation coefficient which makes the theoretical price of a tranche equal to the market price when this model is used) of the tranches are different each other [1]. Therefore, many researchers have proposed various kinds of CDO pricing models to fit with the market prices of the tranches well. One of them is the Hull and White's implied copula model.

The implied copula model does not use a parametric copula such as Gaussian copula, Clayton copula, and so on. Its essences are the conditional independence (defined later) of the default times of the entities and the comonotonic feature of the their default probabilities. In the model, stochastic variable N, $N \in \{1, \cdots, K\}$, denotes the credit state of the future environment, and on the N-th state, the hazard rate of the i-th entity of the CDO at time t is given by $h_{i,N}(t) = a_N(t)h_i(t)$ where $a_j(t)$ and $h_i(t)$ are some functions of i, j and time t. Given N, the term structures of the default probability for i-th entity are given as $1 - \exp\{-\int_0^t h_{i,N}(s)\mathrm{d}s\}$, which means the comonotonic default probabilities, and each asset defaults independently according its default probability. Therefore, it can be said that the implied copula model belongs to the category of the conditionally independent default models. The calibration of this model is possible from the market data. The hazard rates $h_i(t)$ are calibrated from the single-name CDS spreads, and the distribution of $a_N(t)$ can be from the liquid CDO (Index Tranches) spreads. Hull and White[10] assumed that the hazard rates are the same for all entities, and described the distribution of the hazard rates as a non-parametric manner. They calibrated numerically the distribution so that the theoretical prices of the CDO tranches could be consistent with the market prices. The calibrated distribution of the hazard rates had a fat tail, and it was remarkable that there was a small probability mass in the very high range [2].

[1] This fact is well-known and called as "correlation smile" or "correlation skew".

[2] If you would like to see the calibration results in the crisis for comparing those before the crisis, we

Another important tool is Kijima and Muromachi's framework, which was proposed for constructing synthetic risk evaluation models of credit and market risks of a portfolio. In their framework, the risk factors such as interest rates and stock prices are described by the stochastic differential equations, and the no-arbitrage prices are used as the valuation of assets at present and at future, so that the risk measures such as Value at Risks (VaRs) and Expected Shortfalls (ESs) are calculated from the distribution of the future price of the portfolio at a certain future time T, which is called the risk horizon. The remarkable feature of this framework is the use of the two probability measures; the physical (or statistical) probability measure and the pricing probability measure such as the risk-neutral probability measure. This is because the physical probability measure is needed for generating a lot of future scenarios from the present state, and the pricing probability measure is needed for pricing assets at present and at future. The image of the use of two measures is described in Figure 1. For example, consider a risk evaluation of a CDO tranche at a future time T. The present no-arbitrage price of the tranche is calculated as the expectation of the discounted present values of the future cashflows under the risk-neutral probability measure. On the other hand, the default loss up to time t is evaluated under the physical probability measure, and the no-arbitrage price of the tranche at T is also calculated as the conditional expectation of the discounted values at T of all the cashflows after T under the risk-neutral probability measure.

Probably, there exist some critics that the risk evaluation model used for the risk management should be constructed based on the historical data under the physical probability measure, especially some practitioners would think so rigidly. In this context, our proposed model cannot be accepted. However, in order to construct more desirable risk evaluation models, it is desirable that we will use not only the historical data of the risk factors but also the market prices of derivatives which include some forward-looking information or insights. Remember that a small probability of severe loss is always detected in the CDO market prices before this financial crisis (although the correlation smile of the CDO tranches might be created by other reasons, for example, different kinds of investors buy different tranches). By using liquid market price data, for example, prices of derivatives, in order to derive information not included in the usual historical data, we can propose new risk evaluation models including implicitly some catastrophic events against which most of the traders feel fears. Such events might be called as "market-implied stress scenarios," and additionally, it might be possible that their probabilities would be evaluated by some theoretically consistent ways. Then, we can include their effects on the existing statistical risk evaluation models. We think

recommend to read Brigo et al.[3].

that this might be one of the possible ways to connect the statistical models with the stress tests.

This article is organized as follows. Section 2 describes the basic setting of the proposed risk evaluation model. Especially, in order to show a single-period model which can be used easily in practice, we make some assumptions. In Section 3 and Section 4, we show two numerical examples; the former is a risk evaluation of CDO tranches, and the latter is of a bond portfolio. Section 5 concludes this article with some remarks.

2 The Model

In this article, we consider risk evaluation models of a portfolio subject to the market risk and the credit risk. In the risk evaluation models there exist two types; one is based on the future value of the portfolio, and the other is based on the potential loss. The market risk evaluation models belong to the former, and many credit risk evaluation models belong to the latter. The proposed model belongs to the former, in which the risk measures such as VaR (Value at Risk) and ES (Expected Shortfall) are calculated based on the distribution of the future value of the portfolio.

In order to construct such evaluation models, we use the framework proposed by Kijima and Muromachi[16]. They gave the stochastic differential equations (hereafter, abbreviated by SDEs) to describe the dynamics of the state variables in the financial market, for example, interest rates, stock prices and default probabilities, and they imposed some relations between the SDEs under the physical probability measure and the pricing measure, such as the risk-neutral probability measure. In their framework, many future scenarios are generated up to the risk horizon according to the SDEs under the physical probability measure by the Monte Carlo simulation, and all the assets are evaluated by the no-arbitrage prices at the present and at the risk horizon.

In this section, we describe the stochastic structure and main assumptions for a simple single-period model [3]. Here, we use a "sub-filtration approach" called in credit risk modeling.

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where P is the physical (or statistical) probability measure. The time is denoted by t, $t \geq 0$. Here, $t = 0$ means the present, and the risk horizon is T, $T > 0$. Suppose that there are n assets, and consider the credit and interest rate risks of the portfolio consisting of these n assets, with the holding amount w_j of the j-th asset, $j = 1, \cdots, n$, up to the time horizon T. We assume that there uniquely exists the risk-neutral probability measure $\widetilde{P}$, which is equivalent to the physical

[3] We are now preparing an article describing more generalized version of our model.

probability measure P up to time T^*, $T^* \geq \max_j T_j$, where T_j is the maturity of the j-th asset if it exists. The price processes of the assets without maturities such as stocks are assumed to be $\mathcal{F}_t$-adapted up to T^*. These settings are necessary to evaluate the prices of all assets.

The default time of the j-th asset is denoted by τ_j, $j = 1, \cdots, n$ and $\tau_j > 0$. The default process is defined by $H_j(t) = 1_{\{\tau_j \leq t\}}$, $j = 1, \cdots, n$ where 1_A is an indicator function, that is, $1_A = 1$ when the event A is true and $1_A = 0$ otherwise. A filtration generated by the default process $H_j(t)$ is denoted by $\mathcal{H}_t^j$, that is, $\mathcal{H}_t^j = \sigma(H_j(s), 0 \leq s \leq t)$, and the filtration $\mathcal{H}_t$ is defined by $\mathcal{H}_t = \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^n$.

The filtration $\mathcal{F}$ is divided into $\mathcal{G}$ and $\mathcal{H}$ such that $\mathcal{F} = \mathcal{G} \vee \mathcal{H}$, i.e., $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$ for any $t \in [0, T^*]$. As defined before, $\mathcal{H}$ is the filtration which corresponds to the information about the default times of the n assets, and $\mathcal{G}$ is the filtration which corresponds to the information other than the default times. This setting means that the information about the default probabilities are included in $\mathcal{G}$. We describe the detailed contents of the filtration $\mathcal{G}$ below. And, in this article, we assume that all the filtrations satisfy the usual conditions. We say that a filtration $\mathcal{K} = (\mathcal{K}_t)_{t\geq 0}$ satisfies the usual conditions if it is right-continuous and $\mathcal{K}_0$ contains all the P-negligible events in $\mathcal{K}$.

2.1 Default-free interest rate process

In our setting, we can use any kinds of interest rate processes proposed before, for example, Heath-Jarrow-Morton's forward rate processes[6], market models such as Brace et al.[2] and Jamshidian[12], and so on. However, for simplicity, we use a Hull-White model[8], that is, the extended Vasicek model. Suppose that the default-free instantaneous short rate at time t, $r(t)$, under the physical probability measure P follows the SDE

$$\mathrm{d}r(t) = (b_0(t) - a_0 r(t))\mathrm{d}t + \sigma_0 \mathrm{d}z_0(t), \qquad t \geq 0 \tag{2.1}$$

where a_0 and σ_0 are positive constants, $b_0(t)$ are deterministic functions of time t, and $z_0(t)$ is a standard Brownian motion under P. By introducing the market price of risk $\beta(t)$, the short rate process under the risk-neutral probability measure $\widetilde{P}$ can be obtained. Let

$$\phi(t) = b_0(t) - \beta(t)\sigma_0,$$

then, the resulting SDE under $\widetilde{P}$ is given by

$$\mathrm{d}r(t) = (\phi(t) - a_0 r(t))\mathrm{d}t + \sigma_0 \mathrm{d}\widetilde{z}_0(t), \qquad 0 \leq t \leq T^*$$

where $\widetilde{z}(t)$, defined by

$$\mathrm{d}\widetilde{z}_0(t) = \mathrm{d}z_0(t) + \beta(t)\mathrm{d}t$$

is a standard Brownian motion under $\widetilde{P}$.

The time t price of the default-free discount bond maturing at time τ, $0 \le t \le \tau$, is given by

$$v_0(t,\tau) = \widetilde{E}_t\left[\exp\left\{-\int_t^\tau r(s)\mathrm{d}s\right\}\right] = A(t,\tau)e^{-B(a_0,t,\tau)r(t)} \tag{2.2}$$

where $\widetilde{E}_t$ is the conditional expectation operator given $\mathcal{G}_t$ under $\widetilde{P}$, and

$$A(t,\tau) = \exp\left\{\frac{\sigma_0^2}{2a_0^2}\left(\tau - t - 2B(a_0,t,\tau) + B(2a_0,t,\tau)\right) - \int_t^\tau \phi(u)B(a_0,u,\tau)\mathrm{d}u\right\}$$

and

$$B(a,t,\tau) = \frac{1-e^{-a(\tau-t)}}{a}.$$

2.2 Hazard rate and default processes

The hazard rate (intensity) for the default of the j-th asset at time t, $t \ge 0$, is defined by

$$h_j(t) \equiv \lim_{\Delta t \downarrow 0} \frac{P\{\tau_j \le t + \Delta t | \tau_j > t\}}{\Delta t}$$

and we use Hull-White model as the hazard rate process. That is, suppose that the hazard rates $h_j(t)$ follow the SDEs

$$\mathrm{d}h_j(t) = (b_j(t) - a_j h_j(t))\mathrm{d}t + \sigma_j \mathrm{d}z_j(t), \qquad t \ge 0, \quad j = 1, \cdots, n \tag{2.3}$$

where a_j and σ_j are positive constants, $b_j(t)$ are deterministic functions of time t, and $z_j(t)$ are standard Brownian motions under P. The correlations between the Brownian motions under P are assumed to be constant, $\mathrm{d}z_j(t)\mathrm{d}z_k(t) = \rho_{jk}\mathrm{d}t$, $j,k = 0,\cdots,n$, and for simplicity, $\rho_{j0} = 0$ for all j [4].

Under the risk-neutral probability measure $\widetilde{P}$, we suppose that the hazard rate $\widetilde{h}_j(t)$ follow the SDEs

$$\mathrm{d}\widetilde{h}_j(t) = (\phi_j(t) - a_j\widetilde{h}_j(t))\mathrm{d}t + \sigma_j \mathrm{d}\widetilde{z}_j(t), \qquad t \ge 0, \quad j = 1, \cdots, n$$

[4] The setting $\rho_{j0} = 0$ can be relaxed easily. See Kijima and Muromachi[17].

where $\phi_j(t)$ are deterministic functions of time t, and $\widetilde{z}_j(t)$ are standard Brownian motions under $\widetilde{P}$. Then, there exist predictable processes $\ell_j(t)$ satisfying

$$\widetilde{h}_j(t) = h_j(t) + \ell_j(t), \qquad t \geq 0, \quad j = 1, \cdots, n \tag{2.4}$$

and we call $\ell_j(t)$ as the risk premia adjustments. Notice that $\ell_j(0) \neq 0$ in general.

Consider a defaultable discount bond with maturity τ, $0 \leq \tau \leq T^*$, issued by firm j. Let δ_j be the recovery rate of this bond, and δ_j be constant. We assume that at time τ the holder of this bond receives either δ if it defaults before τ or 1 if it does not. This assumption is consistent with the setting used in Jarrow and Turnbull[13]. Then, the present price of this discount bond, $v_j(0, \tau)$, is given by

$$\begin{aligned} v_j(0,\tau) &= \widetilde{E}\left[\exp\left\{-\int_0^\tau r(s)\mathrm{d}s\right\}\left(1_{\{\tau_j>\tau\}} + \delta_j 1_{\{\tau_j\leq\tau\}}\right)\right] \\ &= v_0(0,\tau)\left[\delta_j + (1-\delta_j)\widetilde{P}\{\tau_j > \tau\}\right] \end{aligned} \tag{2.5}$$

where $\widetilde{P}\{\tau_j > \tau\} = \widetilde{P}_0\{\tau_j > \tau\}$ and

$$\widetilde{P}_t\{\tau_j > \tau\} = \widetilde{E}_t\left[\exp\left\{-\int_t^\tau \widetilde{h}_j(s)\mathrm{d}s\right\}\right] = A_j(t,\tau)e^{-B(a_j,t,\tau)\widetilde{h}_j(t)} \tag{2.6}$$

and

$$A_j(t,\tau) = \exp\left\{\frac{\sigma_j^2}{2a_j^2}\left(\tau - 2B(a_j,0,\tau) + B(2a_j,0,\tau)\right) - \int_0^\tau \phi_j(u)B(a_j,u,\tau)\mathrm{d}u\right\}.$$

More generally, the time t price of the discount bond, $v_j(t, \tau)$, $0 \leq t \leq \tau$, is given by

$$\begin{aligned} v_j(t,\tau) &= 1_{\{\tau_j\leq t\}}\delta_j v_0(t,\tau) + 1_{\{\tau_j>t\}}\widetilde{E}_t\left[\exp\left\{-\int_t^\tau r(s)\mathrm{d}s\right\}\left(1_{\{\tau_j>\tau\}} + \delta_j 1_{\{\tau_j\leq\tau\}}\right)\right] \\ &= v_0(t,\tau)\left[\delta_j + 1_{\{\tau_j>t\}}(1-\delta_j)\widetilde{P}_t\{\tau_j > \tau\}\right]. \end{aligned} \tag{2.7}$$

Notice that the distribution function of the default time, $\widetilde{P}\{\tau_j \leq \tau\} = 1 - \widetilde{P}\{\tau_j > \tau\}$, is what is called the (cumulative) default probability under $\widetilde{P}$ in practice.

Here, define the filtration $\mathcal{G}'$ as the information of the above mentioned stochastic processes, that is, $\mathcal{G}'_t = \sigma(z_0(s), (z_j(s), \ell_j(s), j = 1, \cdots, n); 0 \leq s \leq t)$. Given $\mathcal{G}'_t$, the sample paths of the short rate $r(s)$, $0 \leq s \leq t$ and the hazard rates $h_j(s)$, $\widetilde{h}_j(s)$, $0 \leq s \leq t$, $j = 1, \cdots, n$ are determined uniquely. In Sections 2.1 and 2.2 we can replace $\mathcal{G}$ to $\mathcal{G}'$. For example, we can interpret $\widetilde{E}_t[\cdot] = \widetilde{E}[\cdot|\mathcal{G}_t] = \widetilde{E}[\cdot|\mathcal{G}'_t]$. However, in Section 2.3 we define the filtration $\mathcal{G}$ definitely, and after the definition we have to distinguish the two filtrations, $\mathcal{G}$ and $\mathcal{G}'$, clearly.

2.3 Main assumptions for a simplified model

In this subsection, we describe the main assumptions for our risk evaluation model based on Hull and White's implied copula model and Kijima and Muromachi's framework. Remember that the original implied copula model is for pricing CDO tranches, and that the remarkable feature of Kijima and Muromachi's framework is, as described in Section 1, the use of two probability measures: the physical probability measure P for generating future scenarios, and the risk-neutral probability measure $\widetilde{P}$ for pricing at present and at future.

Here we construct a simplified model. Main principles for the simplification are the followings. First, we consider a single-period model. In the single-period model, it is not necessary to consider the transitions between the different states, so that the mathematical descriptions of the basic equations of the model and the calibrations of model parameters become very simple. Second, we assume the conditional independence of the default times. This assumption is used in the implied copula model. Adding some constraints to the model parameters, we construct a model in which the default times are conditionally independent not only under the risk-neutral probability measure $\widetilde{P}$ but also under the physical probability measure P. Third, in the implied copula model, the default probabilities in each state are described by the mean hazard rates multiplied by a factor (called "multiplier" later) given the state; on the other hand, we give the default probabilities in each state as the mean default probabilities multiplied by a factor given the state. Our setting also simplifies the mathematical descriptions of the model, and make the calculation of the future prices of assets much easy. In the following subsections, we describe many concrete advantages derived from the three principles.

2.3.1 Assumptions under the risk-neutral probability measure $\widetilde{P}$

First, we describe the assumptions for pricing. Suppose that the risk horizon T, $0 \leq T \leq T^*$, is fixed. Under the risk-neutral probability measure $\widetilde{P}$, the unconditional cumulative default probability, that is, the unconditional distribution function of the default time τ_j, is denoted by $\widetilde{F}_j(t) = \widetilde{P}\{\tau_j \leq t\}$, and the forward default probability at time t on the event $\{\tau > s\}$ is given by

$$\widetilde{F}_j(s,t) = \widetilde{P}\{\tau_j \leq t | \tau_j > s\} = \frac{\widetilde{P}\{s < \tau_j \leq t\}}{\widetilde{P}\{\tau_j > s\}}, \ 0 \leq s \leq t \leq T^*.$$

Given $\mathcal{G}'_u$, $0 \leq u \leq T^*$, the conditional cumulative default probability at time t is denoted by $\widetilde{F}_j(t|\mathcal{G}'_u) = \widetilde{P}\{\tau_j \leq t|\mathcal{G}'_u\}$, and the conditional forward default probability at time t on the event $\{\tau > s\}$ is given by

$$\widetilde{F}_j(s,t|\mathcal{G}'_u) = \frac{\widetilde{P}\{s < \tau_j \le t|\mathcal{G}'_u\}}{\widetilde{P}\{\tau_j > s|\mathcal{G}'_u\}},\ 0 \le s \le t \le T^*.$$

Apparently, we have $\widetilde{F}_j(t) = \widetilde{F}_j(t|\mathcal{G}'_0)$, and $\widetilde{F}_j(s,t) = \widetilde{F}_j(s,t|\mathcal{G}'_0)$ on the event $\{\tau_j > s\}$.

Definition 2.1. Suppose $\mathcal{K}$ be a filtration under a probability measure Q. Stochastic variables $\tau_1, \cdots, \tau_n$ are said to be $\mathcal{K}$-conditionally independent under Q when the following equation

$$Q\{\tau_1 \le t_1, \cdots, \tau_n \le t_n|\mathcal{K}_T\} = \prod_{j=1}^{n} Q\{\tau_j \le t_j|\mathcal{K}_T\}, \qquad 0 \le \max_j t_j \le T \tag{2.8}$$

is satisfied.

The conditional independence of default times is assumed in the implied copula model, and we also use the assumption. And, similarly to the implied copula model, we introduce one stochastic variable N.

Assumption 2.1. There exists an integer-valued stochastic variable N, $N \in \{1, \cdots, K\}$, and N is independent of $\mathcal{G}'$.

Here, we denote $\sigma(N)$ as the smallest filtration generated by N, and define a new filtration $\mathcal{G} \equiv \mathcal{G}' \vee \sigma(N)$, that is, $\mathcal{G}_t = \mathcal{G}'_t \vee \sigma(N)$ for any $t \in [0, T^*]$.

Assumption 2.2. Under the risk-neutral probability measure $\widetilde{P}$, the default times are $\mathcal{G}$-conditionally independent.

Assumption 2.3. Given $\mathcal{G}_t$, the conditional forward cumulative default probability up to time τ on $\{\tau_j > t\}$, $j = 1, \cdots, n$ under $\widetilde{P}$ is given by

$$\widetilde{F}_j(t,\tau|\mathcal{G}_u) = \widetilde{F}_j(t,\tau|\mathcal{G}'_u \vee \sigma(N)) = \widetilde{\kappa}(N)\widetilde{F}_j(t,\tau|\mathcal{G}'_u), \qquad 0 \le t \le \tau \le T^*,\ 0 \le u \le T^*$$

where $\widetilde{\kappa}(\cdot)$ is a positive function.

Assumption 2.3 shows that the stochastic variable N displays various kinds of economic states. From Assumptions 2.1-2.3, given $\mathcal{G}'_u$, each asset on survival at time t defaults independently according to its conditional forward cumulative default probability. According to (2.8), we obtain

$$\widetilde{F}_j(t,\tau|\mathcal{G}'_u) = 1 - \exp\left\{-\int_t^\tau \widetilde{h}_j(s)\mathrm{d}s\right\}, \qquad 0 \le t \le \tau \le u \le T^*$$

which is calculated from the sample path $\{\widetilde{h}_j(s),\, 0 \le s \le \tau\}$ [5]. And, according to the tower property of the conditional expectation, we have

$$\widetilde{F}_j(t,\tau|\mathcal{G}'_t) = 1 - \widetilde{E}\left[\exp\left\{-\int_t^\tau \widetilde{h}_j(s)\mathrm{d}s\right\}\middle|\,\mathcal{G}'_t\right], \qquad 0 \le t \le \tau \le T^*.$$

And, given $\mathcal{G}_u$, we obtain

$$\widetilde{F}_j(t,\tau|\mathcal{G}_u) = \widetilde{\kappa}(N)\widetilde{F}_j(t,\tau|\mathcal{G}'_u), \qquad 0 \le t \le \tau \le u \le T^*$$

and, according to the tower property of the conditional expectation, we have

$$\widetilde{F}_j(t,\tau|\mathcal{G}_t) = \widetilde{\kappa}(N)\widetilde{F}_j(t,\tau|\mathcal{G}'_t), \qquad 0 \le t \le \tau \le T^*.$$

The stochastic variable N has a probability function $\widetilde{\eta}(k) \equiv \widetilde{P}\{N = k\}$, which means that the following relations

$$\sum_{k=1}^{K} \widetilde{\kappa}(k)\widetilde{\eta}(k) = 1 \tag{2.9}$$

$$\sum_{k=1}^{K} \widetilde{\eta}(k) = 1 \tag{2.10}$$

$$\widetilde{\eta}(k) \ge 0, \qquad k = 1, \cdots, K \tag{2.11}$$

are satisfied.

Hereafter, the positive function $\widetilde{\kappa}(\cdot)$ is called a "multiplier" under $\widetilde{P}$. Based on Assumption 2.1 and 2.3, we consider K regimes which determine the default probabilities of all assets. Then, $\mathcal{G}'_u$-measurable $\widetilde{F}_j(t,\tau|\mathcal{G}'_u)$ means the conditional forward default probability averaged on N, that is,

$$\widetilde{F}_j(t,\tau|\mathcal{G}'_u) = \widetilde{E}[\widetilde{F}_j(t,\tau|\mathcal{G}_u)|\mathcal{G}'_u] = \widetilde{E}[\widetilde{\kappa}(N)\widetilde{F}_j(t,\tau|\mathcal{G}'_u)|\mathcal{G}'_u] = \sum_{k=1}^{K} \widetilde{\eta}(k)\widetilde{\kappa}(k)\widetilde{F}_j(t,\tau|\mathcal{G}'_u).$$

The stochastic hazard rates in Section 2.2 are used for determining $\widetilde{F}_j(t,\tau|\mathcal{G}'_t)$. However, this single-period model does not belong to the regime-switching models because the transition of N is not considered with time t in this article.

We can assume $\widetilde{\kappa}_j(\cdot)$ instead of $\widetilde{\kappa}(\cdot)$ in Assmption 2.3, that is, the multiplier function differs in each asset. But, such assumption makes it difficult drastically to estimate model parameters from the observed data. Since the estimation of $\widetilde{\kappa}(\cdot)$ is one of the critical issues in our model, here we set this tractable assumption for practical use.

[5] For the detailed features of the conditional independence, see Bielecki and Rutkowski[1].

These assumptions are similar to those used in the implied copula model proposed by Hull and White[10]. The remarkable difference between their model and ours is how to describe the default probabilities. Hull and White[10] assumed that all the hazard rates are equal, and a common distribution of the multiplier $(\widetilde{\kappa}(k), \widetilde{\eta}(k))_{k=1,\cdots,K}$ is used for all entities. In each scenario, they use the mean hazard rates multiplied by a "multiplier" for default probabilities. However, when each asset has a different hazard rates, that is, heterogeneous default probability case, some modifications are needed in their model because the following equations

$$\exp\left\{-\int_t^\tau \widetilde{h}_j(s)\mathrm{d}s\right\} = \sum_{k=1}^K \widetilde{\eta}(k)\exp\left\{-\int_t^\tau \widetilde{\kappa}(k)\widetilde{h}_j(s)\mathrm{d}s\right\}, \qquad j = 1,\cdots,n$$

are not satisfied simultaneously where $\widetilde{h}_j(t)$ is the mean hazard rates averaged on N. On the other hand, as you see above, we use the mean cumulative default probabilities multiplied by a "multiplier" for default probabilities in each scenario, so that we can describe the heterogeneous default probability case quite easily. However, to make up for the advance, the following constraint must be imposed on our setting.

Remark 2.1. Since $\widetilde{F}_j(t,\tau|\mathcal{G}_\tau)$ is a probability, the following relation

$$0 \le \widetilde{F}_j(t,\tau|\mathcal{G}_\tau) \le 1, \qquad j = 1,\cdots,n \tag{2.12}$$

must be satisfied. Or, from (2.12), the multiplier must satisfy

$$0 < \widetilde{\kappa}(k) \le \frac{1}{\max_{\mathcal{G}'_\tau} \widetilde{F}_j(t,\tau|\mathcal{G}'_\tau)}.$$

It might not be so easy to satisfy the constraint in Remark 2.1 rigidly under the stochastic hazard rate modelings because the maximum of all the sample paths is included. For example, in the normally distributed hazard rate case, the maximum of the cumulative hazard rates is not unlimited, so that the maximum of the conditional default probability in each sample path might converge to one. Then, according to the Remark 2.1, it implies $0 < \widetilde{\kappa}(k) \le 1$ for all k.

Hull and White[11] proposed another extended version of the implied copula model, in which the logarithms of the hazard rates is assumed to be subject to the shifted t-distribution. Their model can be adopted not only for the homogeneous case but also for the heterogeneous case by choosing the dependence of some parameters. Although we do not discuss about their extended version in this article, we think that the model is a good reference for extending our model to the parametric version.

In the liquid CDO market, the probability distribution of the multiplier $(\widetilde{\kappa}(k), \widetilde{\eta}(k))_{k=1,\cdots,K}$ can be calibrated from the market prices of tranches similar to the case described in Hull and White[10]. They calibrated the distribution $(\widetilde{\kappa}(k), \widetilde{\eta}(k))_{k=1,\cdots,K}$ nonparametrically so that the market prices of CDO tranches at a fixed maturity might be consistent with the theoretical prices calculated from their model. For the details of the calibration methods, see their paper[10] and, as a reference, Brigo et al.[3].

In evaluating the risk for the arbitrary risk horizon, we propose a simple interpolation of the distributions of the multiplier with different maturities. For example, when we know two probability distributions $(\widetilde{\kappa}(k), \widetilde{\eta}(k;T_1))_{k=1,\cdots,K}$ for maturity T_1 and $(\widetilde{\kappa}(k), \widetilde{\eta}(k;T_2))_{k=1,\cdots,K}$ for maturity T_2, the probability distribution $(\widetilde{\kappa}(k), \widetilde{\eta}(k;T))_{k=1,\cdots,K}$ for maturity T, $T_1 < T < T_2$, given by

$$\widetilde{\eta}(k;T) = \frac{(T_2 - T)\widetilde{\eta}(k;T_1) + (T - T_1)\widetilde{\eta}(k;T_2)}{T_2 - T_1}, \qquad k = 1, \cdots, K$$

always satisfies the necessary conditions (2.9)–(2.11) if the two distributions, $(\widetilde{\kappa}(k), \widetilde{\eta}(k;T_1))_{k=1,\cdots,K}$ and $(\widetilde{\kappa}(k), \widetilde{\eta}(k;T_2))_{k=1,\cdots,K}$, satisfy the same conditions.

2.3.2 Assumptions under the physical probability measure P

Next, we consider the future scenario generation up to the risk horizon T under the physical probability measure P. Similarly under the risk-neutral probability measure $\widetilde{P}$, the unconditional cumulative default probability, that is, the unconditional distribution function of the default time τ_j, is denoted by $F_j(t) = P\{\tau_j \leq t\}$, and the forward default probability at time t on the event $\{\tau > s\}$ is given by

$$F_j(s,t) = P\{\tau_j \leq t | \tau_j > s\} = \frac{P\{s < \tau_j \leq t\}}{P\{\tau_j > s\}}, \; 0 \leq s \leq t \leq T^*.$$

Given $\mathcal{G}'_u$, $0 \leq u \leq T^*$, the conditional cumulative default probability at time t is denoted by $F_j(t|\mathcal{G}'_u) = P\{\tau_j \leq t|\mathcal{G}'_u\}$, and the conditional forward default probability at time t on the event $\{\tau > s\}$ is given by

$$F_j(s,t|\mathcal{G}'_u) = \frac{P\{s < \tau_j \leq t|\mathcal{G}'_u\}}{P\{\tau_j > s|\mathcal{G}'_u\}}, \; 0 \leq s \leq t \leq T^*.$$

Apparently, we have $F_j(t) = F_j(t|\mathcal{G}'_0)$, and $F_j(s,t) = F_j(s,t|\mathcal{G}'_0)$ on the event $\{\tau_j > s\}$.

Similarly to (2.6) under $\widetilde{P}$, we obtain the following relation

$$P_t\{\tau_j > \tau\} = E_t\left[\exp\left\{-\int_t^\tau h_j(s)\mathrm{d}s\right\}\right] = C_j(t,\tau)e^{-B(a_j,t,\tau)h_j(t)}$$

on $\{\tau_j > t\}$ under P, where

$$C_j(t,\tau) = \exp\left\{ \frac{\sigma_j^2}{2a_j^2} (\tau - 2B(a_j,0,\tau) + B(2a_j,0,\tau)) - \int_0^\tau b_j(u)B(a_j,u,\tau)\mathrm{d}u \right\}.$$

Based on the above definitions, we set the following assumption.

Assumption 2.4. Given $\mathcal{G}_t$, the conditional forward cumulative default probability up to time τ on $\{\tau_j > t\}$, $j = 1, \cdots, n$ under P is given by

$$F_j(t,\tau|\mathcal{G}_u) = F_j(t,\tau|\mathcal{G}'_u \vee \sigma(N)) = \kappa(N)F_j(t,\tau|\mathcal{G}'_u), \quad 0 \le t \le \tau \le T^*,\ 0 \le u \le T^* \quad (2.13)$$

where $\kappa(\cdot)$ is a positive function.

Under P, the stochastic variable N has a probability function $\eta(k) \equiv P\{N = k\}$, which means that the following relations

$$\sum_{k=1}^{K} \kappa(k)\eta(k) = 1$$
$$\sum_{k=1}^{K} \eta(k) = 1$$
$$\eta(k) \ge 0, \qquad k = 1, \cdots, K$$

are satisfied.

Similarly under $\widetilde{P}$, the positive function $\kappa(\cdot)$ is called a "multiplier" under P. The $\mathcal{G}'_u$-measurable $F_j(t,\tau|\mathcal{G}'_u)$ means the conditional forward default probability averaged on N, that is,

$$F_j(t,\tau|\mathcal{G}'_u) = E[F_j(t,\tau|\mathcal{G}_u)|\mathcal{G}'_u] = E[\kappa(N)F_j(t,\tau|\mathcal{G}'_u)|\mathcal{G}'_u] = \sum_{k=1}^{K} \eta(k)\kappa(k)F_j(t,\tau|\mathcal{G}'_u).$$

We can assume $\kappa_j(\cdot)$ instead of $\kappa(\cdot)$, however, such assumption makes the calibration difficult drastically.

In general, the property of the conditional independence is not invariant under the change of measure, that is, the conditional independence under $\widetilde{P}$ does not imply the conditional independence under P. For more detail, see Kusuoka[18] and Bielecki and Rutkowski[1]. However, as shown in Appendix A, there exists a class of the change of measures where the conditional independent property is invariant.

Assumption 2.5. The risk premia adjustments $\ell_j(t)$, $j = 1, \cdots, n$ in (2.4) are deterministic functions of time t.

Lemma 2.1. If Assumption 2.2 and Assumption 2.5 are satisfied, the default times become $\mathcal{G}$-conditionally independent under P.

See the proof in Appendix A.

From Assumptions 2.1, 2.2 and 2.5, and thanks to Lemma 2.1, given $\mathcal{G}'_u$, each asset on survival at time t defaults independently according to its conditional forward cumulative default probability under P. According to (2.8), we obtain

$$F_j(t,\tau|\mathcal{G}'_u) = 1 - \exp\left\{-\int_t^\tau h_j(s)\mathrm{d}s\right\}, \qquad 0 \le t \le \tau \le u \le T^*$$

which is calculated from the sample path $\{h_j(s),\, 0 \le s \le \tau\}$. And, according to the tower property of the conditional expectation, we have

$$F_j(t,\tau|\mathcal{G}'_t) = 1 - E\left[\exp\left\{-\int_t^\tau h_j(s)\mathrm{d}s\right\}\middle|\, \mathcal{G}'_t\right], \qquad 0 \le t \le \tau \le T^*.$$

And, from Assumption 2.4, given $\mathcal{G}_u$, we obtain

$$F_j(t,\tau|\mathcal{G}_u) = \kappa(N)F_j(t,\tau|\mathcal{G}'_u), \qquad 0 \le t \le \tau \le u \le T^*$$

and, according to the tower property of the conditional expectation, we have

$$F_j(t,\tau|\mathcal{G}_t) = \kappa(N)F_j(t,\tau|\mathcal{G}'_t), \qquad 0 \le t \le \tau \le T^*.$$

Remark 2.2. Since $F_j(t,\tau|\mathcal{G}_\tau)$ is a probability, the following relation

$$0 \le F_j(t,\tau|\mathcal{G}_\tau) \le 1, \qquad j = 1,\cdots,n \tag{2.14}$$

must be satisfied. Or, from (2.14), the multiplier must satisfy

$$0 < \kappa(k) \le \frac{1}{\max_{\mathcal{G}'_\tau} F_j(t,\tau|\mathcal{G}'_\tau)}.$$

Since Remark 2.1 and 2.2 are undesirable constraints, you should consider carefully which the better is, the original implied copula model setting or ours when modeling.

Additionally, we think that the following assumption is necessary to use our model in practice.

Assumption 2.6. The stochastic variable N is not influenced by the change of measure. That is, the distribution of N under P is identical to that under $\widetilde{P}$. And, the multiplier functions under the two measures are the same, that is, $\kappa(k) = \widetilde{\kappa}(k)$ for all $k \in \{1,\cdots,K\}$.

Assumption 2.6 is very arbitrary and controversial. There are neither supporting evidences nor theoretical suggestions. On the contrary, it is natural that the distribution of the multiplier changes with the probability measure. However, Assumption 2.6 makes the calibration very easy, and also makes it possible to reflect the latent fear of the major market participants on evaluating the financial risk directly.

2.3.3 Assumption for simplifying numerical examples

For simplicity, it is often assumed in modeling credit risk that the default-free interest rates and the default probabilities are independent. Here, we use another assumption in order to make pricing financial instruments much simpler.

Assumption 2.7. The hazard rates $\widetilde{h}_j(t)$, $j = 1, \cdots, n$ are deterministic functions of time t. Or, under the setting in Section 2.2, σ_j, $j = 1, \cdots, n$ in (2.3) are all zeros.

Assumption 2.7 leads that the default probabilities, $\widetilde{F}_j(t, \tau|\mathcal{G}_t)$, $j = 1, \cdots, n$, $0 \leq t \leq \tau \leq T^*$, are deterministic functions of times t and τ. Then, given $\mathcal{G}_t$, the conditional forward cumulative default probabilities on $\{\tau_j > t\}$ under $\widetilde{P}$ are given by

$$\widetilde{F}_j(t, \tau|\mathcal{G}_t) = \frac{\widetilde{F}_j(\tau|N) - \widetilde{F}_j(t|N)}{1 - \widetilde{F}_j(t|N)} = \widetilde{\kappa}(N)\frac{\widetilde{F}_j(\tau) - \widetilde{F}_j(t)}{1 - \widetilde{\kappa}(N)\widetilde{F}_j(t)}.$$

Similarly, given $\mathcal{G}_t$, the conditional forward cumulative default probabilities on $\{\tau_j > t\}$ under P also become a deterministic function of t and τ, and are given by

$$F_j(t, \tau|\mathcal{G}_t) = \frac{F_j(\tau|N) - F_j(t|N)}{1 - F_j(t|N)} = \kappa(N)\frac{F_j(\tau) - F_j(t)}{1 - \kappa(N)F_j(t)}.$$

These results make our numerical examples very simple.

2.4 Pricing a defaultable bond and their portfolio

Without considering a stochastic variable N, pricing functions of a default-free and a defaultable discount bonds are given by (2.2) and (2.5), respectively, or more generally, given by (2.7). These functions are used for pricing at present $t = 0$. Here, we show a pricing function for a corporate bond at future $t > 0$.

Consider the same setting described in Section 2.2 and Assumptions 2.1 – 2.7 in Section 2.3. Given $\mathcal{G}_t$, the time t price of a defaultable discount bond with maturity τ, $0 \leq t \leq \tau$, issued by the j-th firm is given by

$$\begin{aligned} v_j(t, \tau|\mathcal{G}_t) &= 1_{\{\tau_j > t\}}\widetilde{E}\left[\exp\left\{-\int_t^\tau r(s)\mathrm{d}s\right\}\left(1_{\{\tau_j > \tau\}} + \delta_j 1_{\{\tau_j \leq \tau\}}\right)\middle|\, \mathcal{G}_t\right] + 1_{\{\tau_j \leq t\}}\delta_j v_0(t, \tau) \\ &= 1_{\{\tau_j > t\}}\widetilde{E}\left[\exp\left\{-\int_t^\tau r(s)\mathrm{d}s\right\}\left(1_{\{\tau_j > \tau\}} + \delta_j 1_{\{\tau_j \leq \tau\}}\right)\middle|\, \mathcal{G}'_t \vee \sigma(N)\right] + 1_{\{\tau_j \leq t\}}\delta_j v_0(t, \tau). \end{aligned}$$

Considering similarly as in (2.7), we obtain

$$\begin{aligned} v_j(t,\tau|\mathcal{G}'_t \vee \sigma(N)) &= 1_{\{\tau_j>t\}} v_0(t,\tau) \left[\delta_j + (1-\delta_j)\widetilde{P}\{\tau_j > \tau|\mathcal{G}'_t \vee \sigma(N)\} \right] + 1_{\{\tau_j \le t\}} \delta_j v_0(t,\tau) \\ &= \delta_j v_0(t,\tau) + 1_{\{\tau_j>t\}} v_0(t,\tau)(1-\delta_j)\widetilde{P}\{\tau_j > \tau|\mathcal{G}'_t \vee \sigma(N)\} \end{aligned} \quad (2.15)$$

where

$$\widetilde{P}\{\tau_j > \tau|\mathcal{G}'_t \vee \sigma(N)\} = 1 - \widetilde{F}_j(t,\tau|\mathcal{G}'_t \vee \sigma(N)).$$

Notice that (2.15) represents the price not only of a surviving bond at time t but also of a defaulted bond up to time t. A defaultable coupon bond can be evaluated as a portfolio of defaultable discount bonds.

For the risk evaluation of a portfolio consisting of defaultable bonds, we should also consider the defaulted loss of bonds up to the risk horizon T, $0 < T$, under the physical probability measure P. Under our assumptions, given $\mathcal{G}_T$, each asset defaults independently under P according to its conditional cumulative default probability given by

$$F_j(T|\mathcal{G}'_0 \vee \sigma(N)) = \kappa(N)F_j(T),$$

which comes from (2.13). And, from (2.15), the future value of each bond is given by

$$v_j(T,\tau|\mathcal{G}'_T \vee \sigma(N)) = \delta_j v_0(T,\tau) + 1_{\{\tau_j>T\}} v_0(T,\tau)(1-\delta_j)\widetilde{P}\{\tau_j > \tau|\mathcal{G}'_T \vee \sigma(N)\} \quad (2.16)$$

where τ, $\tau \ge T$, is the maturity and

$$\widetilde{P}\{\tau_j > \tau|\mathcal{G}'_T \vee \sigma(N)\} = 1 - \widetilde{F}_j(T,\tau|\mathcal{G}'_T \vee \sigma(N)) = 1 - \widetilde{\kappa}(N)\widetilde{F}_j(T,\tau|\mathcal{G}'_T).$$

Additionally, notice that when the portfolio includes coupon bonds, it is also necessary to evaluate the time T values of coupons received up to T under P.

2.5 Pricing a CDO tranche and its defaulted loss

Next, we consider a no-arbitrage price of a CDO tranche. Since our framework is based on the implied copula model proposed by Hull and White[10], the present price of each tranche can be evaluated similarly to their model. Defaults of the assets are conditionally independent under the risk-neutral probability measure $\widetilde{P}$, and the loss distribution of the asset pool at future is obtained as the weighted average of the conditional loss distributions in various kinds of economic states shown by N. Based on the calculated loss distributions of the asset pool at the payment dates and the maturity, we obtain the expected losses of the tranche which has fixed attachment and detachment points, and calculate the present value of the premium legs and the default leg including accrued interests, and finally we obtain the no-arbitrage price of the tranche.

Suppose that there are M CDSs with the same notional principal G and the same recovery rate δ [6]. And, consider a synthetic CDO which provides protection against a subset of the total loss on the above CDS portfolio, and its tranche with an attachment point a_L and a detachment point a_H. Let $P_j(a_L, a_H)$ be the remaining notional principal when j-th default already occurs in the CDS portfolio. Then, the remaining tranche notional after j-th default is given by

$$P_j(a_L, a_H) = \begin{cases} (a_H - a_L)GM, & j < m(n_L) \\ a_H GM - j(1-\delta)G, & m(n_L) \leq j < m(n_H) \\ 0, & m(n_H) \leq j \end{cases}$$

where $n_L = a_L M/(1-\delta)$, $n_H = a_H M/(1-\delta)$, and $m(x)$ is the smallest integer greater than x.

The no-arbitrage price of the tranche is given as the expectation of the sum of the discounted cashflows, and the cashflows are (1) the premium legs, (2) the default leg, and (3) the accrural payment. Let $P(t)$ be the remaining notional principal of the tranche at time t, s be the premium (or spread) of the tranche, and t_i, $i = 1, \cdots, I$, be the dates when periodic payments are made and $t_0 = 0$. For simplicity, assume that the default can occur only at the midpoints of the periodic payment dates, and that the default leg and accrued payment are paid just the time when the default occurs, that is, only at the midpoints of the payment dates. Then, the present value of the premium legs is given by

$$A = s \sum_{i=1}^{I} (t_i - t_{i-1}) \widetilde{E} \left[\exp\left\{ -\int_0^{t_i} r(u) \mathrm{d}u \right\} P(t_i) \right],$$

the present value of the default leg is given by

$$B = (1-R) \sum_{i=1}^{I} \widetilde{E} \left[\exp\left\{ -\int_0^{(t_i + t_{i-1})/2} r(u) \mathrm{d}u \right\} (P(t_{i-1}) - P(t_i)) \right],$$

the present value of the accrual payment is given by

$$C = \frac{s}{2} \sum_{i=1}^{I} (t_i - t_{i-1}) \widetilde{E} \left[\exp\left\{ -\int_0^{(t_i + t_{i-1})/2} r(u) \mathrm{d}u \right\} (P(t_{i-1}) - P(t_i)) \right],$$

[6] These homogeneous settings can be relaxed easily.

and the present price of the tranche is given by $V = A - B + C$ for the protection seller. From Assumption 2.7, the present price V is given by

$$
\begin{aligned}
V = {} & s\sum_{i=1}^{I}(t_i - t_{i-1})v_0(0,t_i)\widetilde{E}\left[P(t_i)\right] \\
& + \sum_{i=1}^{I}\left[\frac{s(t_i - t_{i-1})}{2} - (1-R)\right] v_0\left(0, \frac{t_i + t_{i-1}}{2}\right)\left(\widetilde{E}\left[P(t_{i-1})\right] - \widetilde{E}\left[P(t_i)\right]\right). \qquad (2.17)
\end{aligned}
$$

From (2.17), the present price of the tranche can be calculated from the the expectations of the remaining notional principal of the tranche at the periodic payment dates, $\widetilde{E}[P(t_i)]$, $i = 1, \cdots, I$. The expectation $\widetilde{E}[P(t_i)]$ is rewritten as $\widetilde{E}[P_{J(t_i)}(a_L, a_H)]$ where a stochastic variable $J(t)$ is the number of defaults which occurs up to time t. The loss distribution of the tranche is calculated from the loss distribution of the asset pool, which is given as a weighted sum of the conditional loss distributions given $\mathcal{G}_t$. Notice that, given $\mathcal{G}_t$, the term structures of the forward cumulative default probabilities of the j-th asset at time t, denoted by $\widetilde{F}_j(t, s|\mathcal{G}_t)$, $s \geq t$, that is, the conditional distribution function of τ_j on $\{\tau_j > t\}$, are obtained. We can use some effective numerical methods for calculating a conditional loss distribution, for example, the bucketing method proposed by Hull and White[9]. Although the bucketing method does not give a strictly exact distribution, it provides a quick calculation and the obtained distribution is accurate enough for pricing CDO tranches under appropriate settings.

About pricing a tranche, there are two main differences between at present ($t = 0$) and at future ($t > 0$) under our assumptions. First, the future loss distribution for pricing at present is given as the weighted average of the conditional loss distributions in K economic states, on the other hand, the future loss distribution for pricing at future is given as the conditional distribution in the realization of N selected on each future scenario. If we consider the transition of the state variable N with time t in a multi-period model, which we think as our next work, pricing a CDO tranche in future becomes more difficult. Second, since some reference assets might be defaulted up to the risk horizon T in each scenario, we must take into account the effects of the defaulted assets; (1) the defaulted loss, and (2) the influence on the tranche price. The former is evaluated under the physical probability measure P, and the latter is evaluated under the risk-neutral probability measure $\widetilde{P}$. The change of measure between P and $\widetilde{P}$ on the interest rates and the hazard rates are described by the market price of risk $\lambda(t)$ and the risk-premia adjustments $\ell_j(t)$, $j = 1, \cdots, n$.

2.6 Future scenario generation by Monte Carlo simulation

Based on the stochastic structures and assuptions, we generate a lot of future scenarios up to the risk horizon T by the Monte Carlo simulation. The procedure is given as follows.

1. Generate a realization of $\mathcal{G}_T$ under P. In detail, generate a sample path of $\{(r(s),\ h_j(s),\ j = 1, \cdots, n),\ 0 \leq s \leq T\}$ according to (2.1) and (2.3), and generate a sample of N according to its probability distribution η.

2. Generate a default scenario at time T. That is, judge the states (default or survival) of all assets on time T. Each asset defaults independently with the probability $F_j(T|\mathcal{G}_T) = F_j(0, T|\mathcal{G}_T)$ given by (2.13).

3. Evaluate the future value of each asset at time T.

   (a) For a corporate bond, its future value is given by (2.16), which can be used both in survival and in default at time T. If necessary, add the time T values of the cashflows (for example, coupons) received up to T.

   (b) For a CDO tranche, its future price at time T is evaluated as a no-arbitrage price by the methods described in Section 2.5, taking into account the effect of the reference assets defaulted up to time T. The defaulted assets should be excluded from the pool of the reference assets. If necessary, add the time T values of the cashflows received up to T, and subtract the time T values of the payments against the defaulted assets.

4. Sum up the future values of all assets, then it gives the future value of the portfolio at time T on the scenario.

5. If enough numbers of scenarios are obtained, go out of this procedure. Otherwise, return to Step 1 and follow this procedure.

The image of how to use two probability measures, the physical probability measure P and the risk-neutral probability measure $\widetilde{P}$, is shown in Figure 1. The future scenarios up to the risk horizon T must be generated under P, while the scenarios after T must be generated under $\widetilde{P}$ in order to evaluate the future asset prices at time T. If the Monte Carlo simulation is needed to evaluate the asset prices, the scenarios after T must be generated under $\widetilde{P}$, and the starting state at T under $\widetilde{P}$ is given from the state at T under P on each scenario, taking into account the effect of the change of measure.

In this article we consider only the stochastic behaviors of the default probabilities and the interest rates. If you consider models in which other financial variables such as exchange rates and stock prices are also important, such processes must also be generated in the above procedure simultaneously.

3 Numerical example 1: a synthetic CDO

In the following sections, we show simple numerical examples of the single-period model described in Section 2. First, we consider credit and interest rate risks of synthetic CDO tranches whose reference assets are 125 entities. In order to evaluate the prices of tranches, we use the bucketing method proposed by Hull and White[9]. The bucketing method is one of the efficient calculation methods based on the conditional independence of defaults. However, the detailed explanation of the method is omitted because it is not essential in this article.

3.1 Model parameters and calculation of future values

For numerical examples in this article, the default-free spot rate process is set to be Vasicek model[20] under P, that is, we set $b(t)/a = 0.03$ (3.0%, mean reversion level), $a = 0.1$ and $\sigma = 0.01$ in (2.1). And we set $\sigma_j = 0$ for all j in (2.3) so that the forward default probabilities become deterministic functions of time t.

Consider a simple "unfunded" synthetic CDO tranches with maturity $\tau = 5$ years, and their asset pool consists of equally weighted 125 entities with face values 10 and with identical recovery rates $\delta_j = 0.4$. There exist six credit ratings in these entities, denoted by A1, A2, B, C1, C2 and C3, and their default probabilities under the physical probability measure P are 0.1%, 0.2%, 0.5%, 1.0%, 2.0% and 3.0% per year, respectively. Suppose that A1- and A2-rated are 40 entities, B-rated are 30 entities, and other rated are 5 entities, respectively. The initial forward rate curves are assumed to be flat; the default-free forward rates are 3.0%, and the forward rates of the above credit ratings are 3.1%, 3.2%, 3.5%, 4.0%, 5.0% and 6.0%, respectively. The term structures of the default probabilities of each entity under the risk-neutral probability measure $\widetilde{P}$ are calculated from the forward rate curves and recovery rates.

Suppose 6 CDO tranches, and their detachment points are $3\%, 6\%, 9\%, 12\%, 22\%$ and 100%, respectively, which are the same as the iTraxx tranches. Since the total volume is $125 \times 10 = 1250$, then 1% corresponds to 12.5. For example, the volume of the equity tranche

$[0, 3\%]$ is 37.5.

Figure 2 shows the distribution of the multiplier $\widetilde{\kappa}$ used in this article. The distribution of $\widetilde{\kappa}$ has a right long tail: the probability of $\widetilde{\kappa}$ over 3, $P\{\widetilde{\kappa} > 3\}$, is set to be 4.0%. Although this distribution is not implied from the market data of CDO tranches, its shape has some similar features as the calibration results by Hull and White[10] and Brigo et al.[3]. At present $t = 0$, all the tranche spreads are set to be equal to the fair spreads, that is, the present values of all the contracts are zero at $t = 0$. The fair spreads are calculated from pricing by the implied copula model under the conditions used here, and the values except for the second super-senior tranche $[22, 100\%]$ are 40.7%, 169.0bp, 69.9bp, 21.6bp and 0.3bp, respectively [7]. The fair spread 40.7% for the equity tranche $[0, 3\%]$ is the unfront payment, and the periodical coupon is set to be 5% per year for the equity tranche.

We set the risk horizon $T = 1$ year and the number of simulation runs are 500,000. For CDO tranches, we consider the "future cumulative value" at time T, which is defined as (1) the future price of the unfuded CDO tranche at time T, plus (2) the time T values of the quarterly-coupons received up to time T, which are assumed to be invested to the default-free bonds up to time T, minus (3) the actual default loss up to time T. Therefore, the future value of the equity tranche includes the time T value of the up-front payment. In calculating the future loss distribution of the asset pool at time T, the defaulted loss up to time T is included as an important part of the loss. We calculate the default loss of the asset pool on every payment date on every scenario in order to calculate the coupons proportional to the remaining notional principals at every payment date.

3.2 Distribution of the future cumulative values

Figure 3 (a) and (b) show the distribution functions of the future cumulative values, defined in section 3.1, of the tranches at the risk horizon $T = 1$ year. The calculated future cumulative values are for the protection sellers, therefore, when many default occurs up to time T, the future value of the tranches becomes low (negative). And, on the scenarios with very large $\widetilde{\kappa}$ (and also very large κ), the future cumulative values become low (negative) because the CDO tranche prices at time T drop drastically due to the increase of the conditional default probabilities.

In Figure 3 (a) and (b), all distribution functions have left long-tails. And, except the equity tranche, the left tails begin to grow in less than 0.1 (10.0%) confidence level, and

[7]The fair spreads of the super-senior tranches used in this article are lower than the actual data. This is because the distribution of the multiplier $\widetilde{\kappa}$ used in this article is not so fitted with the actual market CDO fair spreads. The appropriate calibration of $\widetilde{\kappa}$'s distribution is needed for the practical use.

grow drastically under 0.05 (5.0%) level. These results imply that all the tranches except for the most senior (second super-senior) tranche have some small probability that serious loss occurs. The probability with serious loss depends strongly on the assumed distribution of the multiplier $\widetilde{\kappa}$ (and κ). In this numerical example, the distribution of $\widetilde{\kappa}$ with a right long tail is directly reflected on the distribution functions.

It seems important that the serious catastrophic loss events happen in all tranches except for the most senior tranche $[22, 100\%]$, and that the probabilities with catastrophic loss are similar, not so different in the tranches. These results imply that the credit enhancement by the senior-subordinated structures for CDOs do not work well with some small probability. This feature can be clearly shown in Fugure 3 (a), and also in Table 1, in which some risk measures (Standard Deviation, Value at Risk "VaR" and Expected Shortfall "ES") are listed for all tranches except for the most senior tranche $[22, 100\%]$. Here, $100\alpha\%$-VaR is defined as the average minus $100(1-\alpha)$ percentile where $0 < \alpha < 1$. In Table 1, 95%-VaRs are already high, however, the VaRs of the tranches jump up from 95% to 99%. The jump size increases with decreasing the priority, and the drastic jump appears in $[3, 6\%]$ (junior-mezzanine), $[6, 9\%]$ (senior-mezzanine), $[9, 12\%]$ (senior) and $[12, 22\%]$ (first super-senior). On the other hand, the VaRs do not change so drastically over 99% region, especially, the losses are almost saturated in the equity and junior-mezzanine tranches.

The above catastrophic loss in the highest confidence level might be induced "Armageddon factor," which is well-known in pricing CDOs. By analysing the market prices of the CDO tranches, we can catch the effect of this factor with a certain probability on the risk evaluation model. In other words, we might say that this catastrophic loss would be called as the "market-implied stress scenario." The important point is that this market-implied stress scenario has a certain implied "probability," therefore, we can include its effect on the statistical Monte Carlo simulation models. To my regret, the probability strongly depends on the assumptions used in the models, however, we think that this might be one of the possible ways to connect the existing statistical models with the stress tests.

In Table 1 and Figure 3 (a), VaRs of the equity tranche $[0, 3\%]$ are smaller than those of the junior-mezzanine tranche $[3, 6\%]$ and the senior-mezzanine tranche $[6, 9\%]$ over 99% confidence level. This is because the future value of the upfront payment is included in the equity tranche. We might say that before the recent financial crisis our proposed model could detect much higher risk in the mezzanine tranches than expected.

Table 2 shows the distributions of the actual default losses of the most junior three tranches. From this table, we can see that the distributions of the actual default losses differ from those of the total losses clearly, and that the actual default losses are not so much that most of the losses come from the decrease of the future tranche prices, especially in the mezzanine tranches $[3, 6\%]$ and $[6, 9\%]$. These results are consistent with the collapse of the CDO market during the financial crisis in the real world.

4 Numerical example 2: a bond portfolio

In this section, we consider a bond portfolio. It is controversial to apply a multiplier's ($\widetilde{\kappa}$'s) distribution implied from the CDO tranche prices to the risk evaluation of a bond portfolio. However, we think that this is one simple idea in order to construct forward-looking risk evaluation methods in future.

4.1 Model parameters

Suppose that there are one thousand corporate discount bonds issued by different firms. We set that their maturities are all 5 years, the face values are 10, and the recovery rates are 40%. The credit ratings and their features are the same as described in Section 3, and we consider a portfolio consisting of 400 A2-rated bonds, 300 B-rated, 200 C1-rated, and 100 C2-rated bonds. We set that the default-free spot rate is constant, 3.0%, and we set $b(t)/a = 0.03$, $a = 0.1$ and $\sigma = 0.001$ in (2.1). Here we use smaller σ than in Section 3 because we have much more interest on the credit risk than on the interest rate risk.

We set the risk horizon $T = 1$ year, and the number of simulation runs are 1,000,000.

4.2 Distribution of the future price

Figure 4–6 show the distribution of the future value of the bond portfolio at the risk horizon $T = 1$ year. Figure 4 is the histogram, Figure 5 is the distribution function, and Figure 6 is the lowest part $[0\%, 10\%]$ of the distribution function. Some risk measures such as the standard deviation, VaRs and ESs are summarized in Table 3.

In Figure 5 and Figure 6, the left tail grows in less than 10.0% confidence level, and grows suddenly near 4.0% confidence level at the vertical axis. The former and the latter correspond to the small peaks around 2.0 and over 4.0 in Figure 2, respectively. From the estimated values of VaRs in Table 3, the estimated VaR jumps up from 95.0% to 99.0% drastically, and the jump corresponds to the small peak over 4.0 in Figure 2.

On the other hand, the estimated ES (Expected Shortfall) also jumps up from 95.0% to 99.0%, but not so drastically in Table 3. The different features of the ESs from the VaRs are shown by Figure 7, in which the Tail Conditional Expectations (hereafter, abbreviated by TCEs) are plotted for various confidence levels. Here, the TCE with α confidence level is defined as the conditional expectation of the future price under the 100α-percentile. In Figure 7 the TCE changes gradually with the confidence level, therefore, the ES also changes gradually because the ES with $100(1-\alpha)\%$ confidence level is equal to the difference between the average future price and the TCE with $100\alpha\%$ confidence level. Such a gradual change implies that the ES would be a better, and more desirable risk measure than the VaR.

5 Concluding remarks

In this article, combining the implied copula model by Hull and White[10] and a general framework for constructing a risk evaluation model proposed by Kijima and Muromachi[16], we propose a simple and new risk evaluation model for a bond portfolio and a CDO (CDO is thought to be a derivative written on a portfolio) including stress events with probabilities. Most of the previous risk evaluation models use the historical data mainly, but we use not only the historical data but also the risk premiums included in the market prices, so that our model could reflect the latent fear of the major market participants on the estimates of the risk through the risk premiums. Numerical results of the single-period model show that all the tranches except for the most senior tranche have some small probabilities that the huge amount of loss occurs, which might be induced by the so-called "Armageddon factor" in pricing CDOs. It is important that we can estimate a certain probability under which such a "market-implied stress scenario" occurs, and therefore, we can evaluate its effect quantitatively in the statistical risk evaluation models. The implied probability depends strongly on the model assumptions, however, we think that this might be one of the possible ways to connect the statistical models with the stress tests and to obtain useful imformations for the risk management.

However, there remain some doubtful assumptions in our model. Especially, the coincidence of the distributions of multipliers κ and $\widetilde{\kappa}$ is controversial, and it is doubtful whether the distribution of multiplier implied from the CDO prices could be applicable to the bond markets, and exactly speaking, it cannot be guaranteed that the common distribution of $\widetilde{\kappa}$ is useful for the CDOs with different reference assets.

Now, we think that the proposition of the multi-period version and the continuous-time version of our model is our next works. In order to consider such extensions of this approach, it is an undesirable trend that the activities such as the trading volumes and the development of new financial instruments in the credit derivatives and CDO markets are reduced drastically after the recent financial crisis. We hope the recoveries and still more extensions of such markets in future.

A Proof of Lemma 2.1

It is known that the property of conditional independence may not be invariant under an equivalent change of probability measure. See, for example, Kusuoka[18] and Bielecki and Rutkowski[1]. Therefore, the conditional independence of the default times τ_j, $j = 1, \cdots, n$ under $\widetilde{P}$ is not always compatible with the conditional independence under P in a general setting. Here, we show some models compatible with these assumptions.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and the default times of two assets, say, $\tau_1 > 0$ and $\tau_2 > 0$. The default processes are defined by $H_j(t) = 1_{\{\tau_j \leq t\}}$, $j = 1, 2$, and a filtration generated by the default process $H_j(t)$ is denoted by $\mathcal{H}_t^j = \sigma(H_j(s), 0 \leq s \leq t)$, and $\mathcal{H} = \mathcal{H}^1 \vee \mathcal{H}^2$, that is, $\mathcal{H}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2$ for all $t \in [0, T]$. The filtration $\mathcal{F}$ is divided into $\mathcal{H}$ and another filtration $\mathcal{G}$, which corresponds to the information except default times, and $\mathcal{F} = \mathcal{G} \vee \mathcal{H}$ is satisfied, that is, $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$ for all $t \in [0, T]$. Additionally, we define $\mathcal{F}_t^j = \mathcal{H}_t^j \vee \mathcal{G}_T$, $j = 1, 2$, and $\mathcal{F}_t' = \mathcal{H}_t \vee \mathcal{G}_T$. Hereafter, we do not consider the detailed information included in $\mathcal{G}$, and we neglect the influence of the change of measure on $\mathcal{G}$-measurable stochastic processes because they are not essential here.

Let Q be a probability measure equivalent to P, and define a $\mathcal{F}_T$-measurable positive random variable $\eta = \mathrm{d}Q/\mathrm{d}P$ as its Radon-Nikodym derivative, and the density process is denoted by

$$\rho(t) = E^P \left[\left. \frac{\mathrm{d}Q}{\mathrm{d}P} \right| \mathcal{F}_t \right] \tag{A.18}$$

where E^I is the expectation under a probability measure I, $I = P, Q$. Assuming $\rho(t)$ is a locally bounded RCLL (right-continuous with left-limits) process, then $\rho(t)$ is a local martingale under P, and can be uniquely expressed by

$$\rho(t) = 1 + \int_0^t \rho(s-) \left(\xi_1(s)\mathrm{d}M_1(s) + \xi_2(s)\mathrm{d}M_2(s) \right), \qquad 0 \leq t \leq T \tag{A.19}$$

where $\xi_j(t)$, $j = 1, 2$ are $\mathcal{F}_t$-predictable processes, and

$$M_j(t) = H_j(t) - \int_0^t (1 - H_j(s-))h_j(s)\mathrm{d}s, \qquad j = 1, 2$$

are $\mathcal{F}$-martingales under P. The solution of (A.19) is given by

$$\begin{aligned}\rho(t) &= \varepsilon \left(\int_0^t \xi_1(s)\mathrm{d}M_1(s) + \int_0^t \xi_2(s)\mathrm{d}M_2(s) \right) \\ &= \exp\left\{ \int_0^t (\xi_1(s)\mathrm{d}M_1(s) + \xi_2(s)\mathrm{d}M_2(s)) \right\} \prod_{0\leq s\leq t} (1 + \Delta\xi_1(s) + \Delta\xi_2(s))e^{-\Delta\xi_1(s)-\Delta\xi_2(s)}\end{aligned}$$

where $\varepsilon(X_t)$ is the Doléans-Dade exponential of X_t. Under Q, the conditional survival functions of τ_j, $j = 1, 2$ are given by

$$Q\{\tau_j > t_j\} = E^Q\left[1_{\{\tau_j > t_j\}}\right] = E^P\left[1_{\{\tau_j > t_j\}}\frac{\mathrm{d}Q}{\mathrm{d}P}\right]$$

and the conditional joint survival function is given by

$$Q\{\tau_1 > t_1, \tau_2 > t_2\} = E^Q\left[1_{\{\tau_1 > t_1, \tau_2 > t_2\}}\right] = E^P\left[1_{\{\tau_1 > t_1, \tau_2 > t_2\}}\frac{\mathrm{d}Q}{\mathrm{d}P}\right].$$

Assume that τ_1 and τ_2 are $\mathcal{G}$-conditionally independent under P, that is, given $\mathcal{G}_T$, the conditional joint survival probability is given by

$$P\{\tau_1 > t_1, \tau_2 > t_2 | \mathcal{G}_T\} = P\{\tau_1 > t_1 | \mathcal{G}_T\} P\{\tau_2 > t_2 | \mathcal{G}_T\}, \qquad 0 \le t_1, t_2 \le T$$

and assume that $h_j(t)$, $\xi_j(t)$, $j = 1, 2$, are $\mathcal{G}_t$-predictable processes and $P\{\tau_1 = \tau_2\} = 0$. Under these assumptions [8], $\rho(t)$ can be written as

$$\rho(t) = \rho_1(t)\rho_2(t) \tag{A.20}$$

where, for $i = 1, 2$,

$$\rho_i(t) = \varepsilon\left(\int_0^t \xi_i(s)\mathrm{d}M_i(s)\right) = \exp\left\{\int_0^t \xi_i(s)\mathrm{d}M_i(s)\right\} \prod_{0 \le s \le t} (1 + \Delta\xi_i(s))e^{-\Delta\xi_i(s)} \tag{A.21}$$

and $\rho_i(t)$ is also a $\mathcal{F}_t^i$-martingale for $i = 1, 2$.

From the Bayes' rule, the chain rule of the conditional expectation, (A.18), (A.20) and (A.21), the conditional survival function of τ_1, $0 \le t \le T$, is given by

$$\begin{aligned}
Q\{\tau_1 > t | \mathcal{G}_T\} &= E^Q\left[1_{\{\tau_1 > t\}} | \mathcal{G}_T\right] = \frac{E^P\left[1_{\{\tau_1 > t\}}\frac{\mathrm{d}Q}{\mathrm{d}P}\middle| \mathcal{G}_T\right]}{E^P\left[\frac{\mathrm{d}Q}{\mathrm{d}P}\middle| \mathcal{G}_T\right]} = \frac{E^P\left[E^P\left[1_{\{\tau_1 > t\}}\frac{\mathrm{d}Q}{\mathrm{d}P}\middle| \mathcal{F}'_t\right]\middle| \mathcal{G}_T\right]}{E^P\left[E^P\left[\frac{\mathrm{d}Q}{\mathrm{d}P}\middle| \mathcal{F}'_0\right]\middle| \mathcal{G}_T\right]} \\
&= \frac{E^P\left[1_{\{\tau_1 > t\}}\rho(t)\middle| \mathcal{G}_T\right]}{E^P\left[\rho(0)\middle| \mathcal{G}_T\right]} = E^P\left[1_{\{\tau_1 > t\}}\rho_1(t)\rho_2(t)\middle| \mathcal{G}_T\right] \\
&= E^P\left[1_{\{\tau_1 > t\}}\rho_1(t)\middle| \mathcal{G}_T\right] E^P\left[\rho_2(t)\middle| \mathcal{G}_T\right] \\
&= \exp\left\{-\int_0^t (1 + \xi_1(s))h_1(s)\mathrm{d}s\right\} E^P\left[E^P[\rho_2(t)|\mathcal{F}_0^2]|\mathcal{G}_T\right] \\
&= \exp\left\{-\int_0^t (1 + \xi_1(s))h_1(s)\mathrm{d}s\right\} E^P\left[\rho_2(0)|\mathcal{G}_T\right] \\
&= \exp\left\{-\int_0^t (1 + \xi_1(s))h_1(s)\mathrm{d}s\right\}.
\end{aligned} \tag{A.22}$$

[8] About the Doléans-Dade exponential, see, for example, Protter[19]. For semimartingales X and Y with $X_0 = Y_0 = 0$, $\varepsilon(X)\varepsilon(Y) = \varepsilon(X + Y + [X, Y])$ where $[X, Y]$ is the quadratic covariation of X and Y. Under the assumptions used here, we obtain $[X, Y] = 0$ where $X = \int_0^t \xi_1(s)\mathrm{d}M_1(s)$ and $Y = \int_0^t \xi_2(s)\mathrm{d}M_2(s)$. Therefore, we get (A.20).

In the above equalities we also use the conditional independence between τ_1 and τ_2 given $\mathcal{G}_T$ under P. Similarly, the conditional survival function of τ_2 is given by

$$Q\{\tau_2 > t|\mathcal{G}_T\} = \exp\left\{-\int_0^t (1+\xi_2(s))h_2(s)\mathrm{d}s\right\}. \tag{A.23}$$

By using the same techniques, it follows that, for $0 \le u \le t \le T$,

$$\begin{aligned}
Q\{\tau_1 > t, \tau_2 > u|\mathcal{G}_T\} &= E^Q[\tau_1 > t, \tau_2 > u|\mathcal{G}_T] \\
&= \frac{E^P\left[1_{\{\tau_1>t,\tau_2>u\}}\frac{\mathrm{d}Q}{\mathrm{d}P}\middle|\mathcal{G}_T\right]}{E^P\left[\frac{\mathrm{d}Q}{\mathrm{d}P}\middle|\mathcal{G}_T\right]} = \frac{E^P\left[E^P\left[1_{\{\tau_1>t,\tau_2>u\}}\frac{\mathrm{d}Q}{\mathrm{d}P}\middle|\mathcal{F}'_t\right]\middle|\mathcal{G}_T\right]}{E^P\left[E^P\left[\frac{\mathrm{d}Q}{\mathrm{d}P}\middle|\mathcal{F}'_0\right]\middle|\mathcal{G}_T\right]} = \frac{E^P\left[1_{\{\tau_1>t,\tau_2>u\}}\rho(t)\middle|\mathcal{G}_T\right]}{E^P\left[\rho(0)\middle|\mathcal{G}_T\right]} \\
&= E^P\left[1_{\{\tau_1>t\}}1_{\{\tau_2>u\}}\rho_1(t)\rho_2(t)|\mathcal{G}_T\right] = E^P\left[1_{\{\tau_1>t\}}\rho_1(t)|\mathcal{G}_T\right]E^P\left[1_{\{\tau_2>u\}}\rho_2(t)|\mathcal{G}_T\right] \\
&= \exp\left\{-\int_0^t (1+\xi_1(s))h_1(s)\mathrm{d}s\right\}E^P\left[1_{\{\tau_2>u\}}E^P\left[\rho_2(t)|\mathcal{F}^2_u\right]|\mathcal{G}_T\right] \\
&= \exp\left\{-\int_0^t (1+\xi_1(s))h_1(s)\mathrm{d}s\right\}E^P\left[1_{\{\tau_2>u\}}\rho_2(u)|\mathcal{G}_T\right] \\
&= \exp\left\{-\int_0^t (1+\xi_1(s))h_1(s)\mathrm{d}s\right\}\exp\left\{-\int_0^u (1+\xi_2(s))h_2(s)\mathrm{d}s\right\}.
\end{aligned} \tag{A.24}$$

For $0 \le t \le u \le T$, we get a similar result corresponding to (A.24).

From (A.22), (A.23) and (A.24), we obtain

$$Q\{\tau_1 > t, \tau_2 > u|\mathcal{G}_T\} = Q\{\tau_1 > t|\mathcal{G}_T\}Q\{\tau_2 > u|\mathcal{G}_T\}, \qquad 0 \le t, u \le T. \tag{A.25}$$

The above equation (A.25) means the conditional independence between τ_1 and τ_2 under Q. The multivariate version for τ_j, $j = 1, \cdots, n$ is proved similarly.

Acknowledgements

The author appreciates the financial support by the Grand-in-Aid for Scientific Research (A) 21241040 and (C) 24510194.

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Table 1. Estimated risks of future cumulative values of CDO tranches.

[attach, detach]		[0%, 3%]	[3%, 6%]	[6%, 9%]	[9%, 12%]	[12%, 22%]
Initial Face Value		37.5	37.5	37.5	37.5	37.5
Average		0.135	−1.468	−0.621	−0.302	−0.381
Standard Deviation		8.507	6.970	4.584	1.995	1.523
VaR	95.0%	17.224	14.181	2.433	1.232	1.796
	99.0%	20.550	31.268	23.551	10.448	7.493
	99.5%	21.013	31.961	24.891	11.708	7.939
	99.9%	21.877	33.157	27.364	14.398	8.996
ES	95.0%	19.578	27.731	18.319	7.787	6.186
	99.0%	21.210	32.144	25.272	12.170	8.143
	99.5%	21.511	32.696	26.378	13.335	8.595
	99.9%	21.935	33.721	28.415	15.792	9.668

Several statistics of the future cumulative values of CDO tranches. The future cumulative value is defined as the future tranche price at time horizon minus actual default loss plus future values of coupons up to time horizon. $100(1-\alpha)$-% VaR is defined as the average minus 100α-percentile where $0 < \alpha < 1$. $100(1-\alpha)$-% ES (Expected Shortfall) is defined as the average minus the conditional expectation in the region lower than 100α-percentile.

Table 2. Distributions of actual default losses of CDO tranches.

loss	$[0, 3\%]$	$[3, 6\%]$	$[6, 9\%]$
0	299,159	499,778	500,000
4.5	0	162	0
6	142,437	0	0
10.5	0	45	0
12	41,153	0	0
16.5	0	13	0
18	11,202	0	0
22.5	0	2	0
24	3,864	0	0
30	1,460	0	0
36	493	0	0
37.5	222	0	0

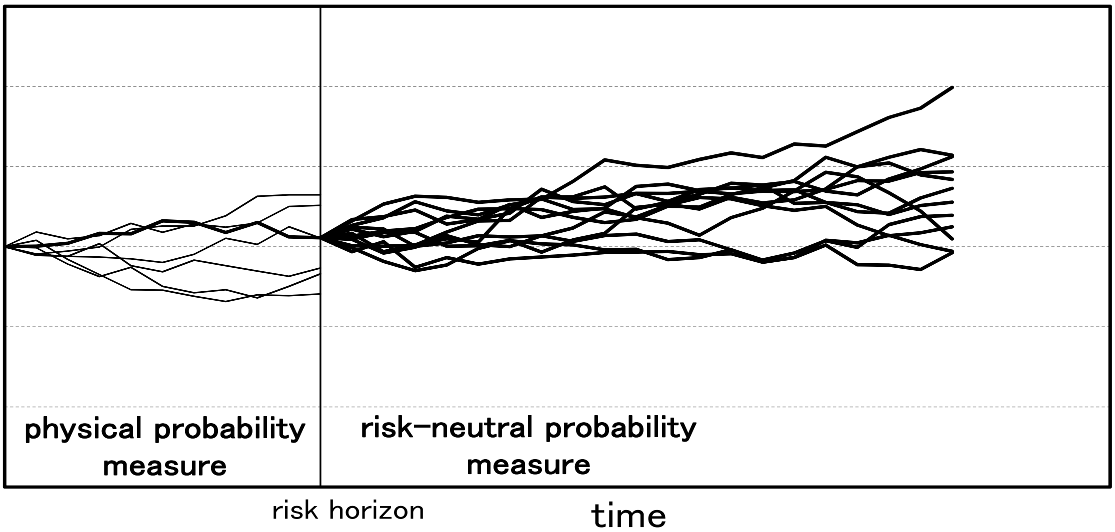
Frequency distributions of actual default losses of CDO tranches in 500,000 simulation runs. The distributions are discretized because the loss given default (LGD) per asset is 6 (fixed).

Table 3. Estimated risks of future price of bond portfolio.

Average		$8,618.4$
Standard Deviation		120.1
VaR	95.0%	186.5
	99.0%	533.7
	99.5%	552.0
	99.9%	581.9
ES	95.0%	450.0
	99.0%	556.3
	99.5%	570.3
	99.9%	596.7

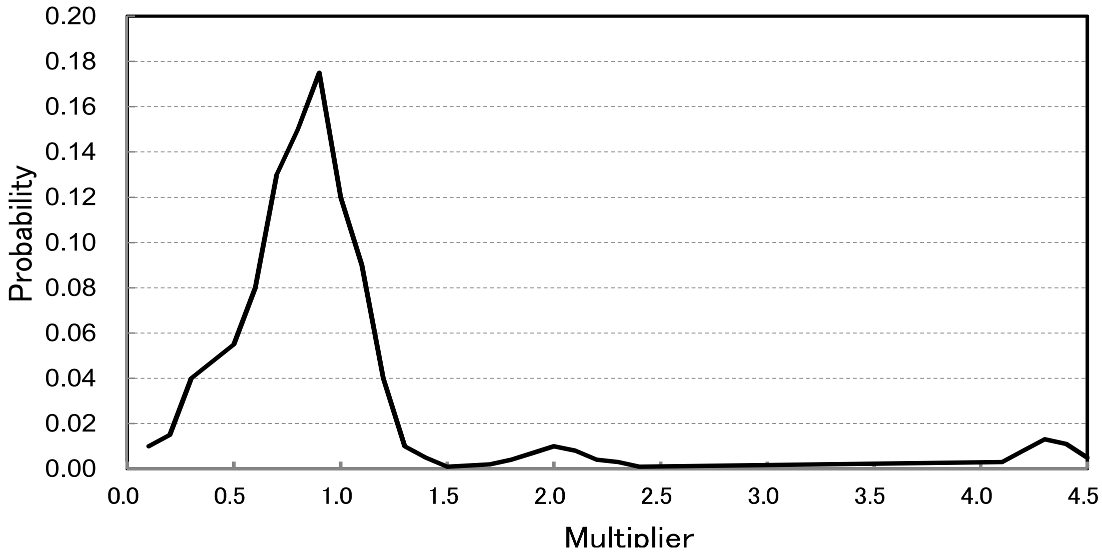
Average and Standard Deviation are those of the future price of the bond portfolio, respectively. $100(1-\alpha)$-% VaR is defined as the average minus 100α-percentile where $0 < \alpha < 1$, and $100(1-\alpha)$-% ES (Expected Shortfall) is defined as the average minus conditional expectation under 100α-percentile.

Figure 1. Image of how to use two probability measures: physical measure and risk-neutral measure.



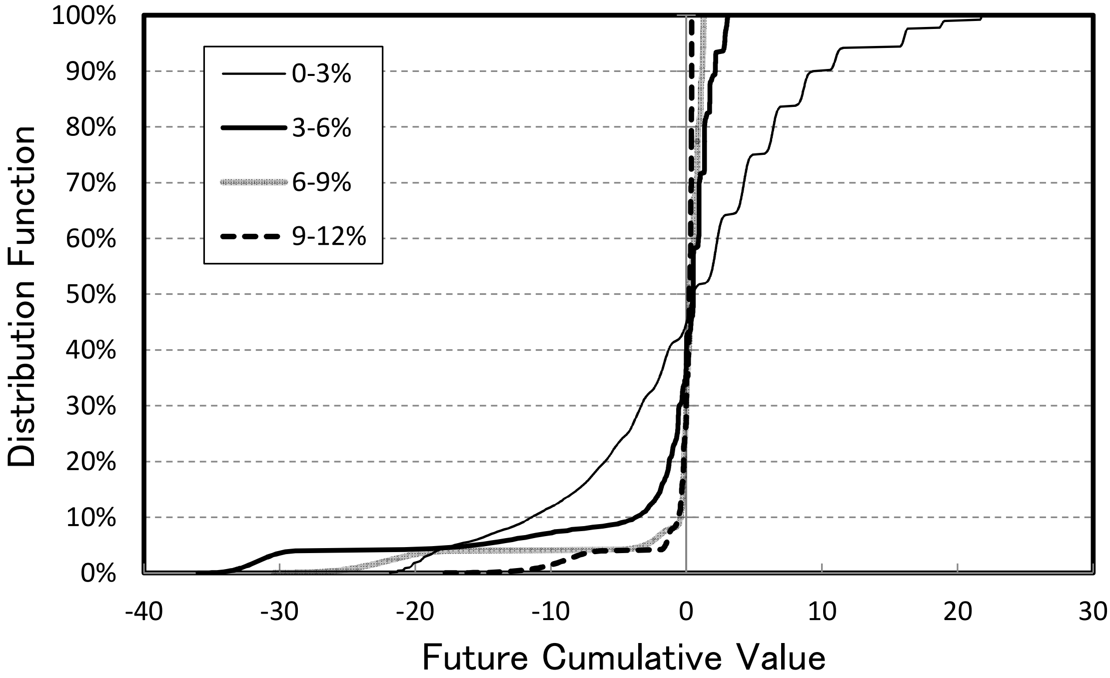
The horizontal axis is time, and the vertical axis is the value of the stochastic process considered. The left end is the present ($t = 0$), and all the sample paths begin from the present value under the physical (statistical) probability measure P. After the risk horizon T, the sample paths, which begin from the value at time T on each sample path under P, are needed under the risk-neutral probability measure $\widetilde{P}$ in order to evaluate asset prices such as derivatives at time T.

Figure 2. Distributions of multiplier $\widetilde{\kappa}$.



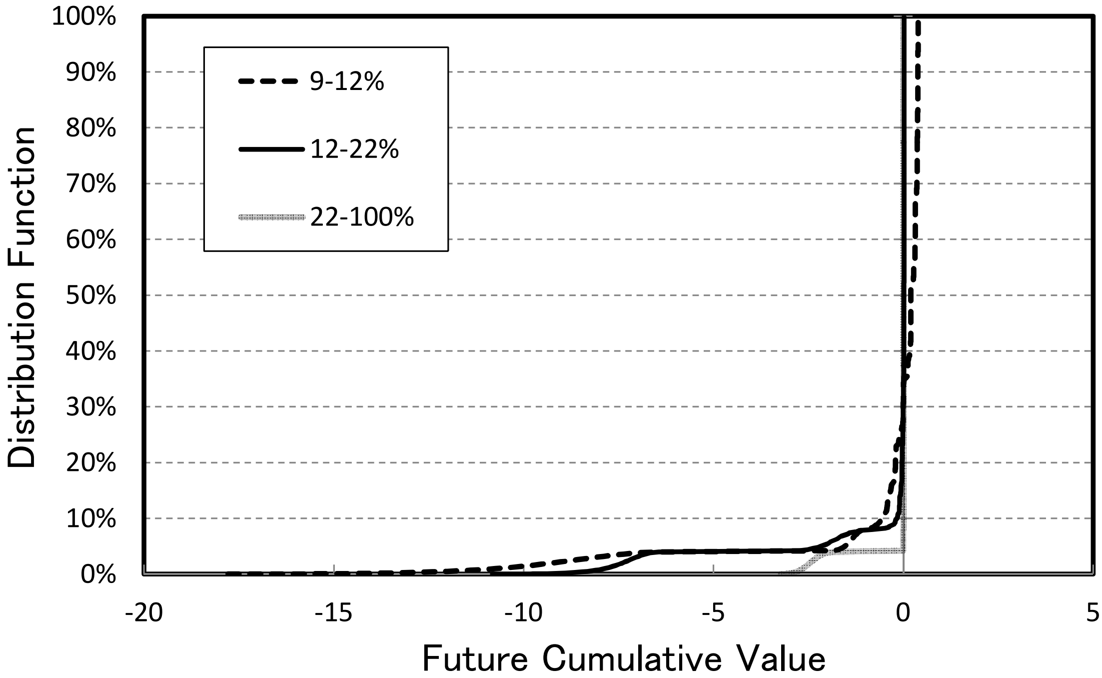
The probability function of the multiplier $\widetilde{\kappa}$. The horizontal axis is the value of $\widetilde{\kappa}$, and the vertical axis is the probability.

Figure 3(a). Distribution functions of future cumulative values of CDO tranches: equity and mezzanine tranches.



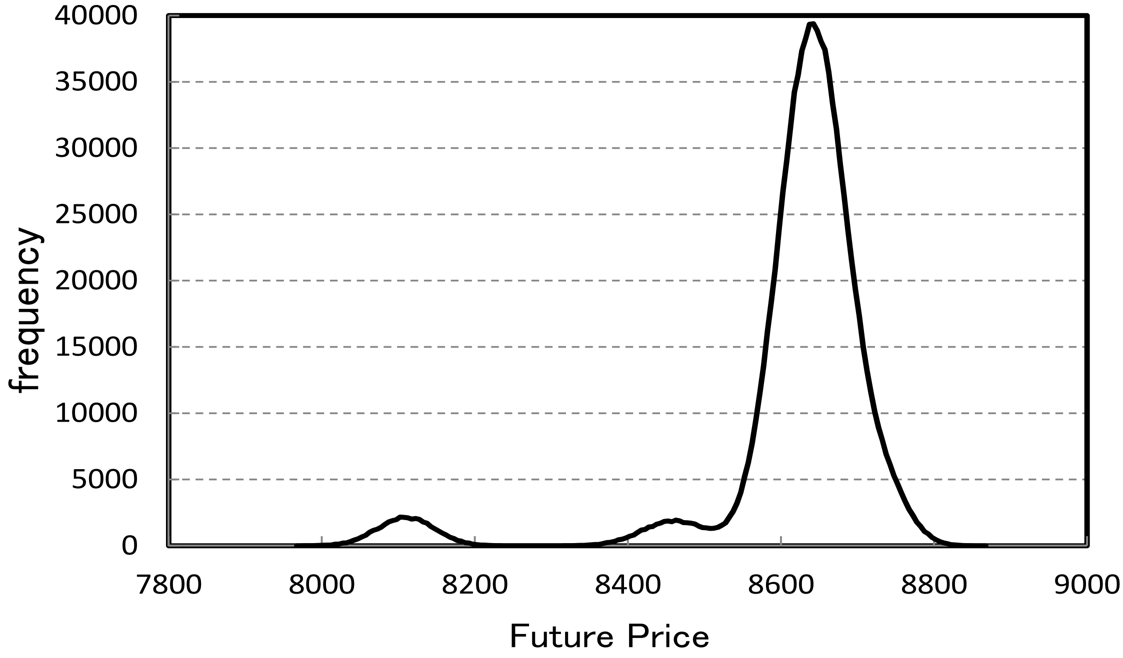
The distribution functions of the future cumulative values of tranches considered in Section 3. The horizontal axis is the future cumulative values of tranche (defined in Section 3.1), and the vertical axis is the value of the distribution function.

Figure 3(b). Distribution functions of future cumulative values of CDO tranches: senior and super-senior tranches.



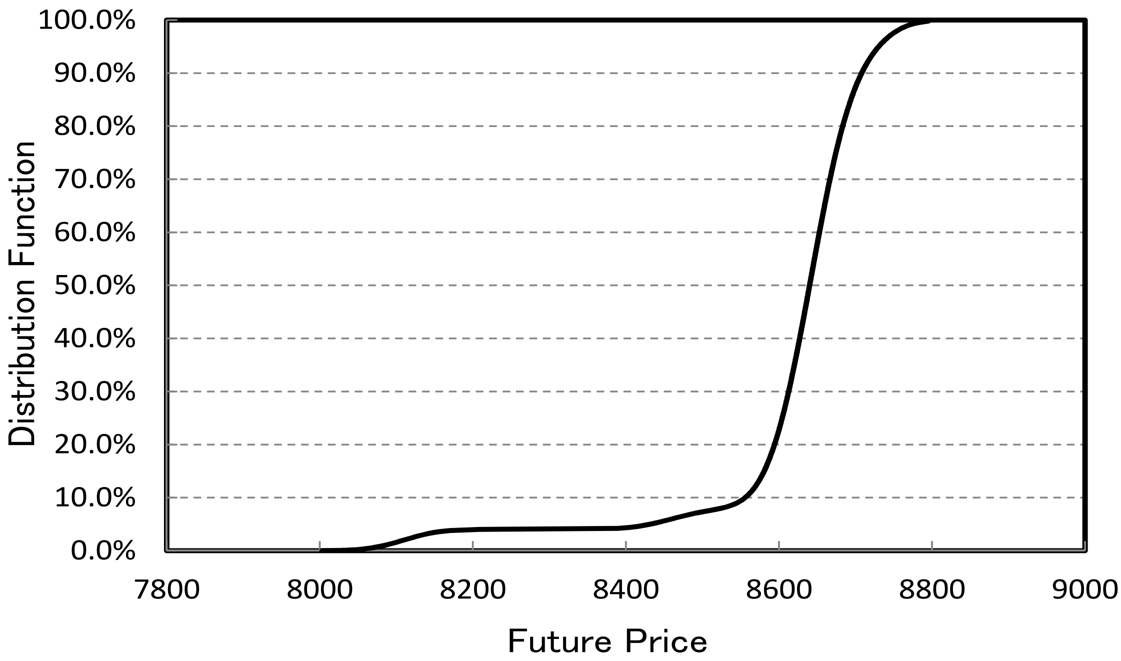
The distribution functions of the future cumulative values of tranches considered in Section 3. The horizontal axis is the future cumulative values of tranche, and the vertical axis is the value of the distribution function.

Figure 4. Distribution of future values of bond portfolio.



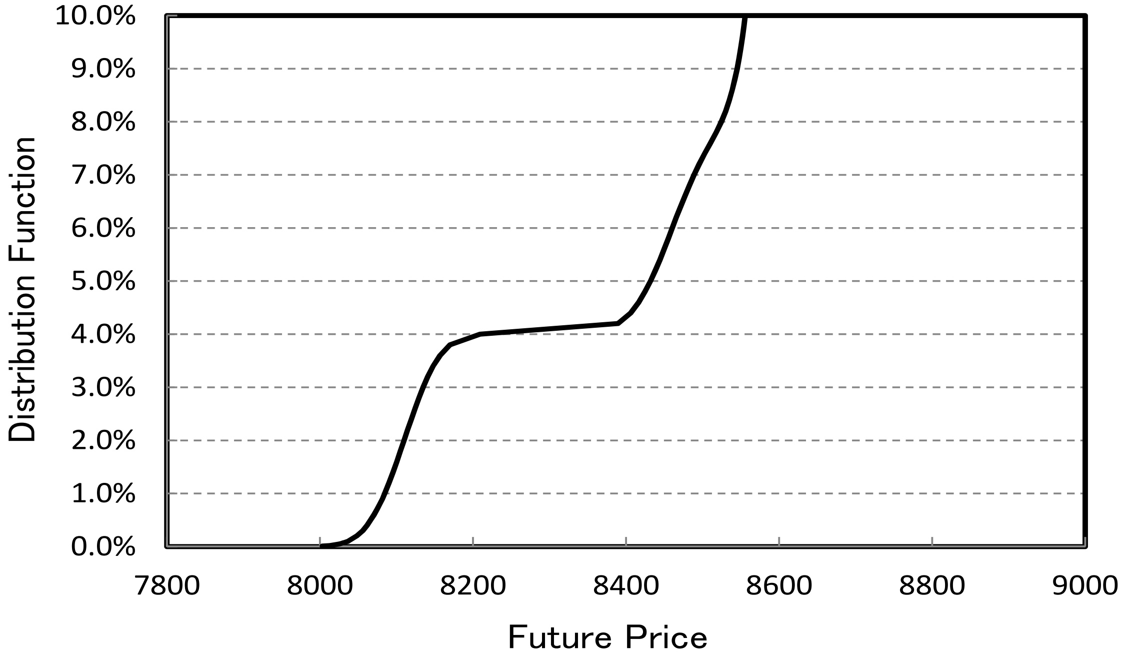
Histogram of the future value of the bond portfolio considered in Section 4. The horizontal axis is the future price of the portfolio, and the vertical axis is the frequency in the 1,000,000 simulation runs.

Figure 5. Distribution function of future values of bond portfolio.



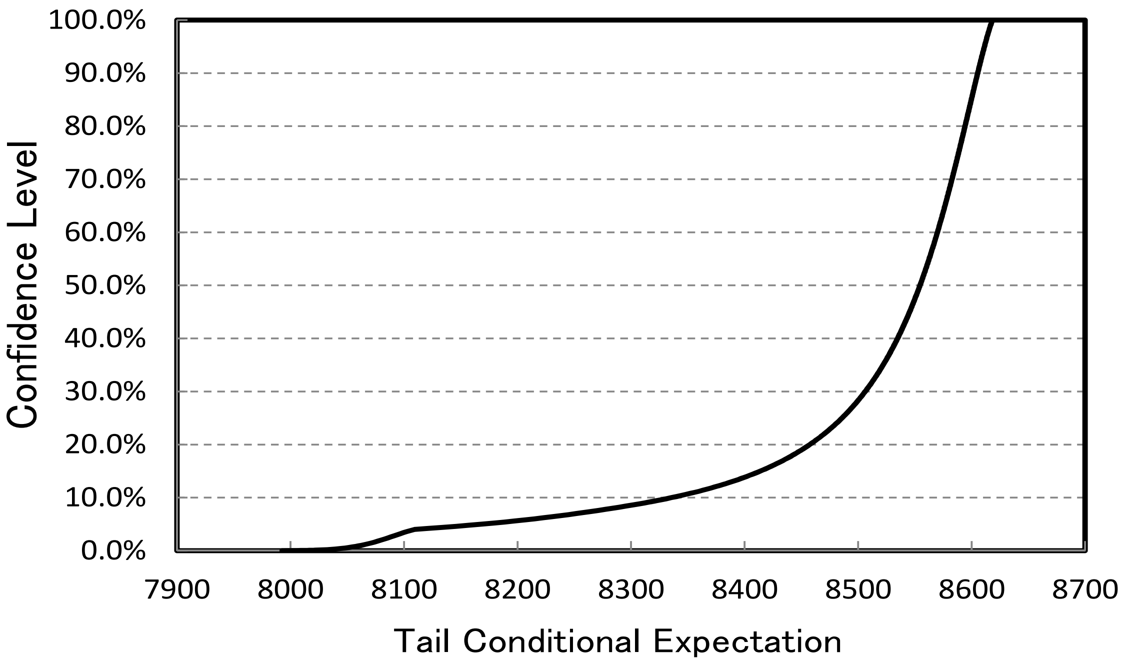
Distribution function of the future value of the bond portfolio considered in Section 4. The horizontal axis is the future price of the portfolio, and the vertical axis is the value of the distribution function.

Figure 6. Distribution function of future values of bond portfolio: left loss tail.



Distribution function of the future value of the bond portfolio considered in Section 4. The horizontal axis is the future price of the portfolio, and the vertical axis is the value of the distribution function. This figure is the lowest 10% part of the confidence level in Figure 5.

Figure 7. Tail conditional expectation of future values of bond portfolio.



Tail conditional expectation of the future value of the bond portfolio considered in Section 4. The horizontal axis is the tail conditional expectation (defined in Section 4.2) of the future value of the portfolio, and the vertical axis is the confidence level.