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Existence of a pure strategy equilibrium in finite symmetric games where payoff functions are integrally concave

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Abstract

In this paper we show that a finite symmetric game has a pure strategy equilibrium if the payoff functions of players are integrally concave (the negative of the integrally convex functions due to Favati and Tardella [Convexity in nonlinear integer programming, *Ricerca Operativa*, 1990, 53:3–44]). Since the payoff functions of any two-strategy game are integrally concave, this generalizes the result of Cheng et al. [Notes on equilibria in symmetric games, *Proceedings of the 6th Workshop On Game Theoretic And Decision Theoretic Agents*, 2004, 23–28]. A simple algorithm to find an equilibrium is also provided.

Keywords: symmetric game, integrally concave function, existence of an equilibrium

JEL Classification: C72 (Noncooperative game)

1 Introduction

The study of symmetric games dates back to Nash's seminal paper (Nash, 1951). There the general existence of a mixed strategy equilibrium was shown, with an additional information for symmetric games that there exists a *symmetric* mixed strategy equilibrium (a situation where every player chooses the same mixed strategy). Several studies extended this result and examined conditions under which symmetric games have symmetric equilibria, for example Dasgupta and Maskin (1986), Reny (1999), Becker and Damianov (2006) and Amir et al.

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(2010). These results, however, are concerned with either infinite games or mixed strategy equilibria in finite games and there are few studies about *pure* strategy equilibria in *finite* symmetric games. Cheng et al. (2004) showed that every symmetric *two-strategy* game has a (not necessarily symmetric) pure strategy equilibrium, which is also verified by the fact that every symmetric two-strategy game is a potential game (see Uno, 2009). They also remarked that the generalization to the strategy sets of more than two strategies is generally impossible, quoting the Rock-Paper-Scissors game as a counterexample.

In this paper we show that a symmetric game in which each player's set of strategies is a finite integer interval has a pure strategy equilibrium if the payoff functions of players are *integrally concave* (the negative of the integrally convex functions due to Favati and Tardella, 1990). Interestingly, the difference between the strategies for any two players in this equilibrium is at most one, irrespectively of the size of strategy sets. We also provide an algorithm for finding an equilibrium. This algorithm can find a solution in quite a few evaluations of a payoff function, whose order is the sum of the number of players and the number of strategies per player.

The class of integrally concave functions is a class of discrete functions having a feature of continuous concave functions: the local maximum coincides with the global maximum. We assume that the payoffs are integrally concave on the entire domain, i.e., on the set of strategy profiles. Thus if someone changes one's own strategy then everyone's payoff varies "concavely". Certainly, this "concavity with respect to the other player's strategy" is a bit too strong condition. It is our conjecture that some discrete analogue of "continuity with respect to the other player's strategy" condition suffices (see our remark at the end of this paper). In any way, this makes the situation more or less like the concave game of Rosen (1965), ensuring the existence of a pure strategy equilibrium. Since the payoff functions of any two-strategy game are integrally concave, our existence result generalizes the result of Cheng et al. (2004).

The results of the present paper are particularly concerned with the following four areas. The first area is the existence of pure strategy equilibria in finite games. A pure strategy equilibrium in a finite game is intuitively appealing in many environments and it is well-known that some classes of finite games, for example, supermodular games (Milgrom and Roberts, 1990; Milgrom and Shannon, 1994) and potential games (Monderer and Shapley, 1996) always have a pure strategy equilibrium. Confining ourselves to the class of symmetric games, we show, in Section 4, that the class of games with integrally concave payoffs is different from the classes of supermodular games and potential games. Thus the class of finite symmetric games with integrally concave payoffs is a new class of games ensuring the existence of pure strategy equilibria.

The second literature related to the present paper is the symmetry of equilibria in symmetric games. Some results on infinite games (Dasgupta and Maskin, 1986; Reny, 1999; Becker and Damianov, 2006) showed that symmetric games have symmetric equilibria under some conditions. Asymmetric equilibria and symmetry-breakings also have gathered attentions in recent studies. Amir et al. (2010) constructed two general classes of infinite symmetric games which always possess only asymmetric pure strategy equilibria. Fey (2011) constructed a two-player symmetric game which has only asymmetric equilibria both in pure and mixed strategies. Focusing on pure strategies in finite games, Amir et al. (2008) showed that any pure strategy equilibrium is symmetric in strictly supermodular symmetric games with one-dimensional strategy sets, but asymmetric equilibria are possible in strictly supermodular symmetric games with multi-dimensional strategy sets. In the present paper, we show that finite symmetric games with integrally concave payoffs possess either a symmetric equilibrium or an asymmetric equilibrium in which the difference of strategies between any two players is at most one. We find it interesting to note that this holds true irrespectively of the size of strategy sets (hence one might say that we have an "almost symmetric" equilibrium given a "huge" set of strategy profiles). Of course, this also applies to the case of two-person two-strategy games, for example, the symmetric equilibrium in Prisoner's Dilemma game and the asymmetric equilibria in Chicken game.

Thirdly, our paper is related to discrete convex analysis (see Murota, 2003) that is recently developed in the field of optimization theory and discrete mathematics. In the theory, several concepts of discrete convexity are proposed in order to extend the usual convex analysis to discrete settings. Integral convexity proposed by Favati and Tardella (1990) is a weak concept of discrete convexity, and covers all the important classes of discrete convexity such as M-convexity and L-convexity proposed in Murota (2003). These latter stronger discrete convexities are required to ensure some properties such as duality and separation which are fundamental in convex optimization (see Murota, 1998, 2003). Our result shows, however, that integral concavity (the negative of integral convexity) is sufficient to consider the existence of equilibria in symmetric games. We also show, in Section 4, that the concave extensibility and Miller's discrete concavity, which are weaker than integrally concavity, do not imply the existence.

Finally, an algorithm to find an equilibrium provided in the present paper is interesting in view of computation of equilibria. Cheng et al. (2004) asserted that the symmetry of a game reduces the burden of the computation to find an equilibrium and proposed some ideas for the computation. Our proof of the existence of an equilibrium is constructive and can be applicable to an algorithm for finding an equilibrium. Our algorithm finds an equilibrium in $O(n+m)$ evaluations of a payoff function, given an n-person symmetric games with m strategies per player.

The paper is organized as follows. Section 2 gives some basic definitions. Section 3 proves our claim, followed by the equilibrium algorithm. In Section 4, we discuss the conditions of our theorem and the relationship of our games to the supermodular games and potential games. Concluding remarks are given in Section 5.

2 Definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{Z}^n$ be the set of its integer points. We denote by e^i the ith unit vector of $\mathbb{R}^n$, $i = 1, \cdots, n$. For any set $X \subseteq \mathbb{R}^n$, $\mathrm{conv}(X)$ denotes the convex hull of X. We say that a set X is an interval of $\mathbb{Z}^n$, if it is written as $X = \{z \in \mathbb{Z}^n \mid a_i \leq z_i \leq b_i \ \forall i = 1, \cdots, n\}$ with some $a_i, b_i \in \mathbb{Z}$, $i = 1, \cdots, n$. Let X be an interval of $\mathbb{Z}^n$ and let $\mathrm{N}(y) := \{z \in \mathbb{Z}^n \mid |z_i - y_i| < 1 \ \forall i = 1, \cdots, n\}$.

Definition 2.1. The *local concave extension* of a discrete function $f\colon X \to \mathbb{R}$ is a piecewise-linear function $\tilde{f}\colon \mathrm{conv}(X) \to \mathbb{R}$ defined for each $y \in \mathrm{conv}(X)$ (using the points $z \in \mathrm{N}(y)$) by

$$\tilde{f}(y) := \max\{\sum \alpha_z f(z) \mid \sum \alpha_z z = y, \ \sum \alpha_z = 1, \ \alpha_z \geq 0 \text{ for all } z \in \mathrm{N}(y)\}.$$

Note that the local concave extension is locally concave on any unit cube, but not necessarily concave on the entire domain.

Definition 2.2 (Favati and Tardella (1990))**.** A discrete function $f\colon X \to \mathbb{R}$ is *integrally concave* if its local concave extension is concave (on the entire domain).

A game is a three-tuple $(N, \{S_i\}_{i\in N}, \{P_i\}_{i\in N})$, where $N := \{1, \cdots, n\}$ is the set of players, S_i is the set of strategies of $i \in N$, and P_i is the payoff function of $i \in N$ defined on the set of strategy profiles $S := S_1 \times \cdots \times S_n$. A game is finite if all the strategy sets are finite sets. For a finite game, we assume that S_i are intervals of $\mathbb{Z}$. According to Nash (1951), we define the symmetry of a game as follows. Let Π be the set of all the bijections $\pi\colon N \to N$. We call any $\pi \in \Pi$ a *permutation of players* and define a *permutation of strategy profile* $\varphi_\pi\colon S \to S$ by

$$\varphi_\pi(s) = (s_{\pi^{-1}(1)}, \cdots, s_{\pi^{-1}(n)}),$$

where $s = (s_1, \cdots, s_n)$.

Definition 2.3. A game $(N, \{S_i\}_{i \in N}, \{P_i\}_{i \in N})$ is *symmetric* if $S_1 = \cdots = S_n$ and

$$P_i(s) = P_{\pi(i)}(\varphi_\pi(s)), \quad \forall \pi \in \Pi,\ \forall s \in S,\ \forall i \in N.$$

A strategy profile $s = (s_1, \cdots, s_n) \in S$ is also denoted by $s = (s_i, s_{-i}) \in S$, where s_i is the strategy of player i and s_{-i} is the $(n-1)$-tuple of the strategies of the other players.

Definition 2.4. A strategy profile $s \in S$ is an *equilibrium* of a game $(N, \{S_i\}_{i \in N}, \{P_i\}_{i \in N})$ if $P_i(s) \geq P_i(s_i', s_{-i})$ for all $s_i' \in S_i$ for every $i \in N$.

3 Main results

Let $\Gamma := (N, \{S_i\}_{i \in N}, \{P_i\}_{i \in N})$ be a finite symmetric game, where P_i are integrally concave. We let $n := |N|$, i.e., Γ is an n-person game ($n \geq 2$). We will show the existence of an equilibrium of Γ by showing the existence of an $s \in S$ such that "ascent directions" of all the payoff functions vanish at s.

Definition 3.1. For each player $i \in N$, the *ascent direction of* P_i is a function $d_i \colon S \to \{\pm 1, 0\}$ defined for each $s \in S$ by

$$d_i(s) = \begin{cases} +1 & \text{if } P_i(s_i + 1, s_{-i}) > P_i(s), \\ -1 & \text{if } P_i(s_i - 1, s_{-i}) > P_i(s), \\ 0 & \text{otherwise,} \end{cases}$$

where we let $P_i(s \pm e^i) = -\infty$ if $s \pm e^i \notin S$.

Note that d_i *is well-defined*, since if P_i is integrally concave, then the local concave extension $\tilde{P}_i$ is concave on $\mathrm{conv}(S_i) \times \{s_{-i}\}$ given any s_{-i}. Also note that

$$d_i(s) = 0 \text{ for all } i \in N \iff s \text{ is an equilibrium of } \Gamma.$$

We use the next property of ascent directions.

Lemma 3.1. *Let $d_i \colon S \to \{\pm 1, 0\}$ be the ascent direction of P_i. Then*

$$d_i(s) > 0 \implies d_i(s \pm e^j - e^i) > 0 \quad \text{and} \quad d_i(s) < 0 \implies d_i(s \mp e^j + e^i) < 0, \quad \forall j \in N \setminus \{i\},$$

whenever the arguments are in S.

Proof. To see that the first implication holds true, suppose by way of contradiction that $d_i(s) > 0$ and $d_i(s \pm e^j - e^i) \leq 0$. Then since $d_i(s) > 0 \iff P_i(s + e^i) > P_i(s)$ and $d_i(s \pm e^j - e^i) \leq 0 \iff P_i(s \pm e^j - e^i) \geq P_i(s \pm e^j)$, we have

$$\frac{P_i(s + e^i) + P_i(s \pm e^j - e^i)}{2} > \frac{P_i(s) + P_i(s \pm e^j)}{2}. \tag{1}$$

For the right-hand side, $\mathrm{N}\Big(\frac{s+(s\pm e^j)}{2}\Big) = \{s, s\pm e^j\}$ implies

$$\frac{P_i(s) + P_i(s\pm e^j)}{2} = \tilde{P}_i\Big(\frac{s+(s\pm e^j)}{2}\Big),$$

where $\tilde{P}_i$ is the local concave extension of P_i.

Now, since

$$\frac{s+(s\pm e^j)}{2} = \frac{(s+e^i) + (s\pm e^j - e^i)}{2},$$

the inequality (1) is rewritten as

$$\frac{\tilde{P}_i(s+e^i) + \tilde{P}_i(s\pm e^j - e^i)}{2} > \tilde{P}_i\Big(\frac{(s+e^i)+(s\pm e^j - e^i)}{2}\Big).$$

However, the concavity of $\tilde{P}_i$ implies

$$\frac{\tilde{P}_i(s+e^i) + \tilde{P}_i(s\pm e^j - e^i)}{2} \le \tilde{P}_i\Big(\frac{(s+e^i)+(s\pm e^j - e^i)}{2}\Big),$$

which is a contradiction. Hence we must have $d_i(s) > 0 \implies d_i(s\pm e^j - e^i) > 0$.

To see that the second implication holds true, suppose by way of contradiction that $d_i(s) < 0$ and $d_i(s \mp e^j + e^i) \ge 0$. Then since $d_i(s) < 0 \iff P_i(s-e^i) > P_i(s)$ and $d_i(s\mp e^j + e^i) \ge 0 \iff P_i(s\mp e^j + e^i) \ge P_i(s\mp e^j)$, we have

$$P_i(s\mp e^j + e^i) + P_i(s-e^i) > P_i(s\mp e^j) + P_i(s).$$

But if all the arguments are shifted by $\pm e^j$, this yields (1), which resulted in a contradiction. Hence we must have $d_i(s) < 0 \implies d_i(s\pm e^j + e^i) < 0$. □

As a corollary to this, we have the next lemma.

Lemma 3.2. *Let $d_i\colon S \to \{\pm 1, 0\}$ be the ascent direction of P_i. Then*

$$d_i(s) > 0 \implies d_i(s\pm e^j) \ge 0 \quad \textit{and} \quad d_i(s) < 0 \implies d_i(s\mp e^j) \le 0, \quad \forall j \in N\setminus\{i\},$$

whenever the arguments are in S.

Proof. We have $d_i(s\pm e^j) \ge 0$ if $s\pm e^j - e^i \notin S$, by the definition of d_i. If $s\pm e^j - e^i \in S$, then $d_i(s) > 0 \implies d_i(s\pm e^j - e^i) > 0$ by Lemma 3.1, and $d_i(s\pm e^j - e^i) > 0$ implies $d_i(s\pm e^j) \not< 0$, since otherwise $P_i(s\pm e^j - e^i) < P_i(s\pm e^j)$ and $P_i(s\pm e^j - e^i) > P_i(s\pm e^j)$, a contradiction. Hence $d_i(s) > 0 \implies d_i(s\pm e^j) \ge 0$. That $d_i(s) < 0 \implies d_i(s\mp e^j) \le 0$ is similarly shown. □

When dealing with d_i's under the symmetry, note that

$$d_i(s) = d_{\pi(i)}(\varphi_\pi(s)), \quad \forall \pi \in \Pi,\ \forall s \in S,\ \forall i \in N.$$

We let $\pi_{i,j} \in \Pi$ be a special type of permutation of players, the *transposition* of players i and j such that $\pi_{i,j}(i) = j$, $\pi_{i,j}(j) = i$, and $\pi_{i,j}(h) = h$ for all $h \neq i, j$. Then

$$d_i(s) = d_j(\varphi_{\pi_{i,j}}(s)).$$

Let $z \in \mathbb{Z}$ and let $V_z \subseteq S$ be the set of vertices of a unit cube such that

$$V_z = \{s \in S \mid s_i \in \{z, z+1\},\ i = 1, \cdots, n\}.$$

Lemma 3.3. *Let $s = (z, \cdots, z)$ and $s' = (z+1, \cdots, z+1)$. If $d_1(s) = 1$ and $d_1(s') = -1$, then there exists a pure strategy equilibrium of Γ in V_z.*

Proof. Note that $d_1(s) = 1$ and $d_1(s') = -1$ imply $d_i(s) = 1$ and $d_i(s') = -1$ for any $i \in N$ since $\varphi_{\pi_{1,i}}(s) = s$ and $\varphi_{\pi_{1,i}}(s') = s'$ for any $i \in N$. We let $(s^0, s^1, \cdots, s^n)$ be a sequence of points in V_z such that $s^0 = s$, $s^k = s^{k-1} + e^k$ for $k = 1, \cdots, n$, and $s^n = s'$.

To prove the lemma, we observe the following two facts. First, for any $i \leq k$, $\varphi_{\pi_{1,i}}(s^k) = s^k$ because $s^k_i = s^k_1 = z+1$. Hence $d_i(s^k) = d_1(\varphi_{\pi_{1,i}}(s^k)) = d_1(s^k)$. Similarly, for any $i > k$, $\varphi_{\pi_{n,i}}(s^k) = s^k$ because $s^k_i = s^k_n = z$, and $d_i(s^k) = d_n(\varphi_{\pi_{n,i}}(s^k)) = d_n(s^k)$. In summary we obtain

$$d_i(s^k) = \begin{cases} d_1(s^k) & \text{if } i \leq k, \\ d_n(s^k) & \text{if } i > k. \end{cases} \tag{2}$$

Second, we find that, for any $k = 1, \cdots, n$,

$$d_n(s^{k-1}) > 0 \quad \Longrightarrow \quad d_1(s^k) \geq 0. \tag{3}$$

To see this, note that $d_n(s^{k-1}) > 0$ implies $d_k(s^{k-1}) > 0$ since $d_k(s^{k-1}) = d_n(s^{k-1})$ by (2). If $d_1(s^k) < 0$ then $d_k(s^k) < 0$ by (2), so $d_k(s^k)d_k(s^{k-1}) < 0$, but this contradicts Lemma 3.2 since $d_k(s^k) = d_k(s^{k-1} + e^k)$. Thus we must have $d_1(s^k) \geq 0$ if $d_n(s^{k-1}) > 0$.

Now, we will prove the lemma. Let $\psi(s) := (d_1(s), d_n(s))$. Since $\psi(s^0) = (1,1)$ and $\psi(s^n) = (-1,-1)$, and since $d_1(s^{k-1}) > 0 \implies d_1(s^k) \geq 0$ and $d_n(s^{k-1}) > 0 \implies d_n(s^k) \geq 0$ for any $k \geq 1$ by Lemma 3.2, there exists the greatest integer k^* such that $\psi(s^{k^*}) = (0,0)$ or $\psi(s^{k^*}) = (1,0)$ or $\psi(s^{k^*}) = (0,1)$. That is, the greatest integer k^* such that both $d_1(s^{k^*})$ and $d_n(s^{k^*})$ are nonnegative, but not both ones. Observe that $k^* \neq n$ (since $\psi(s^n) = (-1,-1)$). If $k^* = n-1$ then $\psi(s^{k^*}) \neq (1,0)$ and $\psi(s^{k^*}) \neq (0,1)$, since both $\psi(s^{k^*}) = (1,0)$ and $\psi(s^{k^*}) = (0,1)$ contradict $\psi(s^{k^*+1}) = \psi(s^n) = (-1,-1)$ under Lemma 3.2. In the rest of the proof, we show that $\psi(s^{k^*}) \neq (1,0)$ and $\psi(s^{k^*}) \neq (0,1)$ also if $k^* < n-1$.

Suppose first $\psi(s^{k^*}) = (0,1)$. Then $d_1(s^{k^*}) = 0$ and $d_n(s^{k^*}) > 0$, and the maximality of k^* implies that $d_1(s^{k^*+1}) < 0$ or $d_n(s^{k^*+1}) < 0$. Here $d_1(s^{k^*+1}) < 0$ is impossible, since $d_n(s^{k^*}) > 0 \implies d_1(s^{k^*+1}) \geq 0$ by (3). Also $d_n(s^{k^*+1}) < 0$ is impossible, since $d_n(s^{k^*}) > 0 \implies d_n(s^{k^*+1}) \geq 0$ by Lemma 3.2. Hence $\psi(s^{k^*}) \neq (0,1)$.

Suppose next $\psi(s^{k^*}) = (1,0)$. Then $d_1(s^{k^*}) > 0$ and $d_n(s^{k^*}) = 0$, and the maximality of k^* implies that $d_1(s^{k^*+1}) < 0$ or $d_n(s^{k^*+1}) < 0$. Here $d_1(s^{k^*+1}) < 0$ is impossible, since $d_1(s^{k^*}) > 0 \implies d_1(s^{k^*+1}) \geq 0$ by Lemma 3.2. Also $d_n(s^{k^*+1}) < 0$ is impossible. To see this, note that $d_n(s^{k^*+1}) < 0 \implies d_n(s^{k^*+1} + e^n - e^1) < 0$ by Lemma 3.1, and $d_n(s^{k^*+1} + e^n - e^1) = d_1(s^{k^*+1})$ since $s^{k^*+1} + e^n - e^1 = \varphi_{\pi_{1,n}}(s^{k^*+1})$. Hence, $d_n(s^{k^*+1}) < 0 \implies d_1(s^{k^*+1}) < 0$. But since $d_1(s^{k^*}) > 0 \implies d_1(s^{k^*+1}) \geq 0$ by Lemma 3.2, this is impossible. Hence $\psi(s^{k^*}) \neq (1,0)$.

Therefore, we conclude that $\psi(s^{k^*}) = (0,0)$. Then s^{k^*} is an equilibrium because (2) implies $d_i(s^{k^*}) = 0$ for any $i \in N$. This completes the proof. $\square$

Theorem 3.1. *A finite symmetric game has a pure strategy equilibrium if the payoff functions of players are integrally concave. In particular, there exists an equilibrium $s^* \in S$ such that $|s_i^* - s_j^*| \leq 1$ for any $i, j \in N$.*

Proof. Let $S_1 = \cdots = S_n = \{1, \cdots, m\}$. If there exists an $s = (z, \cdots, z) \in S$ such that $d_1(s) = 0$, then $d_i(s) = 0$ for any $i \in N$ by the symmetry, and $s^* := s$ is a symmetric equilibrium. Clearly, $|s_i^* - s_j^*| = 0$ for any $i, j \in N$.

Otherwise, since $d_1(1, \cdots, 1) = 1$ and $d_1(m, \cdots, m) = -1$, there exists an integer z such that $d_1(s) = 1$ and $d_1(s') = -1$ for $s := (z, \cdots, z)$ and $s' := (z+1, \cdots, z+1)$. Then there exists a pure strategy equilibrium s^* in V_z by Lemma 3.3. Clearly, this s^* also satisfies $|s_i^* - s_j^*| \leq 1$ for any $i, j \in N$. $\square$

We note that if $s \in S$ is an equilibrium of a symmetric game, then $\varphi_\pi(s) \in S$ is also an equilibrium given any $\pi \in \Pi$. This is so, since we have $d_{\pi(i)}(\varphi_\pi(s)) = d_i(s) = 0$ for any $i \in N$ given any $\pi \in \Pi$ if s is an equilibrium.

An equilibrium algorithm is shown by the next pseudocode, which consumes up to $2(m+n-1)$ times of evaluations of ascent directions d_1 and d_n, where n is the number of players and m is the number of strategies per player. Since $d_i(s)$ is determined by two evaluations of P_i, this says that an equilibrium is found within $4(m+n-1)$ evaluations of P_1 and P_n. Actually we only need d_1 (hence P_1) since $d_n(s) = d_1(\varphi_{\pi_{1,n}}(s))$. Note, however, that this finds only a sample equilibrium s (up to its equivalents $\varphi_\pi(s)$).

Algorithm 1 Find and return an equilibrium

Require: $d_1(1, \cdots, 1) \geq 0$ and $d_1(m, \cdots, m) \leq 0$

for $z = 1$ **to** $m - 1$ **do**
  if $d_1(z, \cdots, z) = 0$ **then**
    return $(z, \cdots, z)$
  else if $d_1(z + 1, \cdots, z + 1) < 0$ **or** $z = m - 1$ **then**
    $s \leftarrow (z, \cdots, z)$
    for $i = 1$ **to** n **do**
      $s \leftarrow s + e^i$
      if $d_1(s) = 0$ **and** $d_n(s) = 0$ **then**
        return s
      end if
    end for
  end if
end for

4 Discussion

In the previous section, we proved that a finite game has a pure strategy equilibrium if (i) the payoff functions of players are integrally concave and (ii) the game is symmetric. In this section, we first discuss these two conditions.

Let us begin with the following example.

Example 4.1. Let $\tilde{\Gamma}_1 = (\{1, 2\}, \{\tilde{S}_1, \tilde{S}_2\}, \{\tilde{P}_1, \tilde{P}_2\})$ be a game with compact and convex strategy sets $\tilde{S}_1 = \tilde{S}_2 = [0, 1] \subset \mathbb{R}$ and piecewise-linear concave payoff functions

$$\tilde{P}_1(s_1, s_2) = 1 - 2|s_1 + s_2 - 1|,$$
$$\tilde{P}_2(s_1, s_2) = 1 - 2|s_2 - s_1|.$$

For any $i = 1, 2$, the linearity domains of $\tilde{P}_i$ are integral, i.e., the sets on which $\tilde{P}_i$ is linear are spanned by integer points ($\tilde{P}_i$ has two linearity domains, whose border is $\mathrm{conv}\{(0, 1), (1, 0)\}$ for $i = 1$ and $\mathrm{conv}\{(0, 0), (1, 1)\}$ for $i = 2$). This game has a pure strategy equilibrium $(\frac{1}{2}, \frac{1}{2})$.

Now, let $\Gamma_1 = (\{1, 2\}, \{S_1, S_2\}, \{P_1, P_2\})$ be a game with strategy sets $S_1 = S_2 = \{0, 1\} \subset \mathbb{Z}$ and the payoff P_i, which are the restrictions of $\tilde{P}_i$ to $S_1 \times S_2$, $i = 1, 2$. Then Γ_1 is a Matching Pennies game described in Figure 1. Γ_1 does not have a pure strategy equilibrium. ■

This example tells us about two important things. First, the existence of a pure strategy equilibrium in an infinite game does not imply the existence of a pure strategy equilibrium

1 \ 2	0	1
0	(-1 , 1)	(1 , -1)
1	(1 , -1)	(-1 , 1)

Figure 1: A Matching Pennies game Γ_1, which is the restriction of $\tilde{\Gamma}_1$ to $S_1 \times S_2$.

in the game restricted to the integer lattice. Second, the payoff functions of both players in a Matching Pennies game are integrally concave, since $\tilde{P}_1$ and $\tilde{P}_2$ are also the local concave extensions of P_1 and P_2, respectively. (Note that *any* discrete function defined on the vertices of a unit cube is integrally concave, since its local concave extension concave on the cube is concave on the entire domain.) Thus the integral concavity of payoff functions alone is not sufficient to ensure the existence of a pure strategy equilibrium. At the same time, the symmetry alone is also insufficient, as the Rock-Paper-Scissors game suggests. It is instructive here to have a brief look at the payoff function of Rock-Paper-Scissors game

$$
\begin{array}{lll}
P_1(0,0) = 0 & P_1(0,1) = -1 & P_1(0,2) = 1 \\
P_1(1,0) = 1 & P_1(1,1) = 0 & P_1(1,2) = -1 \\
P_1(2,0) = -1 & P_1(2,1) = 1 & P_1(2,2) = 0,
\end{array}
$$

where the Rock, Paper, and Scissors are numbered 0, 1, and 2, respectively. As we can see, the player 1's payoff fails to satisfy concavity of any sort with respect to his own strategy when player 2 chooses the Scissors. The situation is similar under any numbering of strategies: there is always a strategy of player 2 against which player 1's payoff loses concavity. This suggests that we need some sort of discrete concavity uniformly for the payoff functions, in addition to the symmetry, in order for a game to have a pure strategy equilibrium.

We have generalized the result of Cheng et al. (2004) to the symmetric game with more than two strategies by assuming the integral concavity of payoffs. But, it remains to clarify whether or not there exists a wider class of discrete concavity than integral concavity that also works well. There are some definitions of concavity for discrete functions (see Murota, 2003), among which the *concave extensibility* and *Miller's discrete concavity* (Miller, 1971) are known to be weaker than integral concavity: any integrally concave function is concave extensible, and also discretely concave in Miller's sense. Let us examine these two conditions, in turn.

A function f from an interval $X \subset \mathbb{Z}^n$ to $\mathbb{R}$ is said to be *concave extensible* if $\overline{f}(x) = f(x)$

for any $x \in X$, where $\overline{f}\colon \mathrm{conv}(X) \to \mathbb{R}$ is the *concave closure* of f defined by

$$\overline{f}(y) := \inf_{p \in \mathbb{R}^n, \alpha \in \mathbb{R}} \{p \cdot y + \alpha \mid p \cdot x + \alpha \geq f(x) \text{ for all } x \in X\}$$

("dot" denotes the inner product). The following example shows that a symmetric game may not have a pure strategy equilibrium, if the payoff functions are just concave extensible.

Example 4.2. Let $\overline{P}_1 : [0,2]^2 \to \mathbb{R}$ be a function defined by

$$\overline{P}_1(s_1, s_2) = \begin{cases} s_1 + 5s_2 - 2 & \text{if } s_1 + 2s_2 \leq 2, \\ s_1 - 4s_2 + 7 & \text{if } -s_1 + s_2 \geq 1, \\ -2s_1 - s_2 + 4 & \text{otherwise.} \end{cases}$$

Then $\overline{P}_1$ is a piecewise-linear concave function, whose linearity domains are integral ($\overline{P}_1$ has three linearity domains separated by two borders $\mathrm{conv}\{(0,1),(2,0)\}$ and $\mathrm{conv}\{((0,1),(1,2)\}$). By this integrality of linearity domains, its restriction P_1 to $\{0,1,2\}^2$ is a concave extensible function whose concave closure equals $\overline{P}_1$.

Let P_2 be the symmetric counterpart of P_1, and let $\Gamma_2 = (\{1,2\}, \{S_1, S_2\}, \{P_1, P_2\})$, where $S_1 = S_2 = \{0,1,2\}$. Then Γ_2 is a finite symmetric game with concave extensible payoff functions (Figure 2). However, Γ_2 has no pure strategy equilibrium. ■

1 \ 2	0	1	2
0	(−2, −2)	(3 , −1)	(−1, 0)
1	(−1, 3)	(1 , 1)	(0, −1)
2	(0 , −1)	(−1, 0)	(−2, −2)

Figure 2: Concave extensible game Γ_2 of Example 4.2 does not have a pure strategy equilibrium.

A function f from an interval $X \subset \mathbb{Z}^n$ to $\mathbb{R}$ is called a *Miller's discrete concave function* if

$$\max\{f(z) \mid z \in \mathrm{N}(\alpha x + (1-\alpha)y)\} \geq \alpha f(x) + (1-\alpha) f(y), \quad \forall x, y \in X,\ 0 \leq \alpha \leq 1$$

(see, Murota (2003); Miller (1971); recall that $\mathrm{N}(y) := \{z \in \mathbb{Z}^n \mid |z_i - y_i| < 1\ \forall i = 1, \cdots, n\}$). We observe that the payoff function P_1 (hence P_2) of Γ_2 also satisfies this condition; i.e., this is a Miller's discrete concave function that is not integrally concave. As we have observed, however, Γ_2 has no pure strategy equilibrium.

Hence, we claim that the integral concavity of payoff functions is a minimum requirement of discrete concavity to ensure a pure strategy equilibrium in a symmetric game with more than two strategies.

Now let us discuss the relationships of our games with the supermodular games and potential games, in turn. As the results of Milgrom and Roberts (1990) and Milgrom and Shannon (1994), any supermodular game (finite or infinite) has a pure strategy equilibrium. A game $\Gamma = (N, \{S_i\}_{i \in N}, \{P_i\}_{i \in N})$ is said to be a supermodular game if for any $i \in N$, (a) S_i is a complete lattice, (b) $P_i(s_i, s_{-i})$ is supermodular in s_i, and (c) $P_i(s_i, s_{-i})$ satisfies increasing differences in s_i and s_{-i}, i.e.,

$$P_i(s_i, s_{-i}) - P_i(s'_i, s_{-i}) \geq P_i(s_i, s'_{-i}) - P_i(s'_i, s'_{-i}) \quad \forall s_i \geq s'_i \quad \forall s_{-i} \geq s'_{-i}.$$

If S_i is an interval of $\mathbb{Z}$, both (a) and (b) are always satisfied. Hence, in our settings, Γ is a supermodular game if Γ satisfies (c).

A game with integrally concave payoff functions seems to be close to a supermodular game (recall our Algorithm; we could find an equilibrium by chasing an increasing sequence of points). However, the following two examples show that neither of the classes of games includes the other.

Example 4.3. Let $\tilde{P}_1 : [-1, 1]^2 \to \mathbb{R}$ be a function defined by

$$\tilde{P}_1(s_1, s_2) = \begin{cases} 1 - (|s_1| + |s_2|) & \text{if } |s_1| + |s_2| \geq 1, \\ 0 & \text{if } |s_1| + |s_2| \leq 1, \end{cases}$$

and P_1 be the restriction of $\tilde{P}_1$ to $\{-1, 0, 1\}^2$. Since $\tilde{P}_1$ is the local concave extension of P_1 and $\tilde{P}_1$ is concave, P_1 is integrally concave. Let $S_1 = S_2 = \{-1, 0, 1\}$, P_2 the symmetric companion of P_1, and $\Gamma_3 = (\{1, 2\}, \{S_1, S_2\}, \{P_1, P_2\})$. The game Γ_3 is described in Figure 3 (there are one symmetric and four asymmetric equilibria).

The game Γ_3 is a symmetric game with integrally concave payoffs, but P_1 does not satisfy increasing differences in s_1 and s_2, because

$$P_1(0, -1) - P_1(-1, -1) > P_1(0, 0) - P_1(-1, 0).$$

Hence Γ_3 is not a supermodular game. Note that P_1 does not also satisfy decreasing differences in s_1 and s_2, because

$$P_1(0, 0) - P_1(-1, 0) < P_1(0, 1) - P_1(-1, 1).$$

This says that Γ_3 cannot be a supermodular game even by changing the order of the strategies for either or both of players. ■

1 \ 2	-1	0	1
-1	$(-1, -1)$	$(0, 0)$	$(-1, -1)$
0	$(0, 0)$	$(0, 0)$	$(0, 0)$
1	$(-1, -1)$	$(0, 0)$	$(-1, -1)$

Figure 3: A game with integrally concave payoffs which is not a supermodular game.

Example 4.4. Let $\Gamma_4 = (\{1, 2\}, \{S_1, S_2\}, \{P_1, P_2\})$, where $S_1 = S_2 = \{0, 1, 2\}$, P_1 is defined by

$$\begin{array}{lll} P_1(0,0) = 0 & P_1(0,1) = 0 & P_1(0,2) = 0 \\ P_1(1,0) = -1 & P_1(1,1) = 0 & P_1(1,2) = 1 \\ P_1(2,0) = 0 & P_1(2,1) = 2 & P_1(2,2) = 4, \end{array}$$

and P_2 is its symmetric counterpart. The game Γ_4 is described in Figure 4 (there are two symmetric equilibria). We can easily check that P_1 satisfies increasing differences so that Γ_4 is a supermodular game.

However, P_1 is not integrally concave. To see this, let $\tilde{P}_1$ be the local concave extension of P_1. $\tilde{P}_1$ is not concave because $\tilde{P}_1(0, 0) = \tilde{P}_1(2, 0) = 0$ and $\tilde{P}_1(1, 0) = -1$. ■

1 \ 2	0	1	2
0	$(0, 0)$	$(0, -1)$	$(0, 0)$
1	$(-1, 0)$	$(0, 0)$	$(1, 2)$
2	$(0, 0)$	$(2, 1)$	$(4, 4)$

Figure 4: A supermodular game whose payoff functions are not integrally concave.

As was introduced and proved by Monderer and Shapley (1996), potential games also admit a pure strategy equilibrium. A game $\Gamma = (N, \{S_i\}_{i \in N}, \{P_i\}_{i \in N})$ is said to be an ordinal potential game if there exists an ordinal potential function $G\colon S \to \mathbb{R}$ such that

$$P_i(x_i, s_{-i}) > P_i(y_i, s_{-i}) \iff G(x_i, s_{-i}) > G(y_i, s_{-i}), \qquad \forall s \in S,\ i \in N,\ x_i, y_i \in S_i.$$

It is called a potential game if there exists a potential function G such that

$$P_i(x_i, s_{-i}) - P_i(y_i, s_{-i}) = G(x_i, s_{-i}) - G(y_i, s_{-i}), \qquad \forall s \in S,\ i \in N,\ x_i, y_i \in S_i.$$

Clearly, any potential game is an ordinal potential game. The next two examples are a finite symmetric game with integral concave payoffs that cannot be a potential game, and a finite symmetric potential game that does not have integrally concave payoffs, respectively.

Example 4.5. Consider a symmetric game $\Gamma_5 = (\{1,2\}, \{S_1, S_2\}, \{P_1, P_2\})$, where $S_1 = S_2 = \{0,1,2\}$, P_1 is defined by

$$\begin{array}{lll} P_1(0,0) = 0 & P_1(0,1) = 0 & P_1(0,2) = 0 \\ P_1(1,0) = 0 & P_1(1,1) = 1 & P_1(1,2) = 0 \\ P_1(2,0) = 0 & P_1(2,1) = 1 & P_1(2,2) = 0, \end{array}$$

and P_2 is its symmetric counterpart. The game Γ_5 is described in Figure 5 (there are three symmetric and four asymmetric equilibria). Then the local concave extension $\tilde{P}_1\colon [0,2]^2 \to \mathbb{R}$ of P_1 is given by

$$\tilde{P}_1(s_1, s_2) = \begin{cases} s_1 & \text{if } s_2 \geq s_1 \text{ and } s_2 \leq 2 - s_1, \\ s_2 & \text{if } s_2 \leq s_1 \text{ and } s_2 \leq 1, \\ 2 - s_2 & \text{if } s_2 \geq 2 - s_1 \text{ and } s_2 \geq 1, \end{cases}$$

which is concave, so this is a symmetric game with integrally concave payoff functions.

Suppose Γ_5 has an ordinal potential function G. Then since

$$P_2(1,1) > P_2(1,0), \quad P_1(1,0) = P_1(2,0), \quad P_2(2,0) = P_2(2,1), \quad P_1(2,1) = P_1(1,1),$$

we have a contradiction

$$G(1,1) > G(1,0) = G(2,0) = G(2,1) = G(1,1).$$

This says that Γ_5 is not an ordinal potential game, nor a potential game. ■

Example 4.6. Consider a symmetric game $\Gamma_6 = (\{1,2\}, \{S_1, S_2\}, \{P_1, P_2\})$, where $S_1 = S_2 = \{0,1,2\}$, P_1 is defined by

$$\begin{array}{lll} P_1(0,0) = 0 & P_1(0,1) = 0 & P_1(0,2) = 0 \\ P_1(1,0) = 0 & P_1(1,1) = 0 & P_1(1,2) = 0 \\ P_1(2,0) = 0 & P_1(2,1) = 0 & P_1(2,2) = 1, \end{array}$$

1 \ 2	0	1	2
0	(0 , 0)	(0 , 0)	(0 , 0)
1	(0 , 0)	(1 , 1)	(0 , 1)
2	(0 , 0)	(1 , 0)	(0 , 0)

Figure 5: A symmetric game with integrally concave payoffs which is neither an ordinal potential game nor a potential game.

and P_2 is its symmetric counterpart. The game Γ_6 is described in Figure 5 (there are three symmetric and two asymmetric equilibria). Then the local concave extension $\tilde{P}_1\colon [0,2]^2 \to \mathbb{R}$ of P_1 is given by

$$\tilde{P}_1(s_1, s_2) = \begin{cases} 0 & \text{if } s_1 + s_2 \leq 3, \\ (s_1 + s_2) - 3 & \text{otherwise.} \end{cases}$$

This is not concave, so this is not a game with integrally concave payoff functions.

However, the function $G\colon \{0,1,2\}^2 \to \mathbb{R}$ defined by $G(s_1, s_2) = \frac{1}{2}(P_1(s_1, s_2) + P_2(s_1, s_2))$ is a potential function of Γ_6. Hence Γ_6 is a finite symmetric potential game that is not a game of integrally concave payoffs. ■

1 \ 2	0	1	2
0	(0 , 0)	(0 , 0)	(0 , 0)
1	(0 , 0)	(0 , 0)	(0 , 0)
2	(0 , 0)	(0 , 0)	(1 , 1)

Figure 6: A symmetric potential game which is not a game with integrally concave payoffs.

5 Concluding Remarks

In this paper we showed that a finite symmetric game has a pure strategy equilibrium if the payoff functions of players are integrally concave. Concave extensibility and Miller's discrete concavity, which are weaker requirements for discrete concavity than integrally concavity, do not imply the existence. The class of supermodular games does not include the class of games with integrally concave payoffs and vice versa. The same is true for the relationships with the class of potential games.

We note some remarks on our results, mostly for the purpose of future studies. First, the property that appears in Lemma 3.2 looks very much like the *direction preserving* or (more general) *locally gross direction preserving* properties studied by Iimura et al. (2005); Cheng and Deng (2006); van der Laan et al. (2006); Yang (2009), which were deemed to be the analogue of continuity in discrete case (see also Herings et al. (2008) for the application of this kind of property to (dis-)continuous setting; for such a setting we also remark the similarity to Urai and Hayashi (2000) that focused on the "local directions of mappings"). This suggests that there could be another proof of existence using a discrete type of fixed point or zero point theorems. In the earlier version of this paper, we have succeeded in the proof for the case where the number of players is less than or equal to three *or* the number of strategies per player is less than or equal to three. The proof for the general case with fixed/zero-point arguments is an open question. Second, our last Example 4.6 suggests the existence of a broader subclass of finite symmetric games than ours that admit a pure strategy equilibrium. If we carefully read the proof of Lemma 3.3, we find that the only things we need are the well-definability of ascent directions and their property in Lemmas 3.1. The equilibrium-free examples in this paper (Examples 4.1, 4.2, and the Rock-Paper-Scissors game) are all violating either of these conditions, though there are examples having an equilibrium also violating either of the conditions (Examples 4.4 of a supermodular game and 4.6 of a potential game). To relax the condition of integral concavity should be another problem to be tackled.

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