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Irreversible Investment with Regime Switching : Revisit with Linear Algebra

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Abstract

We consider irreversible investment problems with regime switching feature under a monopoly setting. Several parameters describing the economic environment vary according to a regime switching with general number of states. We present the derivation of the value function via solving a system of simultaneous ordinary differential equations with knowledge of linear algebra. It is found that the value function is represented by the eigenvalues and the eigenvectors of a coefficient matrix. Furthermore, we obtain an analytical expression of an expectation of a payoff at the first passage time to the stop region by applying the Dirichlet problem and the aforementioned technique. The detailed derivation is provided for the case of two regimes. These results will help us to deepen our understanding of the investment problem.

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1 Introduction

We consider irreversible investment problems with regime switching feature under a monopoly setting. Several parameters describing the economic environment vary according to a regime switching with general number of states. A firm seeks an optimal timing to invest an irreversible project while observing the potential profits. We present a derivation of the value function via solving a system of simultaneous ordinary differential equations with knowledge of linear algebra. It will enable us to investigate a comparative analysis of the investment problem. The contribution of this paper is a natural extension of Guo and Zhang (2004) to a real option problem with the general number of regime states. Furthermore, we obtain an analytical expression of an expectation of a payoff at the first passage time to the stop region by applying the Dirichlet problem and the aforementioned technique.

In the literature, the value function of a typical real option problem is usually calculated by first guessing the form and then solving the unknown coefficients as in Dixit and Pyndick (1994), Guo et al. (2005) for $S = 2$, where S is the number of regimes, and Grenadier and Wang (2007) for general $S \geq 2$. For American put options, Guo and Zhang (2004) take similar approach for $S = 2$. Apparently one of the drawbacks is that there are no clues why such a form is taken. Jobert and Rogers (2006), Jiang and Pistorius (2008) utilize Wiener-Hopf factorization to obtain the value function for $S \geq 2$. They present explicit forms of the value function up to exponential matrix. Due to the complicated form further analysis including comparative analysis seems difficult without an insightful expression of the value function. The optimal stopping time is of threshold type. However, the thresholds are regime-dependent. Our approach is simple and straightforward: solve a system of simultaneous ordinary differential equations with appropriate conditions directly on each interval of the thresholds. Due to the form, knowledge of linear algebra helps a lot in the derivation. We do not rely on a "guess functional form" nor the Wiener-Hopf factorization that is more technical. It is found that the value function is represented by the eigenvalues and the eigenvectors of a coefficient matrix. With the same technique we can carry out a calculation of the expected and discounted value of a payoff at the first passage time by applying the Dirichlet problem in the two-dimensional space. The original value function can be decomposed with these expectations. The smooth pasting conditions at the boundary between the continuation region and the stop region are recovered by the optimality conditions of the thresholds. The detailed derivation is provided for the case of two regimes. These results will help us to deepen our understanding of the investment problem.

This paper is organized as follows. Section 2 is the setup of our model. In section 3 we discuss the derivation of the value function under general number of the regime states. Section 4 provides the detailed results in the case of two regime states. The issues of the first passage time are discussed in section 5. Section 6 concludes. All proofs are found in the Appendix.

2 Setup

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on infinite time horizon. Let $J = \{J(t)\}$ be a continuous-time Markov chain on a finite state space $E = \{1, 2, \cdots, S\}$. $J(t)$ is interpreted as a regime or a state of the economy at time t. The intensity matrix of the regime switching is given by $\mathbf{Q}$

$$\mathbf{Q} = (q_{ij})_{i,j \in E}, \quad q_{ii} = -\sum_{j \in E \setminus \{i\}} q_{ij}. \tag{1}$$

The process $X = \{X(t)\}$ satisfies

$$dX(t) \quad = \quad \mu_{J(t)} X(t) dt + \sigma_{J(t)} X(t) dW_t, \quad X(0) = x, \tag{2}$$

where $W = \{W_t\}$ is a standard Brownian motion, μ_j and $\sigma_j > 0$ are finite constants for each $j \in E$. Denote the filtration generated by (W, J) as $\{\mathcal{F}_t\}$ with $\mathcal{F}_t = \sigma(W_s, J(s), 0 \leq s \leq t)$.

The firm has a chance to start a project to make a product as a monopoly of the product whose revenue depends on the state variables $(X(t), J(t))$ of the economy. We assume that the firm has a technology to enter into the project by paying the cost K_i when the regime state is i, and after the investment the firm obtains the instant revenue of $D_{J(t)}X(t)$ at time t from the project, where $D_i\, K_i (i \in E)$ are positive constants. One may interpret that $X(t)$ is the unit price of the products from the project and D_i be the (potential) demand quantity for the products in the economy.

In this paper a matrix is represented in bold. $\mathbf{O}_n$ denotes the zero matrix of order n and $\mathbf{I}_n$ denotes the identity matrix of order n. An element of a matrix $\mathbf{A} = (a_{ij})$ is denoted by $a_{ij} = \{\mathbf{A}\}_{ij}$. Let us denote vectors, matrices and functions

$$
\begin{aligned}
\mathbf{e}_i &= (0, \cdots, 0, 1, 0, \cdots, 0)^\top \in \mathbb{R}^S, \quad \mathbf{1}_S = (1, \cdots, 1)^\top \in \mathbb{R}^S, \\
\mathbf{D} &= \left(D_1, \cdots, D_S\right)^\top, \quad \mathbf{M} = \mathrm{diag}\left[\mu_1, \cdots, \mu_S\right], \\
\mathbf{\Lambda}(\beta) &= \begin{pmatrix} g_1(\beta) & q_{12} & \cdots & q_{1S} \\ q_{21} & g_2(\beta) & \cdots & q_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{S1} & q_{S2} & \cdots & g_S(\beta) \end{pmatrix}, \qquad (3) \\
g_k(\beta) &= \frac{1}{2}\sigma_k^2\beta^2 + \left(\mu_k - \frac{1}{2}\sigma_k^2\right)\beta + q_{ii} - r. \qquad (4)
\end{aligned}
$$

For each $i \in E$, consider a Lévy process $L_t^{(i)} = \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i W_t$, which has the Lévy exponent $\theta^{(i)}(z) = \left(\mu_i - \frac{1}{2}\sigma_i^2 - r\right)z + \frac{1}{2}\sigma_i^2 z^2 = g_i(z) - q_{ii} + r$. Then $\ln X(t) = L_t^{(J(t))}$ evolves like a diffusion while the regime does not switch, and the matrix (3) is expressed as

$$\mathbf{\Lambda}(\beta) = \mathrm{diag}[\theta^{(k)}(\beta); k \in E] + \mathbf{Q} - r\mathbf{I}_S.$$

For a simple notation it is convenient to introduce a "truncating" operator $\mathbf{H}_n$ on $S \times S$ square matrices $(a_{ij})_{1\le i,j\le S}$ defined by

$$\mathbf{H}_n((a_{ij})_{1\le i,j\le S}) = (a_{ij})_{1\le i,j\le n}. \qquad (5)$$

The truncating operator $\mathbf{H}_n$ reflects our focusing on n regime states in a continuation region among S regimes as discussed later.

We assume the following properties;

Assumption 1.

1. $\mathbf{Q}$ is irreducible.
2. The matrices $\mathbf{H}_n(r\mathbf{I}_S - \mathbf{M} - \mathbf{Q})$ and $\mathbf{H}_n(r\mathbf{I}_S - \mathbf{Q})$ are invertible for all $n \in E$.
3. $r - \mu_i > 0$ for all $i \in E$ and $r > 0$.

The property 1 of Assumption 1 is to restrict our analysis to a non-redundant case of regime switching. Roughly speaking, each regime has a chance to move to another regime and/or to be transferred from another regime. The property 2 is due to a technical reason to make our discussion simple. The property 3 guarantees a convergence of total revenue and other properties.

Note that the matrices mentioned in Assumption 1 are expressed with $\mathbf{\Lambda}(\beta)$ as

$$r\mathbf{I}_S - \mathbf{M} - \mathbf{Q} = -\mathbf{\Lambda}(1), \quad r\mathbf{I}_S - \mathbf{Q} = -\mathbf{\Lambda}(0).$$

3 Value function

The firm seeks the optimal timing of the investment. When the current regime state is i, the value function V_i is defined by

$$V_i(x) \quad = \quad \max_{\tau} \mathbb{E}\left[\int_{\tau}^{\infty} e^{-ru} D_{J(u)} X(u) du - e^{-r\tau} K_{J(\tau)} \mid X(0) = x, J(0) = i\right].$$

The following lemma gives useful expressions For the calculation of the value function.

Lemma 1.

$$\begin{aligned}
&(1) \quad \mathbb{E}\left[e^{-rT} X(T)^{\beta} \mid \mathcal{F}_t\right] = X(t)^{\beta} \mathbf{e}_{J(t)}^{\top} \exp\left(\mathbf{\Lambda}(\beta)(T-t)\right) \mathbf{1}_S, \quad \beta \in \mathbb{R}, \\
&(2) \quad \mathbb{E}\left[\int_t^{\infty} e^{-ru} D_{J(u)} X(u) du \mid \mathcal{F}_t\right] = e^{-rt} \alpha_{J(t)} D_{J(t)} X(t),
\end{aligned}$$

where

$$\alpha_i D_i = \mathbf{e}_i^{\top} \left(r\mathbf{I}_S - \mathbf{M} - \mathbf{Q}\right)^{-1} \mathbf{D}.$$

Proof. See Appendix A.1. □

By Lemma 1, the value function at the regime i is reduced to

$$V_i(x) \quad = \quad \max_{\tau} \mathbb{E}\left[e^{-r\tau}\left(\alpha_{J(\tau)} D_{J(\tau)} X(\tau) - K_{J(\tau)}\right) \mid X(0) = x, J(0) = i\right].$$

As discussed in Jobert and Rogers (2006) and Guo and Zhang (2004), the candidate of the optimal stopping time τ^* must be in a form of

$$\tau^* = \min_{j \in E} \tau_j, \quad \tau_j = \inf\{t > 0 : X(t) \geq x_j,\ J(t) = j\}$$

with some positive x_j ($j \in E$). We will obtain the explicit form of the value function by assuming that the order of the thresholds is

$$x_S \leq x_{S-1} \leq \cdots \leq x_2 \leq x_1 \tag{6}$$

in what follows. Namely the regime state S is the best and the regime state 1 is the worst for starting the project. In case that (6) is not satisfied, the following procedure must be carried out after the regime indices are interchanged appropriately.

Figure 1 illustrates the structure of our problem in the two-dimensional $(X(t), J(t))$-plane. $X(t)$ moves horizontally by the Brownian motion and the regime $J(t)$ jumps vertically among the regimes by the switching. When the regime is n, the continuation region of $X(t)$ ("Wait") is an interval $(0, x_n)$ in red and the stop region ("Start") is $[x_n, \infty)$ in blue.

On the n-th interval (x_{n+1}, x_n) the regimes $1, \cdots, n$ (continuation regimes) are in the continuation region while the regimes $n+1, \cdots, S$ (stop regimes) are in the stop region. Hence, the state will enter into a stop region either if $X(t)$ moves upward gradually beyond $x_{J(t)}$ by the Brownian motion without regime switches ("Creep") or if the regime $J(t)$ is suddenly switched to either of the stop regimes by a regime switch ("Switch").

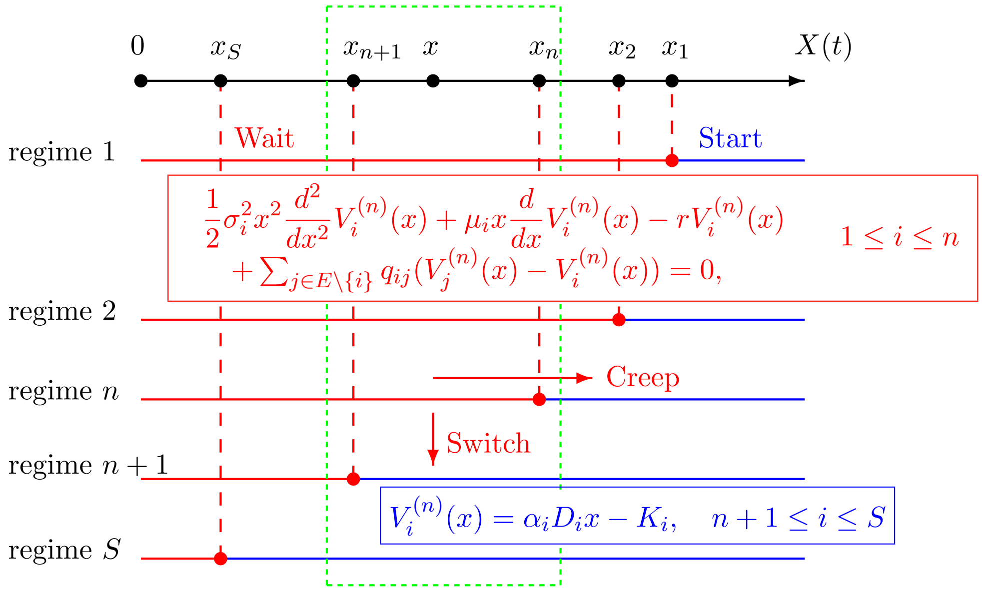
The value function will take a different functional form on each interval of the thresholds as

$$V_i(x) = \begin{cases} V_i^{(0)}(x) & \text{if } x \in [x_1, \infty), \\ V_i^{(n)}(x) & \text{if } x \in [x_{n+1}, x_n), \quad (n = 1, 2, \cdots, S-1), \\ V_i^{(S)}(x) & \text{if } x \in (0, x_S). \end{cases}$$

We will calculate $V_i^{(n)}(x)$ for each i on n-th interval by starting from $n = 0$ and moving on to $n = S$. When $x \in [x_1, \infty)$, the state is in the stop region at any regime and it is optimal for the firm to start the project immediately since the price $X(t)$ is high enough. Hence we have

$$V_i^{(0)}(x) = \alpha_i D_i x - K_i, \quad i \in E \tag{7}$$

Figure 1: Decision making



For $x \in [x_{n+1}, x_n)$ $(n = 1, 2, \cdots, S-1)$, the firm will enter into the project if the regime is either of $n+1, \cdots, S$, otherwise she should wait. Thus, the value function $V_i^{(n)}$ $(1 \le i \le n)$ satisfies

$$\frac{1}{2}x^2\sigma_i^2 \frac{d^2}{dx^2} V_i^{(n)}(x) + x\mu_i \frac{d}{dx} V_i^{(n)}(x) - rV_i^{(n)}(x) + \sum_{j \in E \setminus \{i\}} q_{ij}(V_j^{(n)}(x) - V_i^{(n)}(x)) = 0, \qquad (8)$$

and $V_i^{(n)}(x) = \alpha_i D_i x - K_i$ for $n+1 \le i \le S$. Finally, for $x \in (0, x_S)$, $V_i^{(S)}$ obeys the same ODE as (8) with $n = S$. The first three terms of (8) represents a change of the value function due to a movement of Brownian motion and the last term represents a change due to a regime switching. The optimality condition requires that the sum of these changes must be zero.

In summary, we must solve simultaneous ODEs on an interval $[x_{n+1}, x_n)$, $(n = 1, 2, \cdots, S-1)$,

$$\begin{aligned}
\mathcal{A}_1 V_1^{(n)}(x) + \sum_{1 \le j \le n,\, j \ne 1} q_{1j} V_j^{(n)}(x) &= -\sum_{n+1 \le j \le S} q_{1j} V_j^{(n)}(x) \\
\mathcal{A}_2 V_2^{(n)}(x) + \sum_{1 \le j \le n,\, j \ne 2} q_{2j} V_j^{(n)}(x) &= -\sum_{n+1 \le j \le S} q_{2j} V_j^{(n)}(x) \\
&\vdots \\
\mathcal{A}_n V_n^{(n)}(x) + \sum_{1 \le j \le n,\, j \ne n} q_{nj} V_j^{(n)}(x) &= -\sum_{n+1 \le j \le S} q_{nj} V_j^{(n)}(x),
\end{aligned}$$

with the value matching condition and the smooth pasting conditions at $x = x_n, x_{n+1}$, where $\mathcal{A}_i$ is a differential operator defined by

$$\mathcal{A}_i f(x) = \frac{1}{2}x^2\sigma_i^2 \frac{d^2}{dx^2} f(x) + x\mu_i \frac{d}{dx} f(x) + (q_{ii} - r)f(x).$$

It is symbolic to represent them in a form of matrix as

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} V_1^{(n)}(x) \\ V_2^{(n)}(x) \\ \vdots \\ V_n^{(n)}(x) \end{pmatrix} = - \begin{pmatrix} q_{1,n+1} & q_{1,n+2} & \cdots & q_{1S} \\ q_{2,n+1} & q_{2,n+2} & \cdots & q_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n,n+1} & q_{n,n+2} & \cdots & q_{nS} \end{pmatrix} \begin{pmatrix} V_{n+1}^{(n)}(x) \\ V_{n+2}^{(n)}(x) \\ \vdots \\ V_S^{(n)}(x) \end{pmatrix}. \tag{9}$$

The functions on the LHS are unknown and to be solved while ones on the RHS $V_j^{(n)}(x) = \alpha_j D_j x - K_j$ ($j \in \{n+1, \ldots, S\}$) are known. As for an interval $(0, x_S)$ a similar system of ODEs must be solved

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1S} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{S1} & q_{S2} & \cdots & \mathcal{A}_S \end{pmatrix} \begin{pmatrix} V_1^{(S)}(x) \\ V_2^{(S)}(x) \\ \vdots \\ V_S^{(S)}(x) \end{pmatrix} = \mathbf{0}_S.$$

A set of ODEs to be solved is dependent of the interval of x. We study the equations (9) on $V_i^{(n)}(x)$ ($i = 1, 2, \cdots, n$) defined on the interval (x_{n+1}, x_n) ($n = 1, 2, \cdots, S$) by modifying the RHS and the interval of x appropriately in the case of $n = S$. Since we know the solution $V_i^{(n)}(x) = \alpha_i D_i x - K_i$ for $i = n+1, \cdots, S$, the equations of the remaining $V_i^{(n)}$ for $1 \le i \le n$ are reduced to simultaneous second-order linear ODEs. It follows that the solution $V_i^{(n)}$ is decomposed with the general solution $\widetilde{V}_i^{(n)}$ and the special solution $v_i^{(n)}$ for each $i = 1, 2, \cdots, n$.

The special solution $v_i^{(n)}$ is easily found to be a linear function $v_i^{(n)}(x) = a_i^{(n)} x + b_i^{(n)}$, where the coefficients $\mathbf{a}^{(n)} = (a_1^{(n)}, \cdots, a_n^{(n)})^\top, \mathbf{b}^{(n)} = (b_1^{(n)}, \cdots, b_n^{(n)})^\top$ are given by

$$\begin{aligned} \mathbf{a}^{(n)} &= \mathbf{H}_n (r\mathbf{I}_S - \mathbf{M} - \mathbf{Q})^{-1} \begin{pmatrix} \sum_{j=n+1}^{S} q_{1j}\alpha_j D_j \\ \sum_{j=n+1}^{S} q_{2j}\alpha_j D_j \\ \vdots \\ \sum_{j=n+1}^{S} q_{nj}\alpha_j D_j \end{pmatrix}, \qquad (10) \\ \mathbf{b}^{(n)} &= -\mathbf{H}_n (r\mathbf{I}_S - \mathbf{Q})^{-1} \begin{pmatrix} \sum_{j=n+1}^{S} q_{1j} K_j \\ \sum_{j=n+1}^{S} q_{2j} K_j \\ \vdots \\ \sum_{j=n+1}^{S} q_{nj} K_j \end{pmatrix}, \end{aligned}$$

where the inverse matrices are guaranteed to exist by Assumption 1.

Now we turn our eyes to the general solution $\widetilde{V}_i^{(n)}$. Let us change the variable $y = \ln x$ and introduce auxiliary functions $\overline{V}_i^{(n)}(y) = \widetilde{V}_i^{(n)}(e^y)$, $\overline{W}_i^{(n)}(y) = \frac{d}{dy}\overline{V}_i^{(n)}(y)$. Then the equations for the general solution part of (9) can be rewritten as a system of first-order ODEs,

$$\frac{d}{dy} \begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix} = \mathbf{\Gamma}_n \begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned}
\mathbf{\Gamma}_n &= \begin{pmatrix} \mathbf{O}_n & \mathbf{I}_n \\ \mathbf{R}_n & \mathbf{C}_n \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \quad \mathbf{\Sigma}_n = \frac{1}{2}\mathrm{diag}\left[\sigma_1^2, \cdots, \sigma_n^2\right] \in \mathbb{R}^{n\times n}, \\
\mathbf{R}_n &= \mathbf{\Sigma}_n^{-1}\mathbf{H}_n(r\mathbf{I}_S - \mathbf{Q})) = -2 \begin{pmatrix} \frac{q_{11}-r}{\sigma_1^2} & \frac{q_{12}}{\sigma_1^2} & \cdots & \frac{q_{1n}}{\sigma_1^2} \\ \frac{q_{21}}{\sigma_2^2} & \frac{q_{22}-r}{\sigma_2^2} & \cdots & \frac{q_{2n}}{\sigma_2^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{q_{n1}}{\sigma_n^2} & \frac{q_{n2}}{\sigma_n^2} & \cdots & \frac{q_{nn}-r}{\sigma_n^2} \end{pmatrix} \in \mathbb{R}^{n\times n}, \\
\mathbf{C}_n &= \mathbf{\Sigma}_n^{-1}\mathbf{H}_n(\mathbf{\Sigma}_S - \mathbf{M}) = \mathrm{diag}\left[1-\frac{2\mu_1}{\sigma_1^2}, \cdots, 1-\frac{2\mu_n}{\sigma_n^2}\right] \in \mathbb{R}^{n\times n}.
\end{aligned}$$

Thus, the solution is given by

$$\begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix} = \exp\left((y-y_0)\mathbf{\Gamma}_n\right) \begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y_0) \\ \overline{\mathbf{W}}^{(n)}(y_0) \end{pmatrix}$$

with some y_0 from the boundary conditions when the exponential matrix $\exp\left((y-y_0)\mathbf{\Gamma}_n\right)$ is available. If the coefficient matrix $\mathbf{\Gamma}_n$ is diagonalizable, it is straightforward to solve and obtain an explicit representation of the solution of the system of ODEs (11). Otherwise, one can proceed similarly by making use of the Jordan normal form that is guaranteed to exist for any square matrix by the theory.

By the knowledge of linear algebra the characteristic function of of $\mathbf{\Gamma}_n$ is obtained as

$$\det\begin{pmatrix} \mathbf{O}_n - \beta\mathbf{I}_n & \mathbf{I}_n \\ \mathbf{R}_n & \mathbf{C}_n - \beta\mathbf{I}_n \end{pmatrix} = f_n(\beta)\prod_{j=1}^{n}\left(\frac{1}{2}\sigma_j^2\right)^{-1},$$

where

$$f_n(\beta) = \det\left(\mathbf{\Sigma}_n\left(\mathbf{I}_n\beta^2 - \mathbf{C}_n\beta - \mathbf{R}_n\right)\right) = \det\mathbf{H}_n(\mathbf{\Lambda}(\beta)). \tag{12}$$

Thus, the eigenvalues are the solutions of $f_n(\beta) = 0$. In this paper we make the following assumption for simple and useful results.

Assumption 2. 1. For $n = 1, 2, \cdots, S-1$, $\mathbf{\Gamma}_n$ has $2n$ distinct eigenvalues $\beta_1^{(n)}, \cdots, \beta_{2n}^{(n)}$.

2. $\mathbf{\Gamma}_S$ has $2S$ distinct eigenvalues such that $\beta_1^{(S)}, \cdots, \beta_S^{(S)}$ are strictly positive and $\beta_{S+1}^{(S)}, \cdots, \beta_{2S}^{(S)}$ are strictly negative.

If the eigenvalues become complex numbers or duplicated so that the above assumption is not satisfied, the following discussion can be accordingly modified by considering the Jordan normal form as mentioned before.[1]

By Assumption 2 there exist distinct eigenvalues $\beta_j^{(n)}$ $(1 \leq j \leq 2n)$. Since the upper right block of $\mathbf{\Gamma}_n$ is $\mathbf{I}_n$, the eigenvector for the eigenvalue $\beta_j^{(n)}$ must be in the form

$$\tilde{\mathbf{u}}_j^{(n)} = \begin{pmatrix} \mathbf{u}_j^{(n)} \\ \beta_j^{(n)}\mathbf{u}_j^{(n)} \end{pmatrix} \in \mathbb{R}^{2n},$$

with some non-zero vector $\mathbf{u}_j^{(n)} \in \mathbb{R}^n$ satisfying

$$\mathbf{H}_n(\mathbf{\Lambda}(\beta))\mathbf{u}_j^{(n)} = \mathbf{0}_n. \tag{13}$$

[1] Due to the duplicated eigenvalues, a Jordan normal form appears in the value function in Grenadier and Wang (2007) in a context of hyperbolic discounting which may be able to be modeled with a regime switching and our discussion may be applicable to.

Such a vector $\mathbf{u}_j^{(n)}$ exists for each j because the determinant of the coefficient matrix on the LHS of (13) is equal to $f_n\left(\beta_j^{(n)}\right) = 0$ by definition of $\beta_j^{(n)}$. Thus, $\mathbf{\Gamma}_n$ is diagonalized as

$$\mathbf{\Gamma}_n = \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix} \mathrm{diag}\left[\beta_1^{(n)}, \cdots, \beta_{2n}^{(n)}\right] \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix}^{-1},$$

where

$$\mathbf{U}^{(n)} = \begin{pmatrix} \mathbf{u}_1^{(n)} & \mathbf{u}_2^{(n)} & \cdots & \mathbf{u}_{2n}^{(n)} \end{pmatrix} \in \mathbb{R}^{n\times 2n}, \quad \mathbf{B}^{(n)} = \mathrm{diag}\left[\beta_1^{(n)}, \cdots, \beta_{2n}^{(n)}\right] \in \mathbb{R}^{2n\times 2n}.$$

The matrix

$$\begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1^{(n)} & \mathbf{u}_2^{(n)} & \cdots & \mathbf{u}_{2n}^{(n)} \\ \beta_1^{(n)}\mathbf{u}_1^{(n)} & \beta_2^{(n)}\mathbf{u}_2^{(n)} & \cdots & \beta_{2n}^{(n)}\mathbf{u}_{2n}^{(n)} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{u}}_1^{(n)} & \tilde{\mathbf{u}}_2^{(n)} & \cdots & \tilde{\mathbf{u}}_{2n}^{(n)} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$$

is invertible since the eigenvalues of $\mathbf{\Gamma}_n$ are distinct so that the corresponding eigenvectors $\tilde{\mathbf{u}}_j^{(n)}$ are linearly independent.

Then we can solve the system of ODEs (11) as

$$\begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix} = \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix} \mathrm{diag}\left[e^{\beta_1^{(n)} y}, \cdots, e^{\beta_{2n}^{(n)} y}\right] \mathbf{A}^{(n)},$$

with some constant vector $\mathbf{A}^{(n)} \in \mathbb{R}^{2n}$. By adding the special solutions, we have the vector of the value functions $\mathbf{V}^{(n)}(x) = (V_1^{(n)}(x), \cdots, V_n^{(n)}(x))^\top$ at each regime on the interval $[x_{n+1}, x_n)$ given as

$$\mathbf{V}^{(n)}(x) \quad = \quad \mathbf{U}^{(n)}\mathbf{X}^{(n)}(x)\mathbf{A}^{(n)} + \mathbf{v}^{(n)}(x), \tag{14}$$

where

$$\mathbf{X}^{(n)}(x) \quad = \quad \mathrm{diag}\left[x^{\beta_1^{(n)}}, \cdots, x^{\beta_{2n}^{(n)}}\right], \quad \mathbf{v}^{(n)}(x) = \mathbf{a}^{(n)}x + \mathbf{b}^{(n)}.$$

Unknown boundaries $x_S \le \cdots \le x_1$ and unknown vectors $\mathbf{A}^{(1)}, \cdots, \mathbf{A}^{(S)}$ will be determined by the value matching conditions, the smooth pasting conditions and the values at $x = 0$. We will investigate them by looking at x_1 first and moving downward to x_S as follows.

The value matching conditions at $x = x_n$, $V_i^{(n)}(x_n) = V_i^{(n-1)}(x_n)$ for $i = 1, \cdots, n$ are represented by n-dimensional vectors as

$$\mathbf{U}^{(n)}\mathbf{X}^{(n)}(x_n)\mathbf{A}^{(n)} + \mathbf{v}^{(n)}(x_n) = \begin{pmatrix} \mathbf{U}^{(n-1)}\mathbf{X}^{(n-1)}(x_n)\mathbf{A}^{(n-1)} + \mathbf{v}^{(n-1)}(x_n) \\ \alpha_n D_n x_n - K_n \end{pmatrix}. \tag{15}$$

Similarly, the smooth pasting conditions $x_n \frac{d}{dx} V_i^{(n)}(x_n) = x_n \frac{d}{dx} V_i^{(n-1)}(x_n)$ for $i = 1, \cdots, n$ require

$$\mathbf{U}^{(n)}\mathbf{dX}^{(n)}(x_n)\mathbf{A}^{(n)} + \mathbf{a}^{(n)}x_n = \begin{pmatrix} \mathbf{U}^{(n-1)}\mathbf{dX}^{(n-1)}(x_n)\mathbf{A}^{(n-1)} + \mathbf{a}^{(n-1)}x_n \\ \alpha_n D_n x_n \end{pmatrix}, \tag{16}$$

where

$$\mathbf{dX}^{(n)}(x) = \mathrm{diag}\left[\beta_1^{(n)}x^{\beta_1^{(n)}}, \cdots, \beta_{2n}^{(n)}x^{\beta_{2n}^{(n)}}\right] = \mathbf{B}^{(n)}\mathbf{X}^{(n)}(x), \quad \mathbf{a}^{(S)} = \mathbf{0}_S.$$

By coupling these conditions (15), (16) into one vector and making use of a relationship

$$\begin{pmatrix} \mathbf{U}^{(n)}\mathbf{X}^{(n)}(x) \\ \mathbf{U}^{(n)}\mathbf{dX}^{(n)}(x) \end{pmatrix} = \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix} \mathbf{X}^{(n)}(x),$$

$\mathbf{A}^{(n)}$ is represented with a function of x_n and $\mathbf{A}^{(n-1)}$ as

$$\begin{aligned}\mathbf{A}^{(n)} &= \mathbf{X}^{(n)}\left(x_n^{-1}\right)\begin{pmatrix}\mathbf{U}^{(n)}\\ \mathbf{U}^{(n)}\mathbf{B}^{(n)}\end{pmatrix}^{-1}\\ &\quad\times\left[\begin{pmatrix}\mathbf{U}^{(n-1)}\mathbf{X}^{(n-1)}(x_n)\mathbf{A}^{(n-1)}+\mathbf{v}^{(n-1)}(x_n)\\ \alpha_n D_n x_n - K_n\\ \mathbf{U}^{(n-1)}\mathbf{dX}^{(n-1)}(x_n)\mathbf{A}^{(n-1)}+\mathbf{a}^{(n-1)}x_n\\ \alpha_n D_n x_n\end{pmatrix}-\begin{pmatrix}\mathbf{v}^{(n)}(x_n)\\ \mathbf{a}^{(n)}x_n\end{pmatrix}\right].\end{aligned} \tag{17}$$

for $n = 2, \cdots, S-1$, and

$$\mathbf{A}^{(1)} = \mathbf{X}^{(1)}\left(x_1^{-1}\right)\begin{pmatrix}\mathbf{U}^{(1)}\\ \mathbf{U}^{(1)}\mathbf{B}^{(1)}\end{pmatrix}^{-1}\begin{pmatrix}\alpha_1 D_1 x_1 - K_1 - v^{(1)}(x_1)\\ \alpha_1 D_1 x_1 - a^{(1)}x_1\end{pmatrix}, \tag{18}$$

$$\mathbf{A}^{(S)} = \mathbf{X}^{(S)}\left(x_S^{-1}\right)\begin{pmatrix}\mathbf{U}^{(S)}\\ \mathbf{U}^{(S)}\mathbf{B}^{(S)}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{U}^{(S-1)}\mathbf{X}^{(S-1)}(x_S)\mathbf{A}^{(S-1)}+\mathbf{v}^{(S-1)}(x_S)\\ \alpha_S D_S x_S - K_S\\ \mathbf{U}^{(S-1)}\mathbf{dX}^{(S-1)}(x_S)\mathbf{A}^{(S-1)}+\mathbf{a}^{(S-1)}x_S\\ \alpha_S D_S x_S\end{pmatrix}. \tag{19}$$

Therefore, we can represent unknown vectors $\mathbf{A}^{(1)}, \cdots, \mathbf{A}^{(S)}$ as functions of $x_1, \cdots, x_S$ recursively.

Furthermore, on $(0, x_S]$, we want to impose another condition $\lim_{x\to 0} V_i^{(S)}(x) = 0$ for all i in order to make the value function finite. It implies that the coefficient of $\mathbf{A}^{(S)}$ corresponding to negative eigenvalues $\beta_{S+1}^{(S)}, \cdots, \beta_{2S}^{(S)}$ must be zero,

$$\begin{pmatrix}\mathbf{O}_S & \mathbf{I}_S\end{pmatrix}\mathbf{A}^{(S)} = \mathbf{0}_S. \tag{20}$$

This is a set of S equations that S unknown constants $x_1, \cdots, x_S$ must satisfy. Apparently (20) is a system of complicated algebraic equations, hence they must be solved numerically. In case that the numerical solution doesn't satisfy the order condition (6), the indices of the regimes must be interchanged.

Then, by noting that $\mathbf{v}^{(S)}(x) = 0$, the value function on $(0, x_S)$ can be expressed with terms with positive eigenvalues as

$$\mathbf{V}^{(S)}(x) = \mathbf{U}_S\mathbf{X}_S(x)\mathbf{A}_S, \tag{21}$$

where

$$\mathbf{U}_S = (\mathbf{u}_1^{(S)} \cdots \mathbf{u}_S^{(S)}), \quad \mathbf{X}_S(x) = \mathrm{diag}\left[x^{\beta_1^{(S)}}, \cdots, x^{\beta_S^{(S)}}\right], \quad \mathbf{A}_S = \begin{pmatrix}\mathbf{I}_S & \mathbf{O}_S\end{pmatrix}\mathbf{A}^{(S)}.$$

As a summary, we obtain the main result.

Proposition 1. *Suppose that Assumption 1 and 2 hold, and $x_1, \cdots, x_S$ satisfy (6) and (20). Then the value function is given by (14) and (21).*

4 Examples: Case of two regimes

4.1 Case of two regimes

We consider a case of two states $S = 2$ in this section.
Let us change notations as $-q_{11} = q_{12} = q_1 \geq 0$, $-q_{22} = q_{21} = q_2 \geq 0$.

For $n = 1$, the eigenvalues $\beta_1^{(1)}, \beta_2^{(1)}$ of $\mathbf{\Gamma}_1$ are solutions of $g_1(\beta) = 0$. They exist and satisfy $\beta_2^{(1)} < 0 < 1 < \beta_1^{(1)}$ because $g_1(1) = -(r - \mu_1 + q_1) < 0$ and $g_1(0) = -(r + q_1) < 0$ by Assumption 1. It follows that $\mathbf{u}_1^{(1)} = \mathbf{u}_2^{(1)} = 1$ so that

$$\begin{pmatrix} \mathbf{U}^{(1)} \\ \mathbf{U}^{(1)}\mathbf{B}^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \beta_1^{(1)} & \beta_2^{(1)} \end{pmatrix}.$$

Since $\mathbf{v}^{(1)}(x) = K_2(p_1 \widetilde{\alpha}_2 x - p_2)$, $\mathbf{A}^{(1)}$ in (18) is represented as a function of x_1,

$$\mathbf{A}^{(1)} \quad = \quad K_1 \mathbf{X}^{(1)} \left(x_1^{-1}\right) \begin{pmatrix} \gamma_{11}(x_1) \\ \gamma_{12}(x_1) \end{pmatrix}, \tag{22}$$

where

$$\begin{aligned} \begin{pmatrix} \gamma_{11}(x_1) \\ \gamma_{12}(x_1) \end{pmatrix} &= \frac{1}{K_1} \begin{pmatrix} 1 & 1 \\ \beta_1^{(1)} & \beta_2^{(1)} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 D_1 x_1 - K_1 - p_1 \alpha_2 D_2 x_1 + p_2 K_2 \\ \alpha_1 D_1 x_1 - p_1 \alpha_2 D_2 x_1 \end{pmatrix} \\ &= \frac{1}{\beta_1^{(1)} - \beta_2^{(1)}} \begin{pmatrix} (1 - \beta_2^{(1)})(1 - k_K k_\alpha p_1)\widetilde{\alpha}_1 x + \beta_2^{(1)}(1 - k_K p_2) \\ -(1 - \beta_1^{(1)})(1 - k_K k_\alpha p_1)\widetilde{\alpha}_1 x - \beta_1^{(1)}(1 - k_K p_2) \end{pmatrix}, \\ p_1 &= \frac{q_1}{r - \mu_1 + q_1}, \quad p_2 = \frac{q_1}{r + q_1}, \quad k_K = \frac{K_2}{K_1}, \quad k_\alpha = \frac{\widetilde{\alpha}_2}{\widetilde{\alpha}_1}, \quad \widetilde{\alpha}_i = \frac{\alpha_i D_i}{K_i} \quad (i = 1, 2). \end{aligned} \tag{23}$$

For $n = 2$, the eigenvalues $\beta_j^{(2)}$ are the solutions of the quartic equation[2]

$$f_2(\beta) = \det \begin{pmatrix} g_1(\beta) & q_1 \\ q_2 & g_2(\beta) \end{pmatrix} = g_1(\beta)g_2(\beta) - q_1 q_2 = 0.$$

Note that f_2 is continuous and

$$\begin{aligned} &\lim_{\beta \to -\infty} f_2(\beta) > 0, \quad f_2(\beta_2^{(1)}) = -q_1 q_2 \leq 0, \quad f_2(0) = r^2 + r(q_1 + q_2) > 0, \\ &f_2(1) = (r - \mu_1)(r - \mu_2) + q_2(r - \mu_1) + q_1(r - \mu_2) > 0, \quad f_2(\beta_1^{(1)}) = -q_1 q_2 \leq 0, \quad \lim_{\beta \to \infty} f_2(\beta) > 0 \end{aligned}$$

by Assumption 1.

If $q_1 q_2 \neq 0$, it follows that the solutions are distinct and there exist two positive solutions and two negative solutions

$$\beta_4^{(2)} < \beta_2^{(1)} < \beta_3^{(2)} < 0 < 1 < \beta_2^{(2)} < \beta_1^{(1)} < \beta_1^{(2)},$$

so that Assumption 2 is implied by Assumption 1. In this case, the relevant vectors satisfying (13) are taken as

$$\mathbf{u}_j^{(2)} = \begin{pmatrix} g_2(\beta_j^{(2)}) \\ -q_2 \end{pmatrix}$$

for each $j = 1, 2, 3, 4$.

Let us consider a case of $q_1 q_2 = 0$. The roots of $f_2(\beta) = g_1(\beta)g_2(\beta)$ are ones of g_1 and g_2, each of which has one positive root and one negative root. The positive root of g_1 is $\beta_1^{(1)}$ and

[2] The same equation appears on Guo and Zhang (2004).

the negative root is $\beta_2^{(1)}$ by definition. Therefore, in case of $q_1q_2 = 0$, the existence of distinct four eigenvalues of $\mathbf{\Gamma}_2$ is equivalent to a condition of $g_2(\beta_1^{(1)})g_2(\beta_2^{(1)}) \neq 0$. Note that in this case (i) Assumption 2 is satisfied, (ii) either of $\beta_1^{(1)} = \beta_1^{(2)}$ or $\beta_1^{(1)} = \beta_2^{(2)}$ holds, and (iii) either of $\beta_2^{(1)} = \beta_3^{(2)}$ or $\beta_2^{(1)} = \beta_4^{(2)}$ holds.

Hence, the following discussions are divided into three cases; $q_1q_2 \neq 0$, $q_2 = 0$, $q_1 = 0$.

4.1.1 Case of $q_1q_2 \neq 0$

First, let us consider the case of $q_1q_2 \neq 0$. Then, the corresponding eigenvectors are constructed by

$$\mathbf{u}_j^{(2)} = \begin{pmatrix} g_2(\beta_j^{(2)}) \\ -q_2 \end{pmatrix}, \quad j = 1, 2, 3, 4.$$

For the case of two states, both $\mathbf{A}^{(1)}$ and $\mathbf{A}_2$ have dimension of 2 that is the same as the number of conditions of value matching and smooth pasting. Therefore, it is easier to solve to rearrange the value matching conditions and the smooth pasting conditions at $x = x_2$ as

$$V_1: \quad \begin{pmatrix} g_2(\beta_1^{(2)}) & g_2(\beta_2^{(2)}) \\ g_2(\beta_1^{(2)})\beta_1^{(2)} & g_2(\beta_2^{(2)})\beta_2^{(2)} \end{pmatrix} \mathbf{X}_2(x_2)\mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ \beta_1^{(1)} & \beta_2^{(1)} \end{pmatrix} \mathbf{X}^{(1)}(x_2)\mathbf{A}^{(1)} + \begin{pmatrix} \mathbf{v}^{(1)}(x_2) \\ \mathbf{a}^{(1)}x_2 \end{pmatrix}, \tag{24}$$

$$V_2: \quad \begin{pmatrix} -q_2 & -q_2 \\ -q_2\beta_1^{(2)} & -q_2\beta_2^{(2)} \end{pmatrix} \mathbf{X}_2(x_2)\mathbf{A}_2 = \begin{pmatrix} \alpha_2 D_2 x_2 - K_2 \\ \alpha_2 D_2 x_2 \end{pmatrix}, \tag{25}$$

where $\quad \mathbf{X}_2(x) = \mathrm{diag}[x^{\beta_1^{(2)}}, x^{\beta_2^{(2)}}]$.

Note the difference among (15), (16), (24) and (25). (15) represents the value matching conditions of two functions V_1 and V_2, $V_i^{(2)}(x_2) = V_i^{(1)}(x_2)$ and (16) does the smooth pasting conditions of the two functions, $x_2 \frac{d}{dx} V_i^{(2)}(x_2) = x_2 \frac{d}{dx} V_i^{(1)}(x_2)$. On the other hand (24) is a set of the two value matching and smooth pasting conditions imposed on V_1 and (25) is one on V_2. The first row of (24) and (25) represents the value matching condition of each function at $x = x_2$, $V_i^{(2)}(x_2) = V_i^{(1)}(x_2)$, and the second row does to the smooth pasting condition at $x = x_2$, $x_2 \frac{d}{dx} V_i^{(2)}(x_2) = x_2 \frac{d}{dx} V_i^{(1)}(x_2)$. (25) is equivalent to

$$\mathbf{X}_2(x_2)\mathbf{A}_2 = -\frac{K_2}{q_2} \begin{pmatrix} \gamma_{21}(x_2) \\ \gamma_{22}(x_2) \end{pmatrix}, \tag{26}$$

where

$$\begin{pmatrix} \gamma_{21}(x_2) \\ \gamma_{22}(x_2) \end{pmatrix} = \frac{1}{K_2} \begin{pmatrix} 1 & 1 \\ \beta_1^{(2)} & \beta_2^{(2)} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_2 D_2 - K_2 \\ \alpha_2 D_2 x_2 \end{pmatrix} = \frac{1}{\beta_1^{(2)} - \beta_2^{(2)}} \begin{pmatrix} (1 - \beta_2^{(2)})\widetilde{\alpha}_2 x + \beta_2^{(2)} \\ -(1 - \beta_1^{(2)})\widetilde{\alpha}_2 x - \beta_1^{(2)} \end{pmatrix}.$$

By solving $\mathbf{A}^{(1)}$ in (24), we obtain

$$\mathbf{A}^{(1)} = \mathbf{X}^{(1)}\left(x_2^{-1}\right) \begin{pmatrix} 1 & 1 \\ \beta_1^{(1)} & \beta_2^{(1)} \end{pmatrix}^{-1} \left[\begin{pmatrix} g_2(\beta_1^{(2)}) & g_2(\beta_2^{(2)}) \\ g_2(\beta_1^{(2)})\beta_1^{(2)} & g_2(\beta_2^{(2)})\beta_2^{(2)} \end{pmatrix} \mathbf{X}_2(x_2)\mathbf{A}_2 - \begin{pmatrix} v^{(1)}(x_2) \\ a^{(1)}x_2 \end{pmatrix} \right]. \tag{27}$$

By equating (22) with (27) after plugging (26) to (27), we obtain a vector equation

$$\begin{aligned} &\mathbf{X}^{(1)}(x_1^{-1}) \begin{pmatrix} \gamma_{11}(x_1) \\ \gamma_{12}(x_1) \end{pmatrix} \\ &= k_K \mathbf{X}^{(1)}(x_2^{-1}) \begin{pmatrix} 1 & 1 \\ \beta_1^{(1)} & \beta_2^{(1)} \end{pmatrix}^{-1} \left[\begin{pmatrix} l_1 & l_2 \\ l_1\beta_1^{(2)} & l_2\beta_2^{(2)} \end{pmatrix} \begin{pmatrix} \gamma_{21}(x_2) \\ \gamma_{22}(x_2) \end{pmatrix} - \begin{pmatrix} p_1\widetilde{\alpha}_2 x_2 - p_2 \\ p_1\widetilde{\alpha}_2 x_2 \end{pmatrix} \right], \end{aligned} \tag{28}$$

where $l_j = -g_2(\beta_j^{(2)})/q_2$, $j = 1, 2$, that is a ratio of the eigenvalues. (28) is a system of equations on x_1 and x_2.[3] We need to look for the solution with $0 < x_2 \leq x_1$. The following proposition characterizes the necessary and sufficient condition of the existence.

Proposition 2. *Suppose that* $q_2 \neq 0$ *and*

$$r > \max(\mu_1, \mu_2, 0), \quad (r - \mu_1 + q_1)(r - \mu_2 + q_2) - q_1 q_2 \neq 0, \quad (r + q_1)(r + q_2) - q_1 q_2 \neq 0. \tag{29}$$

Then,
(1) Assumption 1 and 2 hold.
(2) There exists unique solution of (28) with $0 < x_2 \leq x_1$ *if and only if there exists unique solution* λ *satisfying* $0 < \lambda \leq 1$ *and*

$$\begin{aligned} & \left(c_{11}\lambda^{\beta_1^{(1)}-1} - k_K k_\alpha c_{21}\right)\left(d_{12}\lambda^{\beta_2^{(1)}} - k_K d_{22}\right) \\ = \; & \left(c_{12}\lambda^{\beta_2^{(1)}-1} - k_K k_\alpha c_{22}\right)\left(d_{11}\lambda^{\beta_1^{(1)}} - k_K d_{21}\right) < 0, \end{aligned} \tag{30}$$

where

$$\begin{aligned} \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} &= (1 - k_K p_1 k_\alpha)\begin{pmatrix} 1 - \beta_2^{(1)} \\ \beta_1^{(1)} - 1 \end{pmatrix}, \quad \begin{pmatrix} d_{11} \\ d_{12} \end{pmatrix} = (1 - k_K p_2)\begin{pmatrix} \beta_2^{(1)} \\ -\beta_1^{(1)} \end{pmatrix}, \\ \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} &= -\begin{pmatrix} \beta_2^{(1)} & -1 \\ -\beta_1^{(1)} & 1 \end{pmatrix}\left[\frac{1}{\beta_1^{(2)} - \beta_2^{(2)}}\begin{pmatrix} l_1 & l_2 \\ l_1\beta_1^{(2)} & l_2\beta_2^{(2)} \end{pmatrix}\begin{pmatrix} 1 - \beta_2^{(2)} \\ \beta_1^{(2)} - 1 \end{pmatrix} - p_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right], \\ \begin{pmatrix} d_{21} \\ d_{22} \end{pmatrix} &= -\begin{pmatrix} \beta_2^{(1)} & -1 \\ -\beta_1^{(1)} & 1 \end{pmatrix}\left[\frac{1}{\beta_1^{(2)} - \beta_2^{(2)}}\begin{pmatrix} l_1 & l_2 \\ l_1\beta_1^{(2)} & l_2\beta_2^{(2)} \end{pmatrix}\begin{pmatrix} \beta_2^{(2)} \\ -\beta_1^{(2)} \end{pmatrix} + p_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]. \end{aligned}$$

When the above condition (30) is satisfied, the solution (x_1, x_2) *is given by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{-d_{12} + k_K d_{22}\lambda^{-\beta_2^{(1)}}}{c_{12}\widetilde{\alpha}_1 - k_K c_{22}\widetilde{\alpha}_2\lambda^{-\beta_2^{(1)}+1}}\begin{pmatrix} 1 \\ \lambda \end{pmatrix}. \tag{31}$$

Proof. See Appendix A.2. □

Once x_1 and x_2 are obtained from (31), the value function is summarized as follows.

Proposition 3. *Suppose that* $q_1 q_2 \neq 0$ *and (31) holds. Then the value function is given by*

$$\frac{V_1(x)}{K_1} = \begin{cases} \widetilde{\alpha}_1 x - 1, & x \in [x_1, \infty), \\ \gamma_{11}(x_1)\left(\dfrac{x}{x_1}\right)^{\beta_1^{(1)}} + \gamma_{12}(x_1)\left(\dfrac{x}{x_1}\right)^{\beta_2^{(1)}} + k_K\left(p_1\widetilde{\alpha}_2 x - p_2\right), & x \in [x_2, x_1), \\ k_K\left[l_1\gamma_{21}(x_2)\left(\dfrac{x}{x_2}\right)^{\beta_1^{(2)}} + l_2\gamma_{22}(x_2)\left(\dfrac{x}{x_2}\right)^{\beta_2^{(2)}}\right], & x \in (0, x_2), \end{cases}$$

$$\frac{V_2(x)}{K_2} = \begin{cases} \widetilde{\alpha}_2 x - 1, & x \in [x_2, \infty), \\ \gamma_{21}(x_2)\left(\dfrac{x}{x_2}\right)^{\beta_1^{(2)}} + \gamma_{22}(x_2)\left(\dfrac{x}{x_2}\right)^{\beta_2^{(2)}}, & x \in (0, x_2). \end{cases}$$

Proof. See Appendix A.3. □

The semi-analytical form of the value function obtained in this way will help us in a comparative analysis.

[3] For a simpler setting ($D_1 = D_2 = 1$, $K_1 = K_2$) the results (18), (26) and (28) coincide with ones obtained in Guo and Zhang (2004).

4.1.2 Case of $q_2 = 0$

In the case of $q_1 q_2 = 0$, we need to distinguish the roots of g_1 and g_2 from an equation $g_1(\beta)g_2(\beta) = 0$. Therefore, it is convenient to rename $\{\beta_j^{(2)}; j = 1,2,3,4\}$ with $\{\beta_k; k = 1,2,3,4\}$ such that

$$\begin{array}{lll} \beta_2 < 0 < 1 < \beta_1, & g_1(\beta_1) = g_1(\beta_2) = 0, & g_2(\beta_1)g_2(\beta_2) \neq 0, \\ \beta_4 < 0 < 1 < \beta_3, & g_1(\beta_3)g_1(\beta_4) \neq 0, & g_2(\beta_3) = g_2(\beta_4) = 0. \end{array}$$

Namely, $\quad \beta_1 = \beta_1^{(1)}, \quad \beta_2 = \beta_2^{(1)}, \quad \{\beta_1, \beta_3\} = \{\beta_1^{(2)}, \beta_2^{(2)}\}, \quad \{\beta_2, \beta_4\} = \{\beta_3^{(2)}, \beta_4^{(2)}\}.$
The following lemma is necessary to verify Assumption 1 and 2.

Lemma 2. *Suppose that $q_1 q_2 = 0$, $g_2(\beta_1)g_2(\beta_2) \neq 0$ and (29). Then Assumption 1 and 2 hold.*

When $q_1 \neq 0$ and $q_2 = 0$, the regime 2 is the absorbing state; once the regime 2 realizes, there will be no switch to the regime 1. Therefore, it is natural to conjecture that the value function V_2 when the regime 2 will be of a familiar form in the literature.

Under a condition of $q_1 \neq 0$ and $q_2 = 0$ the building blocks for the eigenvectors are taken to be

$$\mathbf{u}_1^{(2)} = \begin{pmatrix} g_2(\beta_1) \\ 0 \end{pmatrix}, \quad \mathbf{u}_2^{(2)} = \begin{pmatrix} -q_1 \\ g_1(\beta_3) \end{pmatrix}, \quad \mathbf{u}_3^{(2)} = \begin{pmatrix} g_2(\beta_2) \\ 0 \end{pmatrix}, \quad \mathbf{u}_4^{(2)} = \begin{pmatrix} -q_1 \\ g_1(\beta_4) \end{pmatrix},$$

corresponding to the eigenvalues $\beta_1, \beta_3, \beta_2, \beta_4$, respectively. Note that the order of the corresponding eigenvalues is changed. Associated with the change of the order, we also change the elements of $\mathbf{X}_2(x)$ and $\mathbf{A}_2$ appropriately and we abuse them in what follows.

Sufficient conditions of the existence of thresholds are provided in Lemma 3. The conditions in the lemma are described in more detail than Proposition 2 due to the condition of $q_2 = 0$.

Lemma 3. *Suppose that $q_1 \neq 0$, $q_2 = 0$, $g_2(\beta_1)g_2(\beta_2) \neq 0$. Let us define*

$$\begin{aligned} h_1(y) \quad &= \quad a_1 y^{-\beta_2+1} + b_1 y^{-\beta_2} + c_1, \\ & \qquad a_1 = \frac{(\beta_1 - 1)\beta_3(1 - k_K p_1 k_\alpha)}{(\beta_3 - 1)k_\alpha}, \quad b_1 = -\beta_1(1 - k_K p_2), \\ & \qquad c_1 = k_K \left(\frac{q_1(\beta_1 - \beta_3)}{g_1(\beta_3)(\beta_3 - 1)} + p_1 \frac{(\beta_1 - 1)\beta_3}{(\beta_3 - 1)} - p_2 \beta_1 \right), \end{aligned}$$

and $z_1 = \max \left[\frac{\beta_2}{1 - \beta_2} \frac{b_1}{a_1}, 0 \right]$.

(1) $h_1(1) < 0$ is equivalent to

$$k_K \left(\frac{(r - \mu_2)/k_D + q_1}{r - \mu_1 + q_1} + \frac{q_1(\beta_1 - \beta_3)}{g_1(\beta_3)(\beta_1 - 1)\beta_3} \right) < \frac{\beta_1(\beta_3 - 1)}{(\beta_1 - 1)\beta_3}. \tag{32}$$

(2) If either of (2-a) $h_1(1) < 0$, or (2-b) $z_1 \leq 1$, $h_1(1) \leq 0$, or (2-c) $z_1 > 1$, $h_1(z_1) = 0$ holds, the thresholds satisfying $x_2 \leq x_1$ exist and they are given by

$$x_1 = \lambda^{-1} x_2, \quad x_2 = \frac{\beta_3}{\beta_3 - 1} \frac{1}{\widetilde{\alpha}_2}, \tag{33}$$

where λ^{-1} is the unique solution of $h_1(\lambda^{-1}) = 0$ such that $1 \leq \lambda^{-1}$.

(3) If either of (3-a) $z_1 \leq 1$, $h_1(1) > 0$ or (3-b) $z_1 > 1$, $h_1(z_1) > 0$ holds, then there does not exist λ satisfying $h_1(\lambda^{-1}) = 0$, $1 \leq \lambda^{-1}$.

(4) If $h_1(1) > 0$, $z_1 > 1$, $h_1(z_1) < 0$, then there exist two distinct solutions of λ satisfying $h_1(\lambda^{-1}) = 0$, $1 \leq \lambda^{-1}$.

Proof. See Appendix A.4. $\square$

When (32) and other trivial conditions are satisfied, the solution λ^{-1} is guaranteed to exist uniquely. One can find the solution of the algebraic equation numerically without difficulties by using a software on a computer. The explicit functional form of the value function is presented in the following proposition.

Proposition 4. *Suppose that $q_1 \neq 0$, $q_2 = 0$, $g_2(\beta_1)g_2(\beta_2) \neq 0$, (29) and (32) hold. Then the value function is given by*

$$\frac{V_1(x)}{K_1} = \begin{cases} \widetilde{\alpha}_1 x - 1, & x \in [x_1, \infty), \\ \gamma_{11}(x_1)\left(\frac{x}{x_1}\right)^{\beta_1} + \gamma_{12}(x_1)\left(\frac{x}{x_1}\right)^{\beta_2} + k_K\left(p_1\widetilde{\alpha}_2 x - p_2\right), & x \in [x_2, x_1), \\ k_K g_2(\beta_1)\gamma_{21}(x_2)\left(\frac{x}{x_2}\right)^{\beta_1} + k_K \frac{-q_1}{g_1(\beta_3)}\frac{1}{1-\beta_3}\left(\frac{x}{x_2}\right)^{\beta_3}, & x \in (0, x_2), \end{cases} \tag{34}$$

$$\frac{V_2(x)}{K_2} = \begin{cases} \widetilde{\alpha}_2 x - 1, & x \in [x_2, \infty), \\ \frac{1}{1-\beta_3}\left(\frac{x}{x_2}\right)^{\beta_3}, & x \in (0, x_2), \end{cases} \tag{35}$$

where

$$\begin{aligned} \gamma_{11}(x_1) &= \frac{1}{\beta_1 - 1}\left[1 - k_K p_2 + (1-\beta_2)\gamma_{12}(x_1)\right], \\ \gamma_{12}(x_1) &= -\frac{c_1}{\beta_1 - \beta_2}\left(\frac{x_1}{x_2}\right)^{\beta_2}, \\ k_K g_2(\beta_1)\gamma_{21}(x_2) &= \frac{1}{\beta_1 - \beta_3}\left[(\beta_1 - \beta_3)\left(\frac{x_2}{x_1}\right)^{\beta_1}\gamma_{11}(x_1) + \frac{\beta_3 - \beta_2}{\beta_1 - \beta_2}c_1\right] \\ &\quad - k_K \frac{(\beta_3 - 1)p_1\widetilde{\alpha}_2 x_2 - p_2\beta_3}{\beta_1 - \beta_3}, \end{aligned}$$

and x_1, x_2 are given by (33).

Proof. See Appendix A.5. □

Apparently, the value function V_2 in (35) is same as the standard result of a real option problem without regime switching as conjectured.

4.1.3 Case of $q_1 = 0$

In the case of $q_1 = 0$, regime 1 is the absorbing state. One can find that

$$\mathbf{u}_1^{(2)} = \begin{pmatrix} g_2(\beta_1) \\ -q_2 \end{pmatrix}, \quad \mathbf{u}_2^{(2)} = \begin{pmatrix} 0 \\ -q_2 \end{pmatrix}, \quad \mathbf{u}_3^{(2)} = \begin{pmatrix} g_2(\beta_2) \\ -q_2 \end{pmatrix}, \quad \mathbf{u}_4^{(2)} = \begin{pmatrix} 0 \\ -q_2 \end{pmatrix}.$$

A similar discussion as the previous case can be carried out and conditions with respect to the unique existence of thresholds are shown in Lemma 4.

Lemma 4. *Suppose that $q_1 = 0$, $q_2 \neq 0$, $g_2(\beta_1)g_2(\beta_2) \neq 0$. Let us define*

$$\begin{aligned} h_2(y) &= a_2 y^{\beta_1} + b_2 y - 1, \\ & \quad a_2 = -\frac{1}{k_K}\frac{q_2}{g_2(\beta_1)}\frac{\beta_1 - \beta_3}{\beta_3(\beta_1 - 1)}, \quad b_2 = \frac{(\beta_3 - 1)\beta_1}{\beta_3(\beta_1 - 1)}\frac{1}{k_K}\frac{(r - \mu_1)k_D + q_2}{r - \mu_2 + q_2}, \end{aligned}$$

and

$$z_2 = \left(\frac{g_2(\beta_1)}{q_2(\beta_1 - \beta_3)}\frac{1}{k_K}\frac{(r-\mu_1)k_D + q_2}{r - \mu_2 + q_2}\right)^{1/(\beta_1 - 1)} > 0.$$

(1) $h_2(1) > 0$ is equivalent to

$$\frac{1}{k_K}\left(\frac{(r-\mu_1)k_D+q_2}{r-\mu_2+q_2} - \frac{q_2(\beta_1-\beta_3)}{g_2(\beta_1)(\beta_3-1)\beta_1}\right) > \frac{\beta_3(\beta_1-1)}{(\beta_3-1)\beta_1}. \tag{36}$$

(2) If either of (2-a) $h_2(1) > 0$, or (2-b) $z_2 \le 1$, $h_2(z_2) = 0$ holds, the thresholds satisfying $x_2 \le x_1$ exist and they are given by

$$x_1 = \frac{\beta_1}{\beta_1-1}\frac{1}{\widetilde{\alpha}_1}, \quad x_2 = \lambda x_1, \tag{37}$$

where λ is the unique solution of $h_2(\lambda) = 0$ such that $0 < \lambda < 1$.

(3) If either of (3-a) $z_2 \ge 1$, $h_2(1) \le 0$ or (3-b) $z_2 \le 1$, $h_2(z_2) < 0$ holds, then there does not exist λ satisfying $h_2(\lambda) = 0$, $0 < \lambda \le 1$.

(4) If $h_2(1) < 0$, $z_2 \le 1$, $h_2(z_2) > 0$, then there exist two distinct solutions of λ satisfying $h_2(\lambda) = 0$, $0 < \lambda \le 1$.

Proof. See Appendix A.6. □

Comparing Lemma 5 with Lemma 3, the basic roles of regime 1 and 2 in the determination of the thresholds are interchanged with each other because of the simple difference, $q_1 = 0$ or $q_2 = 0$. However, in terms of valuation, regime 2 is superior to regime 1 in the sense that the firm is willing to invest at a lower price in regime 2 than regime 1. Furthermore, since regime 1 is the absorbing state ($q_1 = 0$), it follows that when the economy state is in regime 1, the situation is same as the typical investment problem. It implies that the value function V_1 when regime 1 has a familar form as observed in V_2 in case of $q_2 = 0$ in the previous subsection. The intuition is confirmed in the following proposition.

Proposition 5. *Suppose that $q_1 = 0$, $q_2 \neq 0$, $g_2(\beta_1)g_2(\beta_2) \neq 0$, (29) and (36) hold. Then the value function is given by*

$$\frac{V_1(x)}{K_1} = \begin{cases} \widetilde{\alpha}_1 x - 1, & x \in [x_1, \infty), \\ \dfrac{1}{\beta_1-1}\left(\dfrac{x}{x_1}\right)^{\beta_1}, & x \in (0, x_1). \end{cases}$$

$$\frac{V_2(x)}{K_2} = \begin{cases} \widetilde{\alpha}_2 x - 1, & x \in [x_2, \infty), \\ -\dfrac{1}{\beta_1-1}\left(\dfrac{x}{x_1}\right)^{\beta_1}\dfrac{q_2}{k_K g_2(\beta_1)} \\ +\dfrac{1}{\beta_3-1}\left(\dfrac{x}{x_2}\right)^{\beta_3}\left[\dfrac{q_2}{k_K g_2(\beta_1)}\left(\dfrac{x_1}{x_2}\right)^{\beta_1} - \dfrac{\beta_1(\beta_3-1)}{\beta_1-\beta_3}\right], & x \in (0, x_2), \end{cases}$$

where x_1, x_2 are given by (37).

Proof. See Appendix A.7. □

4.2 Numerical example of case of $q_2 = 0$

Let us see a numerical example in case of $q_2 = 0$ for simplicity. We set parameters as follows;

$$r = 0.05, \quad (q_1, q_2) = (0.1, 0), \quad D_1 = 1, \quad (K_1, K_2) = (1, 1)$$
$$(\mu_1, \mu_2) = (0.02, 0.03), \quad (\sigma_1, \sigma_2) = (0.2, 0.3).$$

Then the eigenvalues are obtained as $(\beta_1, \beta_2, \beta_3, \beta_4) = (2.738, -2.738, 1.234, -0.901)$.

By moving D_2, k_α and x_1/x_2 vary as follows.

$k_D = \dfrac{D_2}{D_1}$	0.50	0.67	1.00	1.32	1.50
$k_\alpha = \dfrac{\alpha_2 D_2/K_2}{\alpha_1 D_1/K_1}$	0.93	1.00	1.08	1.13	1.15
x_1/x_2	0.52	0.63	0.82	1.00	1.10
NPV (k_α)	$1 \succ 2$	$1 \sim 2$	$1 \prec 2$	$1 \prec 2$	$1 \prec 2$
Real option (x_1/x_2)	$1 \succ 2$	$1 \succ 2$	$1 \succ 2$	$1 \sim 2$	$1 \prec 2$

The goodness of regime state will depend on the ratio of demand quantity $k_D = D_2/D_1$ since the investment cost is same. NPV method would suggest that the threshold is $k_D = 0.67$ while the real option approach suggests that it is $k_D = 1.32$.

5 First passage time

With the same technique we can carry out a calculation of the expected and discounted value of a payoff at the first passage time by applying the Dirichlet problem in the two-dimensional space.

Once the optimal stopping time τ^* is determined and the thresholds $x_S \le \cdots \le x_1$ are fixed, the value function can be decomposed by functions related to the regime at the first passage time as

$$\begin{aligned} V_i(x) &= \max_\tau \mathbb{E}\left[e^{-r\tau}\left(\alpha_{J(\tau)} D_{J(\tau)} X(\tau) - K_{J(\tau)}\right) \mid X(0)=x, J(0)=i\right] \\ &= \sum_{k\in E} \mathbb{E}\left[1_{\{\tau^*=\tau_k\}} e^{-r\tau^*}\left(\alpha_{J(\tau^*)} D_{J(\tau^*)} X_{\tau^*} - K_{J(\tau^*)}\right) \mid X(0)=x, J(0)=i\right] \\ &= \sum_{k\in E} \left[\alpha_k D_k F_i^k(x) - K_k G_i^k(x)\right], \end{aligned} \tag{38}$$

where $\tau^* = \min_{k\in E} \tau_k$, $\tau_k = \inf\{t>0 : X(t) \ge x_k,\ J(t)=k\}$ and

$$F_i^k(x) = \mathbb{E}\left[e^{-r\tau^*} M_F^k(X_{\tau^*}, J(\tau^*)) \mid X(0)=x, J(0)=i\right], \quad M_F^k(x,j) = x\delta_{jk}, \tag{39}$$

$$G_i^k(x) = \mathbb{E}\left[e^{-r\tau^*} M_G^k(X_{\tau^*}, J(\tau^*)) \mid X(0)=x, J(0)=i\right], \quad M_G^k(x,j) = \delta_{jk}. \tag{40}$$

$F_i^k(x)$ is the discounted expected value of the payoff of $X_{\tau^*}\mathbf{1}_{\{J(\tau^*)=k\}}$ after starting from $(X(0), J(0)) = (x, i)$. Similarly, $G_i^k(x)$ is one of the payoff of $\mathbf{1}_{\{J(\tau^*)=k\}}$.

The purpose of this section is to obtain the explicit form of these decomposing functions F_i^k, G_i^k. Our plan and idea are as follows. The functions defined by the expectation in (39) and (40) are applicable to the Dirichlet problem thanks to the functional form in the definition. However, the domain must be set appropriately in the two-dimensional space. Then we obtain a set of ODEs which can be solved with the same technique as in the previous section. Finally, we verify the decomposition (38) and the smooth pasting conditions at the boundary, which are not imposed in the Dirichlet problem.

Since the stopping time τ^* has the regime-dependent thresholds, we need to rename them on each relevant interval as

$$
\begin{aligned}
F_i^k(x) &= \begin{cases} F_i^{(k,0)}(x), & x \in [x_1, \infty) \\ F_i^{(k,n)}(x), & x \in [x_{n+1}, x_n), \quad 1 \le n \le S-1 \\ F_i^{(k,S)}(x), & x \in (0, x_S) \end{cases} \\
G_i^k(x) &= \begin{cases} G_i^{(k,0)}(x), & x \in [x_1, \infty) \\ G_i^{(k,n)}(x), & x \in [x_{n+1}, x_n), \quad 1 \le n \le S-1 \\ G_i^{(k,S)}(x), & x \in (0, x_S) \end{cases}
\end{aligned}
$$

On the n-th interval, the regimes $i > n$ are in the stop region so that

$$
F_i^{(k,n)}(x) = \delta_{ik} x \equiv M_F^k(x,i), \quad G_i^{(k,n)}(x) = \delta_{ik} \equiv M_G^k(x,i), \quad (i > n). \tag{41}
$$

In order to obtain the explicit expression of the functions $F_i^{(k,n)}$ and $G_i^{(k,n)}$, we apply the following Dirichlet problem.

Lemma 5. *Consider an open set $\mathcal{C} \subset \mathbb{R}^d$ and $\mathcal{D} = \mathbb{R}^d \setminus \mathcal{C}$. Define the first passage time $\tau_{\mathcal{D}} = \inf\{t : Y_t \in \mathcal{D}\}$ of a Markov process Y on $\mathbb{R}^d$ with $Y_0 = x \in \mathcal{C}$, and define $F(x) = \mathbb{E}\left[e^{-r\tau_{\mathcal{D}}} M\left(Y_{\tau_{\mathcal{D}}}\right)\right]$ for a given continuous function $M : \partial\mathcal{C} \to \mathbb{R}$. Then F solves the Dirichlet problem*

$$
\mathbb{A}_Y F = rF \quad \text{in} \quad \mathcal{C}, \qquad F|_{\partial\mathcal{C}} = M
$$

where $\mathbb{A}_Y$ is the infinitesimal operator $\mathbb{A}_Y F(x) = \lim_{t \downarrow 0} \dfrac{\mathbb{E}\left[F(Y_t) \mid Y_0 = x\right] - F(x)}{t}$.

Proof. See, for example, Peskir and Shiraev (2006). □

For a regime switching diffusion $dX(t) = \mu_{J(t)} X(t) dt + \sigma_{J(t)} X(t) dW_t$, it is known that

$$
\mathbb{A}_X F_i(x) = \frac{1}{2}\sigma_i^2 x^2 \frac{d^2 F_i}{dx^2}(x) + \mu_i x \frac{dF_i}{dx}(x) + \sum_{j=1, j \ne i}^{S} q_{ij}\left(F_j(x) - F_i(x)\right).
$$

The continuation region of our problem and the relevant sets are given by

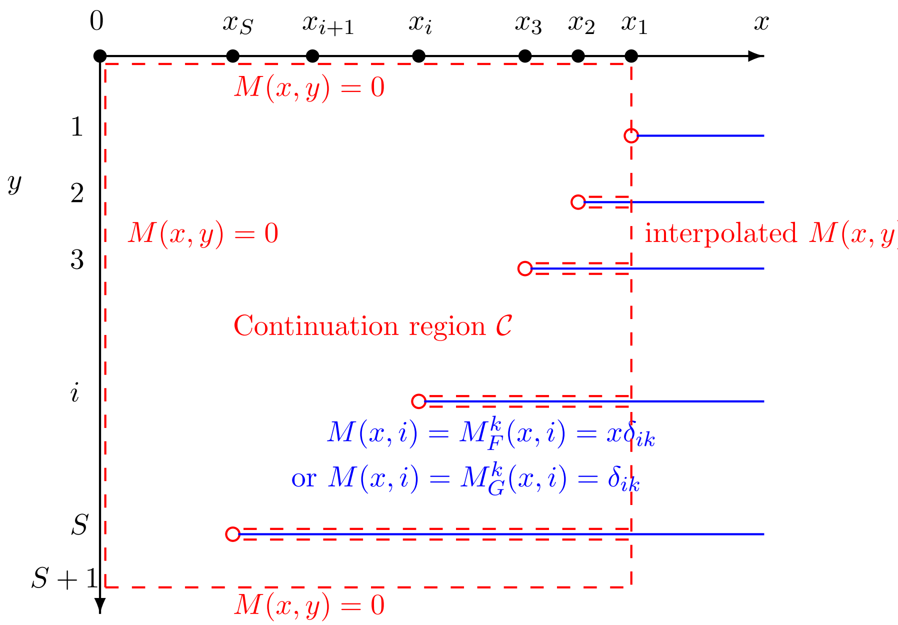
$$
\begin{aligned}
\mathcal{B} &= (0, x_1) \times (0, S+1) \subset \mathbb{R}_{++}^2, \quad \mathcal{C} = \mathcal{B} \setminus \bigcup_{i=1}^{S} \{(x,i) \in \mathbb{R}_{++}^2 \mid x_i \le x < x_1\}, \\
\mathcal{D} &= \mathbb{R}_{++}^2 \setminus \mathcal{C}, \quad \partial\mathcal{C} = \partial\mathcal{B} \cup \left[\bigcup_{i=1}^{S} \{(x,i) \in \mathbb{R}_{++}^2 \mid x_i \le x < x_1\}\right].
\end{aligned}
$$

When we fix k and consider F_i^k $(i \in E)$, the continuous function M on the boundary is defined with $M_F^k(x,i)$ (or $M_G^k(x,i)$)

$$
M = \begin{cases} M_F^k(x,i), & y = i \in E,\ x_i \le x < x_1, \\ ([y] + 1 - y) M_F^k(x_1, [y]) & \\ +(y - [y]) M_F^k(x_1, [y]+1), & [y] \in E,\ x = x_1, \\ 0, & \text{otherwise}. \end{cases}
$$

The function M when considering G_i^k $(i \in E)$ can be constructed similarly by replacing $M_F^k(x,i)$ with $M_G^k(x,i)$ in the above definition. The continuation region $\mathcal{C}$ and the function value $M(x,y)$ are illustrated in Figure 2. Note that the domain is expanded to a dense subset in $\mathbb{R}^2$, though our interest is in a subset in $\mathbb{R} \times \mathbb{Z}$, in order to apply the Dirichlet problem directly. The regimes take values in $\mathbb{Z}$ only so that we don't need to pay attention to the function values in the non-integer

Figure 2: Continuation region



area in the regimes. Thus, the interpolated values of M at $(x_1, y)\,(y \notin E)$ are sufficient to make M continuous on the boundary $\partial\mathcal{C}$.

Then by Dirichlet problem, each of F_i^k and G_i^k satisfies a set of certain ODEs. We must solve simultaneous ODEs on each interval such as

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} F_1^{(k,n)}(x) \\ F_2^{(k,n)}(x) \\ \vdots \\ F_n^{(k,n)}(x) \end{pmatrix} = - \begin{pmatrix} q_{1,n+1} & q_{1,n+2} & \cdots & q_{1S} \\ q_{2,n+1} & q_{2,n+2} & \cdots & q_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n,n+1} & q_{n,n+2} & \cdots & q_{nS} \end{pmatrix} \begin{pmatrix} F_{n+1}^{(k,n)}(x) \\ F_{n+2}^{(k,n)}(x) \\ \vdots \\ F_S^{(k,n)}(x) \end{pmatrix} \tag{42}$$

on $x \in [x_{n+1}, x_n)$, $(n = 1, 2, \cdots, S-1)$ for each $k = 1, 2$, where

$$\mathcal{A}_i f(x) = \frac{1}{2} x^2 \sigma_i^2 \frac{d^2}{dx^2} f(x) + x\mu_i \frac{d}{dx} f(x) + (q_{ii} - r) f(x).$$

The value matching conditions are imposed at x_j in both $\mathcal{C}$ and $\partial\mathcal{C}$. The smooth pasting conditions, however, are imposed at x_j in $\mathcal{C}$ only since a smooth pasting condition at the boundary is meaningless in the Dirichlet problem.

When $n \le k-1$, $F_k^{(k,n)}(x) = x$ but other $F_i^{(k,n)}$ on the RHS of (42) are zero by (41). On the other hand, when $n \ge k$, all of $F_i^{(k,n)}$ on the RHS of (42) are zero. Hence, when $n \le k-1$, (42) becomes

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} F_1^{(k,n)}(x) \\ F_2^{(k,n)}(x) \\ \vdots \\ F_n^{(k,n)}(x) \end{pmatrix} = -x \begin{pmatrix} q_{1k} \\ q_{2k} \\ \vdots \\ q_{nk} \end{pmatrix},$$

and when $n \geq k$, (42) is reduced to

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} F_1^{(k,n)}(x) \\ F_2^{(k,n)}(x) \\ \vdots \\ F_n^{(k,n)}(x) \end{pmatrix} = \mathbf{0}_n.$$

In summary, (42) is equivalent to

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} F_1^{(k,n)}(x) \\ F_2^{(k,n)}(x) \\ \vdots \\ F_n^{(k,n)}(x) \end{pmatrix} = -x1_{\{n<k\}} \begin{pmatrix} q_{1k} \\ q_{2k} \\ \vdots \\ q_{nk} \end{pmatrix}.$$

Similarly we have equations for G_i^k,

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} G_1^{(k,n)}(x) \\ G_2^{(k,n)}(x) \\ \vdots \\ G_n^{(k,n)}(x) \end{pmatrix} = -1_{\{n<k\}} \begin{pmatrix} q_{1k} \\ q_{2k} \\ \vdots \\ q_{nk} \end{pmatrix}.$$

For the following calculation, it is worth of noting that a special solution appears only when $n \leq k-1$, especially $(k,n,i) = (2,1,1)$ if $S = 2$.

We can obtain explicit representations for the case of $S = 2$ by applying the same technique as discussed in the previous section. The following proposition shows the results in case of $q_1 q_2 \neq 0$. One can obtain the results in case of $q_1 q_2 = 0$ in a parallel way.

Proposition 6. *Suppose that $q_1 q_2 \neq 0$.*
$F_i^{(k,n)}, G_i^{(k,n)}$ are given as follows.

$$\begin{aligned}
F_1^{(1,0)}(x) &= x, \quad F_1^{(1,1)}(x) = x_1 \frac{d_1 \left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + d_2 \left(\frac{x}{x_2}\right)^{\beta_2^{(1)}}}{d_1 \left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2 \left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}}, \\
F_1^{(1,2)}(x) &= x_1 \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1 \left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2 \left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}} \left[g_2\left(\beta_1^{(2)}\right) \left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} - g_2\left(\beta_2^{(2)}\right) \left(\frac{x}{x_2}\right)^{\beta_2^{(1)}} \right], \\
F_2^{(1,0)}(x) &= 0, \quad F_2^{(1,1)}(x) = 0, \\
F_2^{(1,2)}(x) &= x_1 \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1 \left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2 \left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}} \left[-q_2 \left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + q_2 \left(\frac{x}{x_2}\right)^{\beta_2^{(1)}} \right], \\
F_1^{(2,0)}(x) &= 0, \quad F_1^{(2,1)}(x) = B_1 \left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + B_2 \left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + p_1 x, \\
F_1^{(2,2)}(x) &= g_2\left(\beta_1^{(2)}\right) B_3 \left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + g_2\left(\beta_2^{(2)}\right) B_4 \left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}, \\
F_2^{(2,0)}(x) &= x, \quad F_2^{(2,1)}(x) = x, \quad F_2^{(2,2)}(x) = -q_2 B_3 \left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - q_2 B_4 \left(\frac{x}{x_2}\right)^{\beta_2^{(2)}},
\end{aligned}$$

and

$$
\begin{aligned}
G_1^{(1,0)}(x) &= 1, \quad G_1^{(1,1)}(x) = \frac{d_1\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}}, \\
G_1^{(1,2)}(x) &= \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}}\left[g_2\left(\beta_1^{(2)}\right)\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} - g_2\left(\beta_2^{(2)}\right)\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}}\right], \\
G_2^{(1,0)}(x) &= 0, \quad G_2^{(1,1)}(x) = 0, \\
G_2^{(1,2)}(x) &= \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}}\left[-q_2\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + q_2\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}}\right], \\
G_1^{(2,0)}(x) &= 0, \quad G_1^{(2,1)}(x) = b_1\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + b_2\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + p_2, \\
G_1^{(2,2)}(x) &= g_2\left(\beta_1^{(2)}\right) b_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + g_2\left(\beta_2^{(2)}\right) b_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}, \\
G_2^{(2,0)}(x) &= 1, \quad G_2^{(2,1)}(x) = 1, \quad G_2^{(2,2)}(x) = -q_2 b_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - q_2 b_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}},
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= -p_1 x_1 \frac{d_1\lambda^{-\beta_1^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} + q_2 x_2 \frac{l_1 l_2\left(\beta_1^{(2)} - \beta_2^{(2)}\right) - p_1\left(l_1\left(\beta_1^{(2)} - 1\right) - l_2\left(\beta_2^{(2)} - 1\right)\right)}{d_1\lambda^{\beta_2^{(1)}} + d_2\lambda^{\beta_1^{(1)}}}, \\
B_3 &= \frac{-p_1 x_1\left(\beta_1^{(1)} - \beta_2^{(1)}\right) + l_2 x_2\left(\left(\beta_2^{(1)} - \beta_2^{(2)}\right)\lambda^{-\beta_1^{(1)}} - \left(\beta_1^{(1)} - \beta_2^{(2)}\right)\lambda^{-\beta_2^{(1)}}\right)}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} \\
&\quad -p_1 x_1 \frac{\left(\beta_2^{(1)} - 1\right)\lambda^{-\beta_1^{(1)}} - \left(\beta_1^{(1)} - 1\right)\lambda^{-\beta_2^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} \\
B_2 &= -p_1 x_1 - B_1, \quad B_4 = -\frac{x_2}{q_2} - B_3, \\
b_1 &= -p_2 \frac{d_1\lambda^{-\beta_1^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} + q_2 \frac{l_1 l_2\left(\beta_1^{(2)} - \beta_2^{(2)}\right) - p_2\left(l_1\beta_1^{(2)} - l_2\beta_2^{(2)}\right)}{d_1\lambda^{\beta_2^{(1)}} + d_2\lambda^{\beta_1^{(1)}}} \\
b_3 &= \frac{-p_2\left(\beta_1^{(1)} - \beta_2^{(1)}\right) + l_2\left(\left(\beta_2^{(1)} - \beta_2^{(2)}\right)\lambda^{-\beta_1^{(1)}} - \left(\beta_1^{(1)} - \beta_2^{(2)}\right)\lambda^{-\beta_2^{(1)}}\right)}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} - p_2 \frac{\beta_2^{(1)}\lambda^{-\beta_1^{(1)}} - \beta_1^{(1)}\lambda^{-\beta_2^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} \\
b_2 &= -p_2 - b_1, \quad b_4 = -\frac{1}{q_2} - b_3.
\end{aligned}
$$

Proof. See Appendix A.8. □

Note that Proposition 6 is valid for any $x_2 \leq x_1$. We don't impose "smooth pasting conditions" of F_i^k, G_i^k at the boundary of the continuation region when solving the Dirichlet problem since "smooth pasting conditions at the boundary" are meaningless in the problem. It follows that, for arbitrary given x_1, x_2, a function constructed with these obtained F_i^k, G_i^k

$$
W_i^{(n)}(x) = \sum_{k \in E}\left(\alpha_k D_k F_i^{(k,n)}(x) - K_k G_i^{(k,n)}(x)\right)
$$

does not necessary satisfy the smooth pasting condition $\frac{d}{dx}W_n^{(n)}(x) = \alpha_n D_n$ at the boundary x_n of the continuation region. The optimality condition involved in $\lambda = x_1/x_2$ plays an important role in veryifying (38) and the smooth pasting conditions at the boundaries.

Lemma 6. *The condition (28) is equivalent to the following condition,*

$$\begin{aligned}
&\frac{\left(1-\beta_2^{(1)}\right)(\alpha_1 D_1 - p_1\alpha_2 D_2)\,x_1 + \beta_2^{(1)}(K_1 - p_2 K_2)}{\beta_1^{(1)} - \beta_2^{(1)}}\lambda^{\beta_1^{(1)}}\begin{pmatrix}1\\ \beta_1^{(1)}\end{pmatrix}\\
&-\frac{\left(1-\beta_1^{(1)}\right)(\alpha_1 D_1 - p_1\alpha_2 D_2)\,x_1 + \beta_1^{(1)}(K_1 - p_2 K_2)}{\beta_1^{(1)} - \beta_2^{(1)}}\lambda^{\beta_2^{(1)}}\begin{pmatrix}1\\ \beta_2^{(1)}\end{pmatrix}\\
= \;& l_1\frac{\left(1-\beta_2^{(2)}\right)\alpha_2 D_2 x_2 + \beta_2^{(2)} K_2}{\beta_1^{(2)} - \beta_2^{(2)}}\begin{pmatrix}1\\ \beta_1^{(2)}\end{pmatrix} - l_2\frac{\left(1-\beta_1^{(2)}\right)\alpha_2 D_2 x_2 + \beta_1^{(2)} K_2}{\beta_1^{(2)} - \beta_2^{(2)}}\begin{pmatrix}1\\ \beta_2^{(2)}\end{pmatrix}\\
&-p_1\alpha_2 D_2 x_2\begin{pmatrix}1\\ 1\end{pmatrix} + p_2 K_2\begin{pmatrix}1\\ 0\end{pmatrix}.
\end{aligned}$$

Proof. See Appendix A.9. □

By making use of Lemma 6, we take the optimality condition into consideration so that we obtain the following result which is seemingly trivial but non-trivial.

Proposition 7. *Suppose that $q_1 q_2 \neq 0$ and x_1, x_2 are given by (31). Then it holds that*

$$V_i^{(n)}(x) = W_i^{(n)}(x), \quad n = 0, 1, 2,\ i = 1, 2,$$

where $V_i^{(n)}$ are given in Proposition 3.

Proof. See Appendix A.10. □

6 Concluding remarks

Our technique of linear algebra provides an insight of the functional form of the value function and it enables us to analyze value function more explicitly under many regime states. Numerical calculation is relatively easy due to expression with eigenvalues and eigenvectors. Comparative analysis is possible due to closed form in some special cases.

As remarks on parameters, we made explicit assumptions on $r, \mathbf{M}, \mathbf{\Sigma}$ and $\mathbf{Q}$ in order to obtain appropriate eigenvalues $\beta_i^{(n)}$ and convergence of income flow multiplier α_i. However, conditions on $\mathbf{D}$ and $\mathbf{K}$ are implicitly involved in calculation of thresholds x_n satisfying the order of $x_S \leq x_{S-1} \leq \cdots \leq x_2 \leq x_1$.

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A Appendix

A.1 Proof of Lemma 1

Due to Asmussen (2003) (Chapter XI, Proposition 2.2) and the Markov property, it holds that for $u \geq t$

$$\begin{aligned} & \mathbb{E}\left[e^{-rT}X(T)^{\beta}1_{\{J(T)=j\}} \mid J(t)=i,\, \mathcal{F}_t\right] \\ = \; & e^{-rT}X(t)^{\beta}\left\{\exp\left(\left(\mathbf{Q}+\mathrm{diag}[\theta^{(k)}(\beta);k\in E]\right)(T-t)\right)\right\}_{ij} \\ = \; & e^{-rt}X(t)^{\beta}\left\{\exp\left(\mathbf{\Lambda}(\beta)(T-t)\right)\right\}_{ij} \end{aligned}$$

since $\mathbf{Q}+\mathrm{diag}[\theta^{(k)}(\beta);k\in E]=\mathbf{\Lambda}(\beta)+r\mathbf{I}_S$. Hence, we obtain the result of (1).

By making use of the result of (a), we see that

$$\begin{aligned} & \mathbb{E}\left[e^{-ru}D_uX(u)^{\beta} \mid J(t)=i,\, \mathcal{F}_t\right] \\ = \; & \sum_{j\in E}\mathbb{E}\left[e^{-ru}D_uX(u)^{\beta}1_{\{J(u)=j\}} \mid J(t)=i,\, \mathcal{F}_t\right] \\ = \; & \sum_{j\in E}\mathbb{E}\left[e^{-ru}X(u)^{\beta}1_{\{J(u)=j\}} \mid J(t)=i,\, \mathcal{F}_t\right]D_j \\ = \; & e^{-rt}X(t)^{\beta}\mathbf{e}_i^{\top}\exp\left(\mathbf{\Lambda}(\beta)(u-t)\right)\mathbf{D}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}\left[\int_t^{\infty}e^{-ru}D_{J(u)}X(u)du \mid \mathcal{F}_t\right] &= \int_t^{\infty}\mathbb{E}\left[e^{-ru}D_{J(u)}X(u) \mid \mathcal{F}_t\right]du \\ &= \int_t^{\infty}e^{-rt}X(t)\mathbf{e}_{J(t)}^{\top}\exp\left(-\left(r\mathbf{I}_S-\mathbf{M}-\mathbf{Q}\right)(u-t)\right)\mathbf{D}du \\ &= e^{-rt}\mathbf{e}_{J(t)}^{\top}\left(\int_t^{\infty}\exp\left(-\left(r\mathbf{I}_S-\mathbf{M}-\mathbf{Q}\right)(u-t)\right)du\right)\mathbf{D}X(t). \end{aligned}$$

Since by Assumption 1 the eigenvalues of $r\mathbf{I}_S-\mathbf{M}-\mathbf{Q}$ are strictly positive, it holds that

$$\int_t^{\infty}\exp\left(-\left(r\mathbf{I}_S-\mathbf{M}-\mathbf{Q}\right)(u-t)\right)du = \left(r\mathbf{I}_S-\mathbf{M}-\mathbf{Q}\right)^{-1}.$$

Therefore, we obtain

$$\begin{aligned} \mathbb{E}\left[\int_t^{\infty}e^{-ru}D_{J(u)}X(u)du \mid \mathcal{F}_t\right] &= e^{-rt}\mathbf{e}_{J(t)}^{\top}\left(r\mathbf{I}_S-\mathbf{M}-\mathbf{Q}\right)^{-1}\mathbf{D}X(t) \\ &= e^{-rt}\alpha_{J(t)}D_{J(t)}X(t). \end{aligned}$$

A.2 Proof of Proposition 2

(28) is equivalent to

$$\begin{aligned} x_1^{-\beta_1^{(1)}}\left(c_{11}\widetilde{\alpha}_1x_1+d_{11}\right) &= k_Kx_2^{-\beta_1^{(1)}}\left(c_{21}\widetilde{\alpha}_2x_2+d_{21}\right), \\ x_1^{-\beta_2^{(1)}}\left(c_{12}\widetilde{\alpha}_1x_1+d_{12}\right) &= k_Kx_2^{-\beta_2^{(1)}}\left(c_{22}\widetilde{\alpha}_2x_2+d_{22}\right). \end{aligned}$$

With $x_2=\lambda x_1$, we obtain

$$x_1=\frac{-d_{11}+k_Kd_{21}\lambda^{-\beta_1^{(1)}}}{c_{11}\widetilde{\alpha}_1-k_Kc_{21}\widetilde{\alpha}_2\lambda^{-\beta_1^{(1)}+1}}=\frac{-d_{12}+k_Kd_{22}\lambda^{-\beta_2^{(1)}}}{c_{12}\widetilde{\alpha}_1-k_Kc_{22}\widetilde{\alpha}_2\lambda^{-\beta_2^{(1)}+1}},$$

which implies an equation for λ

$$\left(c_{11}\widetilde{\alpha}_1\lambda^{\beta_1^{(1)}-1} - k_K c_{21}\widetilde{\alpha}_2\right)\left(d_{12}\lambda^{\beta_2^{(1)}} - k_K d_{22}\right)$$
$$= \left(c_{12}\widetilde{\alpha}_1\lambda^{\beta_2^{(1)}-1} - k_K c_{22}\widetilde{\alpha}_2\right)\left(d_{11}\lambda^{\beta_1^{(1)}} - k_K d_{21}\right),$$

which must be negative in order to ensure the positivity of x_1.

A.3 Proof of Proposition 3

$$\begin{aligned}
\mathbf{V}^{(1)}(x) &= \mathbf{U}^{(1)}\mathbf{X}^{(1)}(x)\mathbf{A}^{(1)} + \mathbf{v}^{(1)}(x) = K_1\mathbf{U}^{(1)}\mathbf{X}^{(1)}(x)\mathbf{X}^{(1)}(x_1^{-1})\begin{pmatrix}\gamma_{11}(x_1)\\ \gamma_{12}(x_1)\end{pmatrix} + \mathbf{v}^{(1)}(x) \\
&= K_1\gamma_{11}(x_1)\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + K_1\gamma_{12}(x_1)\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + K_2(p_1\widetilde{\alpha}_2 x - p_2)
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}^{(2)}(x) &= \mathbf{U}_2\mathbf{X}_2(x)\mathbf{A}_2 = -\frac{K_2}{q_2}\mathbf{U}_2\mathbf{X}_2(x)\mathbf{X}_2(x_2^{-1})\begin{pmatrix}\gamma_{21}(x_2)\\ \gamma_{22}(x_2)\end{pmatrix} \\
&= \begin{pmatrix} K_2 l_1\gamma_{21}(x_2)\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + K_2 l_2\gamma_{22}(x_2)\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}} \\ K_2\gamma_{21}(x_2)\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + K_2\gamma_{22}(x_2)\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}} \end{pmatrix}
\end{aligned}$$

A.4 Proof of Lemma 3

(1) By definition, it holds that

$$\begin{pmatrix}\alpha_1 D_1\\ \alpha_2 D_2\end{pmatrix} = \begin{pmatrix} r-\mu_1+q_1 & -q_1 \\ 0 & r-\mu_2\end{pmatrix}^{-1}\begin{pmatrix}D_1\\ D_2\end{pmatrix} = \frac{1}{(r-\mu_1+q_1)(r-\mu_2)}\begin{pmatrix}(r-\mu_2)D_1 + q_1 D_2\\ (r-\mu_1+q_1)D_2\end{pmatrix}$$

so that

$$k_\alpha = \frac{1}{k_K}\frac{r-\mu_1+q_1}{(r-\mu_2)/k_D + q_1}.$$

Then we see

$$\begin{aligned}
h_1(1) &= a_1 + b_1 + c_1 \\
&= \frac{(\beta_1-1)\beta_3}{\beta_3-1}\left[k_K\left(\frac{(r-\mu_2)/k_D + q_1}{r-\mu_1+q_1} + \frac{q_1(\beta_1-\beta_3)}{g_1(\beta_3)(\beta_1-1)\beta_3}\right) - \frac{\beta_1(\beta_3-1)}{(\beta_1-1)\beta_3}\right].
\end{aligned}$$

Therefore, $h_1(1) < 0$ is equivalent to (32).

(2) The value matching conditions and the smooth pasting conditions at $x = x_2$ are given as

$$V_1: \quad \begin{pmatrix} g_2(\beta_1) & -q_1 \\ g_2(\beta_1)\beta_1 & -q_1\beta_3\end{pmatrix}\mathbf{X}_2(x_2)\mathbf{A}_2 = \begin{pmatrix}\mathbf{U}^{(1)}\mathbf{X}^{(1)}(x_2)\mathbf{A}^{(1)} + v^{(1)}(x_2)\\ \mathbf{U}^{(1)}\mathbf{dX}^{(1)}(x_2)\mathbf{A}^{(1)} + a^{(1)}x_2\end{pmatrix}, \tag{43}$$

$$V_2: \quad \begin{pmatrix} 0 & g_1(\beta_3) \\ 0 & g_1(\beta_3)\beta_3\end{pmatrix}\mathbf{X}_2(x_2)\mathbf{A}_2 = \begin{pmatrix}\alpha_2 D_2 x_2 - K_2\\ \alpha_2 D_2 x_2\end{pmatrix}, \tag{44}$$

where $\mathbf{X}_2(x) = \mathrm{diag}[x^{\beta_1}, x^{\beta_3}]$.

From (44), $\beta_3(\alpha_2 D_2 x_2 - K_2) = \alpha_2 D_2 x_2$, namely

$$x_2 = \frac{\beta_3}{\beta_3-1}\frac{K_2}{\alpha_2 D_2} = \frac{\beta_3}{\beta_3-1}\frac{1}{\widetilde{\alpha}_2}.$$

Then it must be

$$\mathbf{X}_2(x_2)\mathbf{A}_2 = K_2 \begin{pmatrix} \gamma_{21}(x_2) \\ \gamma_{22}(x_2) \end{pmatrix}, \tag{45}$$

where $\gamma_{21}(x_2)$ is unknown while $\gamma_{22}(x_2) = \dfrac{\widetilde{\alpha}_2 x_2}{g_1(\beta_3)\beta_3} = \dfrac{1}{g_1(\beta_3)(\beta_3 - 1)}$.

With (18) and (45), (43) is equivalent to

$$\begin{aligned}
k_K \begin{pmatrix} \gamma_{21}(x_2) \\ \gamma_{22}(x_2) \end{pmatrix} &= \begin{pmatrix} g_2(\beta_1) & -q_1 \\ g_2(\beta_1)\beta_1 & -q_1\beta_3 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \left(\frac{x_2}{x_1}\right)^{\beta_1} \gamma_{11}(x_1) \\ \left(\frac{x_2}{x_1}\right)^{\beta_2} \gamma_{12}(x_1) \end{pmatrix} + k_K \begin{pmatrix} p_1\widetilde{\alpha}_2 x_2 - p_2 \\ p_1\widetilde{\alpha}_2 x_2 \end{pmatrix} \right] \\
&= \frac{1}{g_2(\beta_1)(\beta_1 - \beta_3)} \begin{pmatrix} (\beta_1 - \beta_3)\left(\frac{x_2}{x_1}\right)^{\beta_1} \gamma_{11}(x_1) + (\beta_2 - \beta_3)\left(\frac{x_2}{x_1}\right)^{\beta_2} \gamma_{12}(x_1) \\ -\frac{g_2(\beta_1)}{q_1}(\beta_1 - \beta_2)\left(\frac{x_2}{x_1}\right)^{\beta_2 1} \gamma_{12}(x_1) \end{pmatrix} \\
&\quad - \frac{k_K}{g_2(\beta_1)(\beta_1 - \beta_3)} \begin{pmatrix} (\beta_3 - 1)p_1\widetilde{\alpha}_2 x_2 - p_2\beta_3 \\ \frac{g_2(\beta_1)}{q_1}((\beta_1 - 1)p_1\widetilde{\alpha}_2 x_2 - p_2\beta_1) \end{pmatrix}.
\end{aligned} \tag{46}$$

This is a system of equations on x_1 and $\gamma_{21}(x_2)$. Thus, by looking at the equality for the second row with an observation of $\gamma_{12}(x_1) = \frac{a_1 x_1/x_2 + b_1}{\beta_1 - \beta_2}$, an equation for $\lambda^{-1} = x_1/x_2$ is obtained as

$$k_K \frac{q_1(\beta_1 - \beta_3)}{g_1(\beta_3)(\beta_3 - 1)} = -\left(\frac{x_2}{x_1}\right)^{\beta_2} \left(a_1 \frac{x_1}{x_2} + b_1\right) - k_K((\beta_1 - 1)p_1\widetilde{\alpha}_2 x_2 - p_2\beta_1)$$

or $h_1(\lambda^{-1}) = 0$.

Let us check the monotonicity of h_1. We see that

$$\begin{aligned}
h_1'(y) &= (1 - \beta_2)a_1 y^{-\beta_2} \left(y - \frac{\beta_2 b_1}{(1 - \beta_2)a_1}\right), \\
a_1 &= (\beta_1 - 1)\frac{r - \mu_2}{r - \mu_2 + q_1 k_D}\widetilde{\alpha}_1 > 0.
\end{aligned}$$

It follows that $h_1'(y) < 0$ on $(0, z)$, $h_1'(y) > 0$ on (z, ∞) and $\lim_{y\to\infty} h(y) = +\infty$, where $z \equiv \max\left[\dfrac{\beta_2}{1 - \beta_2}\dfrac{b_1}{a_1}, 0\right]$. Then λ^{-1} exists uniquely satisfying $h_1(\lambda^{-1}) = 0$ and $\lambda^{-1} > 1$ if $h_1(1) < 0$.

It is obvious for other conditions in (2), (3) and (4).

A.5 Proof of Proposition 4

It is straightforward to check that by (**??**), (**??**) and (46),

$$\begin{aligned}
\gamma_{11}(x_1) &= \frac{(1 - \beta_2)\frac{(\beta_1 - \beta_2)\gamma_{12}(x_1) + \beta_1(1 - k_K p_2)}{\beta_1 - 1} + \beta_2(1 - k_K p_2)}{\beta_1 - \beta_2} = \frac{1 - \beta_2}{\beta_1 - 1}\gamma_{12}(x_1) + \frac{1 - k_K p_2}{\beta_1 - 1}, \\
\gamma_{12}(x_1) &= \frac{a_1\lambda^{-1} + b_1}{\beta_1 - \beta_2} = \frac{a_1(\lambda^{-1})^{-\beta_2 + 1} + b_1(\lambda^{-1})^{-\beta_2}}{\beta_1 - \beta_2}\left(\frac{x_1}{x_2}\right)^{\beta_2} = -\frac{c_1}{\beta_1 - \beta_2}\left(\frac{x_1}{x_2}\right)^{\beta_2}, \\
\gamma_{22}(x_2) &= \frac{1}{g_1(\beta_3)(\beta_3 - 1)},
\end{aligned}$$

and

$$\begin{aligned}
&k_K g_2(\beta_1)\gamma_{21}(x_2) \\
&= \frac{1}{\beta_1 - \beta_3}\left[(\beta_1 - \beta_3)\left(\frac{x_2}{x_1}\right)^{\beta_1} \gamma_{11}(x_1) + (\beta_2 - \beta_3)\left(\frac{x_2}{x_1}\right)^{\beta_2} \gamma_{12}(x_1)\right] \\
&\quad - k_K \frac{(\beta_3 - 1)p_1\widetilde{\alpha}_2 x_2 - p_2\beta_3}{\beta_1 - \beta_3}.
\end{aligned}$$

Then it holds that

$$\begin{aligned}
\mathbf{V}^{(1)}(x) &= \mathbf{U}^{(1)}\mathbf{X}^{(1)}(x)\mathbf{A}^{(1)} + \mathbf{v}^{(1)}(x) = K_1\mathbf{U}^{(1)}\mathbf{X}^{(1)}(x)\mathbf{X}^{(1)}(x_1^{-1})\begin{pmatrix}\gamma_{11}(x_1)\\ \gamma_{12}(x_1)\end{pmatrix} + \mathbf{v}^{(1)}(x) \\
&= K_1\gamma_{11}(x_1)\left(\frac{x}{x_1}\right)^{\beta_1} + K_1\gamma_{12}(x_1)\left(\frac{x}{x_1}\right)^{\beta_2} + K_2(p_1\widetilde{\alpha}_2 x - p_2), \\
\mathbf{V}^{(2)}(x) &= \mathbf{U}_2\mathbf{X}_2(x)\mathbf{A}_2 = -\frac{K_2}{q_2}\mathbf{U}_2\mathbf{X}_2(x)\mathbf{X}_2(x_2^{-1})\begin{pmatrix}\gamma_{21}(x_2)\\ \gamma_{22}(x_2)\end{pmatrix} \\
&= \begin{pmatrix} K_2 g_2(\beta_1)\gamma_{21}(x_2)\left(\frac{x}{x_2}\right)^{\beta_1} - K_2 q_1\gamma_{22}(x_2)\left(\frac{x}{x_2}\right)^{\beta_3} \\ K_2 g_1(\beta_3)\gamma_{22}(x_2)\left(\frac{x}{x_2}\right)^{\beta_3}\end{pmatrix}.
\end{aligned}$$

A.6 Proof of Lemma 4

The value matching conditions and the smooth pasting conditions at $x = x_2$ are given as

$$\begin{aligned}
V_1 : &\quad \begin{pmatrix} g_2(\beta_1) & 0 \\ g_2(\beta_1)\beta_1 & 0\end{pmatrix}\mathbf{X}_2(x_2)\mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ \beta_1^{(1)} & \beta_2^{(1)}\end{pmatrix}\mathbf{X}^{(1)}(x_2)\mathbf{A}^{(1)}, && (47)\\
V_2 : &\quad -q_2\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_3\end{pmatrix}\mathbf{X}_2(x_2)\mathbf{A}_2 = \begin{pmatrix}\alpha_2 D_2 x_2 - K_2 \\ \alpha_2 D_2 x_2\end{pmatrix}, && (48)\\
&\quad \text{where} \quad \mathbf{X}_2(x) = \mathrm{diag}[x^{\beta_1}, x^{\beta_3}].
\end{aligned}$$

(48) is equivalent to

$$\mathbf{X}_2(x_2)\mathbf{A}_2 = -\frac{K_2}{q_2}\begin{pmatrix}\gamma_{21}(x_2)\\ \gamma_{22}(x_2)\end{pmatrix}, \tag{49}$$

where

$$\gamma_{21}(x) = \frac{(1-\beta_3)\widetilde{\alpha}_2 x + \beta_3}{\beta_1 - \beta_3}, \quad \gamma_{22}(x) = -\frac{(1-\beta_1)\widetilde{\alpha}_2 x + \beta_1}{\beta_1 - \beta_3}.$$

(47) is rewritten as

$$\mathbf{A}^{(1)} = \mathbf{X}^{(1)}\left(x_2^{-1}\right)\begin{pmatrix} 1 & 1 \\ \beta_1^{(1)} & \beta_2^{(1)}\end{pmatrix}^{-1}\begin{pmatrix} g_2(\beta_1) & 0 \\ g_2(\beta_1)\beta_1 & 0\end{pmatrix}\mathbf{X}_2(x_2)\mathbf{A}_2. \tag{50}$$

By equating (22) with (50) after plugging (49) to (50), we obtain a vector equation

$$\begin{aligned}
\mathbf{X}^{(1)}(x_1^{-1})\begin{pmatrix}\gamma_{11}(x_1)\\ \gamma_{12}(x_1)\end{pmatrix} &= lk_K\gamma_{21}(x_2)\mathbf{X}^{(1)}(x_2^{-1})\begin{pmatrix} 1 & 1 \\ \beta_1^{(1)} & \beta_2^{(1)}\end{pmatrix}^{-1}\begin{pmatrix} 1 \\ \beta_1\end{pmatrix} \\
&= lk_K\gamma_{21}(x_2)\mathbf{X}^{(1)}(x_2^{-1})\begin{pmatrix} 1 \\ 0\end{pmatrix}, && (51)
\end{aligned}$$

since $\beta_1 = \beta_1^{(1)}$, where $l = -g_2(\beta_1)/q_2$.

(51) is equivalent to

$$x_1 = \frac{\beta_1}{\beta_1 - 1}\frac{1}{\widetilde{\alpha}_1}$$

and $h_2(\lambda) = 0$.

$$h_2'(y) = -\frac{1}{k_K}\frac{q_2}{g_2(\beta_1)}\frac{\beta_1(\beta_1-\beta_3)}{\beta_3(\beta_1-1)}\left[y^{\beta_1-1} - \frac{g_2(\beta_1)}{q_2(\beta_1-\beta_3)}\frac{1}{k_K}\frac{(r-\mu_1)k_D + q_2}{r-\mu_2+q_2}\right]$$

By definition of β_1, β_3, we observe that

$$\frac{g_2(\beta_1)}{\beta_1-\beta_3} = \frac{g_2(\beta_1)-g_2(\beta_3)}{\beta_1-\beta_3} > 0.$$

Hence, $h_2'(y) > 0$ on $(0, z_2)$ and $h_2'(y) < 0$ on (z_2, ∞). Since $h_2(0) = -1 < 0$ and $\lim_{y\to\infty} h_2(y) = -\infty$, there exists unique solution of $h_2(\lambda) = 0$ in $(0,1)$ if and only if $h_2(1) > 0$ or

$$\frac{1}{k_K}\left(\frac{(r-\mu_1)k_D+q_2}{r-\mu_2+q_2} - \frac{q_2}{g_2(\beta_1)}\frac{\beta_1-\beta_3}{(\beta_3-1)\beta_1}\right) > \frac{\beta_3(\beta_1-1)}{(\beta_3-1)\beta_1}.$$

A.7 Proof of Proposition 5

$$p_1 = p_2 = 0, \quad v^{(1)}(x) = 0, \quad \gamma_{11}(x) = \frac{(1-\beta_2^{(1)})\widetilde{\alpha}_1 x + \beta_2^{(1)}}{\beta_1^{(1)}-\beta_2^{(1)}}, \quad \gamma_{12}(x) = -\frac{(1-\beta_1^{(1)})\widetilde{\alpha}_1 x + \beta_1^{(1)}}{\beta_1^{(1)}-\beta_2^{(1)}},$$

$$\begin{pmatrix}\alpha_1 D_1\\ \alpha_2 D_2\end{pmatrix} = \begin{pmatrix} r-\mu_1 & 0\\ -q_2 & r-\mu_2+q_2\end{pmatrix}^{-1}\begin{pmatrix}D_1\\ D_2\end{pmatrix} = \frac{1}{(r-\mu_1)(r-\mu_2+q_2)}\begin{pmatrix}(r-\mu_2+q_2)D_1\\ q_2 D_1 + (r-\mu_1)D_2\end{pmatrix},$$

$$k_\alpha = \frac{1}{k_K}\frac{(r-\mu_1)k_D+q_2}{r-\mu_2+q_2}.$$

Since

$$\gamma_{11}(x_1) = \frac{(1-\beta_2)\frac{\beta_1}{\beta_1-1}+\beta_2}{\beta_1-\beta_2} = \frac{1}{\beta_1-1}, \quad \gamma_{12}(x_1) = 0,$$

by (**??**)

$$\gamma_{21}(x_2) = \frac{1}{lk_K}\left(\frac{x_2}{x_1}\right)^{\beta_1}\gamma_{11}(x_1) = -\frac{q_2 K_1}{g_2(\beta_1)K_2(\beta_1-1)}\left(\frac{x_2}{x_1}\right)^{\beta_1},$$

$$\gamma_{22}(x_2) = -\frac{(1-\beta_1)\frac{(\beta_1-\beta_3)\gamma_{21}(x_2)-\beta_3}{1-\beta_3}+\beta_1}{\beta_1-\beta_3} = -\frac{\beta_1-1}{\beta_3-1}\gamma_{21}(x_2) - \frac{\beta_1}{\beta_1-\beta_3},$$

$$\begin{aligned}
\mathbf{V}^{(1)}(x) &= \mathbf{U}^{(1)}\mathbf{X}^{(1)}(x)\mathbf{A}^{(1)} + \mathbf{v}^{(1)}(x) = K_1\mathbf{U}^{(1)}\mathbf{X}^{(1)}(x)\mathbf{X}^{(1)}(x_1^{-1})\begin{pmatrix}\gamma_{11}(x_1)\\ \gamma_{12}(x_1)\end{pmatrix} + \mathbf{v}^{(1)}(x)\\
&= K_1\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\left(\frac{x}{x_1}\right)^{\beta_1} & 0\\ 0 & \left(\frac{x}{x_1}\right)^{\beta_2}\end{pmatrix}\begin{pmatrix}\frac{1}{\beta_1-1}\\ 0\end{pmatrix} = K_1\frac{1}{\beta_1-1}\left(\frac{x}{x_1}\right)^{\beta_1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}^{(2)}(x) &= \mathbf{U}_2\mathbf{X}_2(x)\mathbf{A}_2 = -\frac{K_2}{q_2}\mathbf{U}_2\mathbf{X}_2(x)\mathbf{X}_2(x_2^{-1})\begin{pmatrix}\gamma_{21}(x_2)\\ \gamma_{22}(x_2)\end{pmatrix}\\
&= K_1\frac{1}{\beta_1-1}\left(\frac{x}{x_1}\right)^{\beta_1}\begin{pmatrix}1\\ -\frac{q_2}{g_2(\beta_1)}\end{pmatrix} + K_2\frac{1}{\beta_3-1}\left(\frac{x}{x_2}\right)^{\beta_3}\begin{pmatrix}0\\ \frac{q_2}{k_K g_2(\beta_1)}\left(\frac{x_1}{x_2}\right)^{\beta_1} - \frac{\beta_1(\beta_3-1)}{\beta_1-\beta_3}\end{pmatrix}
\end{aligned}$$

A.8 Proof of Proposition 6

A.8.1 case of $k = 1$ on F_i

First, let us consider F_i with $k = 1$.

For $n = 0$, or on the interval $[x_1, \infty)$, it is obvious that $F_1^{(1,0)}(x) = x$ and $F_2^{(1,0)}(x) = 0$ by definition. For $n = 1, 2$, it holds that

$$\mathcal{A}_1 F_1^{(1,1)}(x) = 0, \quad F_2^{(1,1)}(x) = 0, \quad \begin{pmatrix} \mathcal{A}_1 & q_{12} \\ q_{21} & \mathcal{A}_2 \end{pmatrix} \begin{pmatrix} F_1^{(1,2)}(x) \\ F_2^{(1,2)}(x) \end{pmatrix} = \mathbf{0}_2.$$

Thus, they must be in a form of

$$\begin{aligned}
F_1^{(1,1)}(x) &= A_1 \left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + A_2 \left(\frac{x}{x_1}\right)^{\beta_2^{(1)}}, \\
F_1^{(1,2)}(x) &= g_2\left(\beta_1^{(2)}\right) A_3 \left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + g_2\left(\beta_2^{(2)}\right) A_4 \left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}, \\
F_2^{(1,2)}(x) &= -q_2 A_3 \left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - q_2 A_4 \left(\frac{x}{x_2}\right)^{\beta_2^{(2)}},
\end{aligned}$$

with unknown constants A_1, A_2, A_3, and A_4, by taking the eigenvectors into consideration. Since the necessary value matching conditions and the smooth pasting conditions are

$$F_1^{(1,1)}(x_1) = x_1, \quad F_1^{(1,2)}(x_2) = F_1^{(1,1)}(x_2), \quad \frac{d}{dx}F_1^{(1,2)}(x_2) = \frac{d}{dx}F_1^{(1,1)}(x_2), \quad F_2^{(1,2)}(x_2) = 0,$$

we have a system of equations of these unknown constants

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -\lambda^{\beta_1^{(1)}} & -\lambda^{\beta_2^{(1)}} & g_2\left(\beta_1^{(2)}\right) & g_2\left(\beta_2^{(2)}\right) \\ -\beta_1^{(1)}\lambda^{\beta_1^{(1)}} & -\beta_2^{(1)}\lambda^{\beta_2^{(1)}} & \beta_1^{(2)} g_2\left(\beta_1^{(2)}\right) & \beta_2^{(2)} g_2\left(\beta_2^{(2)}\right) \\ 0 & 0 & -q_2 & -q_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From the first row and the fourth row, we see $A_2 = x_1 - A_1$, $A_4 = -A_3$. Then, the reduced equations are

$$\begin{pmatrix} -\lambda^{\beta_1^{(1)}} + \lambda^{\beta_2^{(1)}} & g_2\left(\beta_1^{(2)}\right) - g_2\left(\beta_2^{(2)}\right) \\ -\beta_1^{(1)}\lambda^{\beta_1^{(1)}} + \beta_2^{(1)}\lambda^{\beta_2^{(1)}} & \beta_1^{(2)} g_2\left(\beta_1^{(2)}\right) - \beta_2^{(2)} g_2\left(\beta_2^{(2)}\right) \end{pmatrix} \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} = x_1 \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix}.$$

Let us call the coefficient matrix on the LHS $\mathbf{W}$,

$$\begin{aligned}
\mathbf{W} &= \begin{pmatrix} -\lambda^{\beta_1^{(1)}} + \lambda^{\beta_2^{(1)}} & g_2\left(\beta_1^{(2)}\right) - g_2\left(\beta_2^{(2)}\right) \\ -\beta_1^{(1)}\lambda^{\beta_1^{(1)}} + \beta_2^{(1)}\lambda^{\beta_2^{(1)}} & \beta_1^{(2)} g_2\left(\beta_1^{(2)}\right) - \beta_2^{(2)} g_2\left(\beta_2^{(2)}\right) \end{pmatrix} \\
&= \begin{pmatrix} -\lambda^{\beta_1^{(1)}} + \lambda^{\beta_2^{(1)}} & -q_2(l_1 - l_2) \\ -\beta_1^{(1)}\lambda^{\beta_1^{(1)}} + \beta_2^{(1)}\lambda^{\beta_2^{(1)}} & -q_2\left(l_1\beta_1^{(2)} - l_2\beta_2^{(2)}\right) \end{pmatrix},
\end{aligned} \tag{52}$$

and check the existence of the inverse matrix $\mathbf{W}^{-1}$. The determinant is evaluated as

$$\begin{aligned}
\det\mathbf{W} &= \left(\lambda^{\beta_1^{(1)}} - \lambda^{\beta_2^{(1)}}\right) q_2 \left(l_1\beta_1^{(2)} - l_2\beta_2^{(2)}\right) - \left(\beta_1^{(1)}\lambda^{\beta_1^{(1)}} - \beta_2^{(1)}\lambda^{\beta_2^{(1)}}\right) q_2(l_1 - l_2) \\
&= -q_2(l_1 - l_2)\lambda^{\beta_2^{(1)}} \left[\left(\beta_1^{(1)} - \frac{l_1\beta_1^{(2)} - l_2\beta_2^{(2)}}{l_1 - l_2}\right)\lambda^{\beta_1^{(1)} - \beta_2^{(1)}} - \left(\beta_2^{(1)} - \frac{l_1\beta_1^{(2)} - l_2\beta_2^{(2)}}{l_1 - l_2}\right)\right].
\end{aligned}$$

Since $0 < \lambda \leq 1$ and $\beta_2^{(1)} < 0 < 1 < \beta_2^{(2)} < \beta_1^{(1)} < \beta_1^{(2)}$, we observe that

$$0 < \lambda^{\beta_1^{(1)} - \beta_2^{(1)}} \leq 1, \quad l_1 = -\frac{g_2\left(\beta_1^{(2)}\right)}{q_2} = -\frac{q_1}{g_1\left(\beta_1^{(2)}\right)} < 0, \quad l_2 = -\frac{g_2\left(\beta_2^{(2)}\right)}{q_2} = -\frac{q_1}{g_1\left(\beta_2^{(2)}\right)} > 0.$$

and

$$\beta_2' \equiv \beta_2^{(1)} - \frac{l_1\beta_1^{(2)} - l_2\beta_2^{(2)}}{l_1 - l_2} < \beta_1' \equiv \beta_1^{(1)} - \frac{l_1\beta_1^{(2)} - l_2\beta_2^{(2)}}{l_1 - l_2}.$$

It follows that

$$0 < \beta_2^{(2)} < \frac{l_1\beta_1^{(2)} - l_2\beta_2^{(2)}}{l_1 - l_2} < \beta_1^{(2)}, \quad \beta_2' < 0.$$

If $\beta_1' \geq 0$, then $\beta_1'\lambda^{\beta_1^{(1)} - \beta_2^{(1)}} - \beta_2' > 0$. In case of $\beta_1' < 0$, we see $\beta_1'\lambda^{\beta_1^{(1)} - \beta_2^{(1)}} - \beta_2' \geq \beta_1' - \beta_2' > 0$. Therefore, in any case, $\det\mathbf{W} > 0$ is confirmed. With the notations

$$d_1 = q_2\left[-\left(-\beta_2^{(1)} + \beta_1^{(2)}\right)l_1 - \left(\beta_2^{(1)} - \beta_2^{(2)}\right)l_2\right], \quad d_2 = q_2\left[-\left(\beta_1^{(1)} - \beta_1^{(2)}\right)l_1 + \left(\beta_1^{(1)} - \beta_2^{(2)}\right)l_2\right],$$

the determinant is rewritten as

$$\det\mathbf{W} = d_1\lambda^{\beta_2^{(1)}} + d_2\lambda^{\beta_1^{(1)}} = \lambda^{\beta_1^{(1)} + \beta_2^{(1)}}\left[d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}\right].$$

Therefore, the coefficients are solvable as

$$\begin{aligned}\begin{pmatrix} A_1 \\ A_3 \end{pmatrix} &= x_1\lambda^{\beta_2^{(1)}}\mathbf{W}^{-1}\begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} = \frac{x_1\lambda^{\beta_2^{(1)}}}{\det\mathbf{W}}\begin{pmatrix} -q_2\left(l_1\beta_1^{(2)} - l_2\beta_2^{(2)}\right) & q_2(l_1 - l_2) \\ \beta_1^{(1)}\lambda^{\beta_1^{(1)}} - \beta_2^{(1)}\lambda^{\beta_2^{(1)}} & -\lambda^{\beta_1^{(1)}} + \lambda^{\beta_2^{(1)}} \end{pmatrix}\begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} \\ &= \frac{x_1\lambda^{\beta_2^{(1)}}}{\det\mathbf{W}}\begin{pmatrix} d_1 \\ (\beta_1^{(1)} - \beta_2^{(1)})\lambda^{\beta_1^{(1)}} \end{pmatrix},\end{aligned}$$

and

$$A_2 = x_1 - A_1 = x_1\frac{d_2\lambda^{\beta_1^{(1)}}}{\det\mathbf{W}}, \quad A_4 = -A_3 = -x_1\frac{(\beta_1^{(1)} - \beta_2^{(1)})\lambda^{\beta_1^{(1)}}}{\det\mathbf{W}}.$$

They are summarized as

$$\begin{aligned} A_1 &= x_1\frac{d_1\lambda^{-\beta_1^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}}, \quad A_2 = x_1\frac{d_2\lambda^{-\beta_2^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}}, \\ A_3 &= x_1\frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}}, \quad A_4 = -x_1\frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}}.\end{aligned}$$

A.8.2 case of $k = 2$ on F_i

As discussed earlier, it holds that

$$F_1^{(2,0)}(x) = 0, \quad F_2^{(2,0)}(x) = x \qquad \mathcal{A}_1F_1^{(2,1)}(x) = -q_{12}x, \quad F_2^{(2,1)}(x) = x, \quad \begin{pmatrix} \mathcal{A}_1 & q_{12} \\ q_{21} & \mathcal{A}_2 \end{pmatrix}\begin{pmatrix} F_1^{(2,2)}(x) \\ F_2^{(2,2)}(x) \end{pmatrix} = \mathbf{0}_2.$$

Thus, they must be in a form of

$$\begin{aligned} F_1^{(2,1)}(x) &= B_1\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + B_2\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + p_1x, \\ F_1^{(2,2)}(x) &= g_2\left(\beta_1^{(2)}\right)B_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + g_2\left(\beta_2^{(2)}\right)B_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}, \\ F_2^{(2,2)}(x) &= -q_2B_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - q_2B_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}, \end{aligned}$$

with unknown constants B_1, B_2, B_3, and B_4. Since the necessary value matching conditions and the smooth pasting conditions are

$$F_1^{(2,1)}(x_1) = 0, \quad F_1^{(2,2)}(x_2) = F_1^{(2,1)}(x_2), \quad \frac{d}{dx}F_1^{(2,2)}(x_2) = \frac{d}{dx}F_1^{(2,1)}(x_2), \quad F_2^{(2,2)}(x_2) = x_2,$$

we have a system of equations of these unknown constants

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -\lambda^{\beta_1^{(1)}} & -\lambda^{\beta_2^{(1)}} & g_2\left(\beta_1^{(2)}\right) & g_2\left(\beta_2^{(2)}\right) \\ -\beta_1^{(1)}\lambda^{\beta_1^{(1)}} & -\beta_2^{(1)}\lambda^{\beta_2^{(1)}} & \beta_1^{(2)}g_2\left(\beta_1^{(2)}\right) & \beta_2^{(2)}g_2\left(\beta_2^{(2)}\right) \\ 0 & 0 & -q_2 & -q_2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \begin{pmatrix} -p_1x_1 \\ p_1x_2 \\ p_1x_2 \\ x_2 \end{pmatrix}.$$

From the first row and the fourth row, we see $B_2 = -p_1x_1 - B_1$, $B_4 = -\frac{x}{q_2} - B_3$. Then, the reduced equations are

$$\begin{aligned} \mathbf{W}\begin{pmatrix} B_1 \\ B_3 \end{pmatrix} &= \begin{pmatrix} -p_1x_1\lambda^{\beta_2^{(1)}} + p_1x_2 - l_2x_2 \\ p_1x_2 - p_1x_1\beta_2^{(1)}\lambda^{\beta_2^{(1)}} - l_2x_2\beta_2^{(2)} \end{pmatrix} \\ &= -p_1x_1\lambda^{\beta_2^{(1)}}\begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} + p_1x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - l_2x_2\begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix}. \end{aligned}$$

Therefore, the coefficients are solvable as

$$\begin{aligned} \begin{pmatrix} B_1 \\ B_3 \end{pmatrix} &= \mathbf{W}^{-1}\left[-p_1x_1\lambda^{\beta_2^{(1)}}\begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} + p_1x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - l_2x_2\begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix}\right] \\ &= -\frac{p_1x_1\lambda^{\beta_2^{(1)}}}{\det\mathbf{W}}\begin{pmatrix} d_1 \\ (\beta_1^{(1)} - \beta_2^{(1)})\lambda^{\beta_1^{(1)}} \end{pmatrix} + \frac{p_1x_2}{\det\mathbf{W}}\begin{pmatrix} -q_2l_1\left(\beta_1^{(2)} - 1\right) + q_2l_2\left(\beta_2^{(2)} - 1\right) \\ \left(\beta_1^{(1)} - 1\right)\lambda^{\beta_1^{(1)}} - \left(\beta_2^{(1)} - 1\right)\lambda^{\beta_2^{(1)}} \end{pmatrix} \\ &\quad -\frac{l_2x_2}{\det\mathbf{W}}\begin{pmatrix} -q_2l_1\left(\beta_1^{(2)} - \beta_2^{(2)}\right) \\ \left(\beta_1^{(1)} - \beta_2^{(2)}\right)\lambda^{\beta_1^{(1)}} - \left(\beta_2^{(1)} - \beta_2^{(2)}\right)\lambda^{\beta_2^{(1)}} \end{pmatrix} \end{aligned}$$

and

$$B_2 = -p_1x_1 - B_1,\ B_4 = -\frac{x}{q_2} - B_3.$$

They are summarized as

$$\begin{aligned} B_1 &= -p_1x_1\frac{d_1\lambda^{-\beta_1^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} + q_2x_2\frac{l_1l_2\left(\beta_1^{(2)} - \beta_2^{(2)}\right) - p_1\left(l_1\left(\beta_1^{(2)} - 1\right) - l_2\left(\beta_2^{(2)} - 1\right)\right)}{d_1\lambda^{\beta_2^{(1)}} + d_2\lambda^{\beta_1^{(1)}}}, \\ B_2 &= p_1x_1\frac{d_2\lambda^{-\beta_2^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} - q_2x_2\frac{l_1(l_2 - p_1)\beta_1^{(2)} - l_2(l_1 - p_1)\beta_2^{(2)} + p_1(l_1 - l_2)}{d_1\lambda^{\beta_2^{(1)}} + d_2\lambda^{\beta_1^{(1)}}}, \\ B_3 &= \frac{-p_1x_1\left(\beta_1^{(1)} - \beta_2^{(1)}\right) + l_2x_2\left(\left(\beta_2^{(1)} - \beta_2^{(2)}\right)\lambda^{-\beta_1^{(1)}} - \left(\beta_1^{(1)} - \beta_2^{(2)}\right)\lambda^{-\beta_2^{(1)}}\right)}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} \\ &\quad -p_1x_1\frac{\left(\beta_2^{(1)} - 1\right)\lambda^{-\beta_1^{(1)}} - \left(\beta_1^{(1)} - 1\right)\lambda^{-\beta_2^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}}, \\ B_4 &= \frac{p_1x_1\left(\beta_1^{(1)} - \beta_2^{(1)}\right) - l_2x_2\left(\left(\beta_2^{(1)} - \beta_2^{(2)}\right)\lambda^{-\beta_1^{(1)}} - \left(\beta_1^{(1)} - \beta_2^{(2)}\right)\lambda^{-\beta_2^{(1)}}\right)}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} \\ &\quad +p_1x_1\frac{\left(\beta_2^{(1)} - 1\right)\lambda^{-\beta_1^{(1)}} - \left(\beta_1^{(1)} - 1\right)\lambda^{-\beta_2^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}} - \frac{x_2}{q_2}. \end{aligned}$$

The functions are expressed as follows,

$$
\begin{aligned}
F_1^{(1,0)}(x) &= x, \quad F_1^{(1,1)}(x) = x_1 \frac{d_1\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}}, \\
F_1^{(1,2)}(x) &= x_1 \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}} \left[g_2\left(\beta_1^{(2)}\right)\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} - g_2\left(\beta_2^{(2)}\right)\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}} \right], \\
F_2^{(1,0)}(x) &= 0, \quad F_2^{(1,1)}(x) = 0, \\
F_2^{(1,2)}(x) &= x_1 \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}} \left[-q_2\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + q_2\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}} \right], \\
F_1^{(2,0)}(x) &= 0, \\
F_1^{(2,1)}(x) &= q_2 x_2 \frac{l_1 l_2\left(\beta_1^{(2)} - \beta_2^{(2)}\right) - p_1\left(l_1\left(\beta_1^{(2)} - 1\right) - l_2\left(\beta_2^{(2)} - 1\right)\right)}{d_1\left(\frac{x_2}{x_1}\right)^{\beta_2^{(1)}} + d_2\left(\frac{x_2}{x_1}\right)^{\beta_1^{(1)}}} \left[\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} - \left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} \right] \\
&\quad + p_1 x + p_1 x_1 \frac{-d_1\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}}, \\
F_1^{(2,2)}(x) &= q_2 B_3 \left[l_1\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - l_2\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}} \right] + l_2\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}, \\
F_2^{(2,0)}(x) &= x, \quad F_2^{(2,1)}(x) = x, \\
F_2^{(2,2)}(x) &= -q_2 B_3 \left[\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - \left(\frac{x}{x_2}\right)^{\beta_2^{(2)}} \right] + x_2\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}.
\end{aligned}
$$

A.8.3 case of $k = 1$ on G_i

The same discussion can be applied as in A.8.1. Since the relevant equations are

$$
G_1^{(1,0)}(x) = 1, \quad G_2^{(1,0)}(x) = 0 \qquad \mathcal{A}_1 G_1^{(1,1)}(x) = 0, \quad G_2^{(1,1)}(x) = 0, \quad \begin{pmatrix} \mathcal{A}_1 & q_{12} \\ q_{21} & \mathcal{A}_2 \end{pmatrix} \begin{pmatrix} G_1^{(1,2)}(x) \\ G_2^{(1,2)}(x) \end{pmatrix} = \mathbf{0}_2,
$$

the functions must be in a form of

$$
\begin{aligned}
G_1^{(1,1)}(x) &= a_1\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + a_2\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}}, \\
G_1^{(1,2)}(x) &= g_2\left(\beta_1^{(2)}\right) a_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + g_2\left(\beta_2^{(2)}\right) a_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}, \\
G_2^{(1,2)}(x) &= -q_2 a_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - q_2 a_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}.
\end{aligned}
$$

Since the necessary value matching conditions and the smooth pasting conditions are

$$
G_1^{(1,1)}(x_1) = 1, \quad G_1^{(1,2)}(x_2) = G_1^{(1,1)}(x_2), \quad \frac{d}{dx}G_1^{(1,2)}(x_2) = \frac{d}{dx}G_1^{(1,1)}(x_2), \quad G_2^{(1,2)}(x_2) = 0,
$$

we have a system of equations of these unknown constants

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -\lambda^{\beta_1^{(1)}} & -\lambda^{\beta_2^{(1)}} & g_2\left(\beta_1^{(2)}\right) & g_2\left(\beta_2^{(2)}\right) \\ -\beta_1^{(1)}\lambda^{\beta_1^{(1)}} & -\beta_2^{(1)}\lambda^{\beta_2^{(1)}} & \beta_1^{(2)}g_2\left(\beta_1^{(2)}\right) & \beta_2^{(2)}g_2\left(\beta_2^{(2)}\right) \\ 0 & 0 & -q_2 & -q_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From the first row and the fourth row, we see $a_2 = 1 - a_1$, $a_4 = -a_3$. Then, the reduced equations are

$$\mathbf{W}\begin{pmatrix} a_1 \\ a_3 \end{pmatrix} = \lambda^{\beta_2^{(1)}}\begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix}.$$

Therefore, the coefficients are solvable as

$$\begin{pmatrix} a_1 \\ a_3 \end{pmatrix} = \lambda^{\beta_2^{(1)}}\mathbf{W}^{-1}\begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} = \frac{x_1\lambda^{\beta_2^{(1)}}}{\det\mathbf{W}}\begin{pmatrix} d_1 \\ (\beta_1^{(1)} - \beta_2^{(1)})\lambda^{\beta_1^{(1)}} \end{pmatrix},$$

and $a_2 = 1 - a_1$, $\quad a_4 = -a_3$. They are summarized as

$$a_1 = \frac{d_1\lambda^{-\beta_1^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}}, \quad a_2 = \frac{d_2\lambda^{-\beta_2^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}},$$
$$a_3 = \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}}, \quad a_4 = -\frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\lambda^{-\beta_1^{(1)}} + d_2\lambda^{-\beta_2^{(1)}}}.$$

Note that $A_i = x_1 a_i$ $(i = 1, 2, 3, 4)$.

A.8.4 case of $k = 2$ on G_i

The same discussion can be applied as in A.8.2. Since the relevant equations are

$$G_1^{(2,0)}(x) = 0, \quad G_2^{(2,0)}(x) = 1 \qquad \mathcal{A}_1 G_1^{(2,1)}(x) = -q_{12}, \quad G_2^{(2,1)}(x) = 1, \quad \begin{pmatrix} \mathcal{A}_1 & q_{12} \\ q_{21} & \mathcal{A}_2 \end{pmatrix}\begin{pmatrix} G_1^{(2,2)}(x) \\ G_2^{(2,2)}(x) \end{pmatrix} = \mathbf{0}_2,$$

the functions must be in a form of

$$\begin{aligned} G_1^{(2,1)}(x) &= b_1\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + b_2\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + p_2, \\ G_1^{(2,2)}(x) &= g_2\left(\beta_1^{(2)}\right)b_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + g_2\left(\beta_2^{(2)}\right)b_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}, \\ G_2^{(2,2)}(x) &= -q_2 b_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - q_2 b_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}. \end{aligned}$$

Since the necessary value matching conditions and the smooth pasting conditions are

$$G_1^{(2,1)}(x_1) = 0, \quad G_1^{(2,2)}(x_2) = G_1^{(2,1)}(x_2), \quad \frac{d}{dx}G_1^{(2,2)}(x_2) = \frac{d}{dx}G_1^{(2,1)}(x_2), \quad G_2^{(2,2)}(x_2) = 1,$$

we have a system of equations of these unknown constants

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -\lambda^{\beta_1^{(1)}} & -\lambda^{\beta_2^{(1)}} & g_2\left(\beta_1^{(2)}\right) & g_2\left(\beta_2^{(2)}\right) \\ -\beta_1^{(1)}\lambda^{\beta_1^{(1)}} & -\beta_2^{(1)}\lambda^{\beta_2^{(1)}} & \beta_1^{(2)}g_2\left(\beta_1^{(2)}\right) & \beta_2^{(2)}g_2\left(\beta_2^{(2)}\right) \\ 0 & 0 & -q_2 & -q_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} -p_2 \\ p_2 \\ 0 \\ 1 \end{pmatrix}.$$

From the first row and the fourth row, we see $b_2 = -p_2 - b_1$, $b_4 = -\frac{1}{q_2} - b_3$. Then, the reduced equations are

$$\mathbf{W}\begin{pmatrix} b_1 \\ b_3 \end{pmatrix} = \begin{pmatrix} -p_1 x_1 \lambda^{\beta_2^{(1)}} + p_1 x_2 - l_2 x_2 \\ p_1 x_2 - p_1 x_1 \beta_2^{(1)} \lambda^{\beta_2^{(1)}} - l_2 x_2 \beta_2^{(2)} \end{pmatrix} = -p_2 \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - l_2 \begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix}.$$

Therefore, the coefficients are solvable as

$$\begin{aligned}
\begin{pmatrix} b_1 \\ b_3 \end{pmatrix} &= \mathbf{W}^{-1}\left[-p_2 \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - l_2 \begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix} \right] \\
&= -\frac{p_2 \lambda^{\beta_2^{(1)}}}{\det \mathbf{W}} \begin{pmatrix} d_1 \\ (\beta_1^{(1)} - \beta_2^{(1)}) \lambda^{\beta_1^{(1)}} \end{pmatrix} + \frac{p_2}{\det \mathbf{W}} \begin{pmatrix} -q_2 l_1 \beta_1^{(2)} + q_2 l_2 \beta_2^{(2)} \\ \left(\beta_1^{(1)} - 1\right) \lambda^{\beta_1^{(1)}} - \left(\beta_2^{(1)} - 1\right) \lambda^{\beta_2^{(1)}} \end{pmatrix} \\
&\quad - \frac{l_2}{\det \mathbf{W}} \begin{pmatrix} -q_2 l_1 \left(\beta_1^{(2)} - \beta_2^{(2)}\right) \\ \left(\beta_1^{(1)} - \beta_2^{(2)}\right) \lambda^{\beta_1^{(1)}} - \left(\beta_2^{(1)} - \beta_2^{(2)}\right) \lambda^{\beta_2^{(1)}} \end{pmatrix}
\end{aligned}$$

and

$$b_2 = -p_2 - b_1, \; b_4 = -\frac{1}{q_2} - b_3.$$

They are summarized as

$$\begin{aligned}
b_1 &= -p_2 \frac{d_1 \lambda^{-\beta_1^{(1)}}}{d_1 \lambda^{-\beta_1^{(1)}} + d_2 \lambda^{-\beta_2^{(1)}}} + q_2 \frac{l_1 l_2 \left(\beta_1^{(2)} - \beta_2^{(2)}\right) - p_2 \left(l_1 \beta_1^{(2)} - l_2 \beta_2^{(2)}\right)}{d_1 \lambda^{\beta_2^{(1)}} + d_2 \lambda^{\beta_1^{(1)}}}, \\
b_2 &= p_2 \frac{d_2 \lambda^{-\beta_2^{(1)}}}{d_1 \lambda^{-\beta_1^{(1)}} + d_2 \lambda^{-\beta_2^{(1)}}} - q_2 \frac{l_1 (l_2 - p_2) \beta_1^{(2)} - l_2 (l_1 - p_2) \beta_2^{(2)}}{d_1 \lambda^{\beta_2^{(1)}} + d_2 \lambda^{\beta_1^{(1)}}}, \\
b_3 &= \frac{-p_2 \left(\beta_1^{(1)} - \beta_2^{(1)}\right) + l_2 \left(\left(\beta_2^{(1)} - \beta_2^{(2)}\right) \lambda^{-\beta_1^{(1)}} - \left(\beta_1^{(1)} - \beta_2^{(2)}\right) \lambda^{-\beta_2^{(1)}} \right)}{d_1 \lambda^{-\beta_1^{(1)}} + d_2 \lambda^{-\beta_2^{(1)}}} \\
&\quad -p_2 \frac{\beta_2^{(1)} \lambda^{-\beta_1^{(1)}} - \beta_1^{(1)} \lambda^{-\beta_2^{(1)}}}{d_1 \lambda^{-\beta_1^{(1)}} + d_2 \lambda^{-\beta_2^{(1)}}}, \\
b_4 &= \frac{p_2 \left(\beta_1^{(1)} - \beta_2^{(1)}\right) - l_2 \left(\left(\beta_2^{(1)} - \beta_2^{(2)}\right) \lambda^{-\beta_1^{(1)}} - \left(\beta_1^{(1)} - \beta_2^{(2)}\right) \lambda^{-\beta_2^{(1)}} \right)}{d_1 \lambda^{-\beta_1^{(1)}} + d_2 \lambda^{-\beta_2^{(1)}}} \\
&\quad +p_2 \frac{\beta_2^{(1)} \lambda^{-\beta_1^{(1)}} - \beta_1^{(1)} \lambda^{-\beta_2^{(1)}}}{d_1 \lambda^{-\beta_1^{(1)}} + d_2 \lambda^{-\beta_2^{(1)}}} - \frac{1}{q_2}.
\end{aligned}$$

The relevant functions are expressed as follows,

$$\begin{aligned}
G_1^{(1,0)}(x) &= 1, \quad G_1^{(1,1)}(x) = \frac{d_1 \left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + d_2 \left(\frac{x}{x_2}\right)^{\beta_2^{(1)}}}{d_1 \left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2 \left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}}, \\
G_1^{(1,2)}(x) &= \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1 \left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2 \left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}} \left[g_2 \left(\beta_1^{(2)}\right) \left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} - g_2 \left(\beta_2^{(2)}\right) \left(\frac{x}{x_2}\right)^{\beta_2^{(1)}} \right], \\
G_2^{(1,0)}(x) &= 0, \quad G_2^{(1,1)}(x) = 0, \\
G_2^{(1,2)}(x) &= \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1 \left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2 \left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}} \left[-q_2 \left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + q_2 \left(\frac{x}{x_2}\right)^{\beta_2^{(1)}} \right],
\end{aligned}$$

$$
\begin{aligned}
G_1^{(2,0)}(x) &= 0,\\
G_1^{(2,1)}(x) &= p_2 \frac{-d_1\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}}\\
&\quad + q_2 \frac{l_1 l_2\left(\beta_1^{(2)} - \beta_2^{(2)}\right) - p_2\left(l_1\beta_1^{(2)} - l_2\beta_2^{(2)}\right)}{d_1\left(\frac{x_2}{x_1}\right)^{\beta_2^{(1)}} + d_2\left(\frac{x_2}{x_1}\right)^{\beta_1^{(1)}}}\left[\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} - \left(\frac{x}{x_1}\right)^{\beta_2^{(1)}}\right] + p_2,\\
G_1^{(2,2)}(x) &= q_2 b_3\left[l_1\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - l_2\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}\right] + l_2\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}},\\
G_2^{(2,0)}(x) &= 1, \quad G_2^{(2,1)}(x) = 1,\\
G_2^{(2,2)}(x) &= -q_2 b_3\left[\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - \left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}\right] + \left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}.
\end{aligned}
$$

A.9 Proof of Lemma 6

By (23) and (27), it holds that

$$
\begin{aligned}
K_1\begin{pmatrix}\gamma_{11}(x_1)\\ \gamma_{12}(x_1)\end{pmatrix} &= \begin{pmatrix}1 & 1\\ \beta_1^{(1)} & \beta_2^{(1)}\end{pmatrix}^{-1}\begin{pmatrix}\alpha_1 D_1 x_1 - K_1 - p_1\alpha_2 D_2 x_1 + p_2 K_2\\ \alpha_1 D_1 x_1 - p_1\alpha_2 D_2 x_1\end{pmatrix},\\
K_2\begin{pmatrix}\gamma_{21}(x_2)\\ \gamma_{22}(x_2)\end{pmatrix} &= \begin{pmatrix}1 & 1\\ \beta_1^{(2)} & \beta_2^{(2)}\end{pmatrix}^{-1}\begin{pmatrix}\alpha_2 D_2 - K_2\\ \alpha_2 D_2 x_2\end{pmatrix}.
\end{aligned}
$$

Thus, (28) is equivalent to

$$
\begin{aligned}
&\begin{pmatrix}1 & 1\\ \beta_1^{(1)} & \beta_2^{(1)}\end{pmatrix}\begin{pmatrix}\lambda^{\beta_1^{(1)}} 0 & \\ 0 & \lambda^{\beta_2^{(1)}}\end{pmatrix}\begin{pmatrix}1 & 1\\ \beta_1^{(1)} & \beta_2^{(1)}\end{pmatrix}^{-1}\begin{pmatrix}\alpha_1 D_1 x_1 - K_1 - p_1\alpha_2 D_2 x_1 + p_2 K_2\\ \alpha_1 D_1 x_1 - p_1\alpha_2 D_2 x_1\end{pmatrix}\\
= &\begin{pmatrix}l_1 & l_2\\ l_1\beta_1^{(2)} & l_2\beta_2^{(2)}\end{pmatrix}\begin{pmatrix}1 & 1\\ \beta_1^{(2)} & \beta_2^{(2)}\end{pmatrix}^{-1}\begin{pmatrix}\alpha_2 D_2 x_2 - K_2\\ \alpha_2 D_2 x_2\end{pmatrix} - \begin{pmatrix}p_1\alpha_2 D_2 x_2 - p_2 K_2\\ p_1\alpha_2 D_2 x_2\end{pmatrix}.
\end{aligned}
$$

Each side can be transformed as follows.

$$
\begin{aligned}
\text{LHS} &= \frac{1}{\beta_1^{(1)} - \beta_2^{(1)}}\begin{pmatrix}-\beta_2^{(1)}\lambda^{\beta_1^{(1)}} + \beta_1^{(1)}\lambda^{\beta_2^{(1)}} & \lambda^{\beta_1^{(1)}} - \lambda^{\beta_2^{(1)}}\\ \beta_1^{(1)}\beta_2^{(1)}\left(-\lambda^{\beta_1^{(1)}} + \lambda^{\beta_2^{(1)}}\right) & \beta_1^{(1)}\lambda^{\beta_1^{(1)}} - \beta_2^{(1)}\lambda^{\beta_2^{(1)}}\end{pmatrix}\\
&\quad \times\left[\left(\alpha_1 D_1 - p_1\alpha_2 D_2\right)x_1\begin{pmatrix}1\\1\end{pmatrix} - (K_1 - p_2 K_2)\begin{pmatrix}1\\0\end{pmatrix}\right]\\
&= \frac{\left(1-\beta_2^{(1)}\right)\left(\alpha_1 D_1 - p_1\alpha_2 D_2\right)x_1 + \beta_2^{(1)}(K_1 - p_2 K_2)}{\beta_1^{(1)} - \beta_2^{(1)}}\lambda^{\beta_1^{(1)}}\begin{pmatrix}1\\ \beta_1^{(1)}\end{pmatrix}\\
&\quad - \frac{\left(1-\beta_1^{(1)}\right)\left(\alpha_1 D_1 - p_1\alpha_2 D_2\right)x_1 + \beta_1^{(1)}(K_1 - p_2 K_2)}{\beta_1^{(1)} - \beta_2^{(1)}}\lambda^{\beta_2^{(1)}}\begin{pmatrix}1\\ \beta_2^{(1)}\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\text{RHS} &= \frac{1}{\beta_1^{(2)}-\beta_2^{(2)}}\begin{pmatrix} l_1 & l_2 \\ l_1\beta_1^{(2)} & l_2\beta_2^{(2)}\end{pmatrix}\begin{pmatrix}-\beta_2^{(2)} & 1 \\ \beta_1^{(2)} & -1\end{pmatrix}\begin{pmatrix}\alpha_2 D_2 x_2 - K_2 \\ \alpha_2 D_2 x_2\end{pmatrix} - \begin{pmatrix} p_1\alpha_2 D_2 x_2 - p_2 K_2 \\ p_1\alpha_2 D_2 x_2\end{pmatrix} \\
&= l_1 \frac{\left(1-\beta_2^{(2)}\right)\alpha_2 D_2 x_2 + \beta_2^{(2)} K_2}{\beta_1^{(2)}-\beta_2^{(2)}}\begin{pmatrix}1 \\ \beta_1^{(2)}\end{pmatrix} - l_2 \frac{\left(1-\beta_1^{(2)}\right)\alpha_2 D_2 x_2 + \beta_1^{(2)} K_2}{\beta_1^{(2)}-\beta_2^{(2)}}\begin{pmatrix}1 \\ \beta_2^{(2)}\end{pmatrix} \\
&\quad - \begin{pmatrix} p_1\alpha_2 D_2 x_2 - p_2 K_2 \\ p_1\alpha_2 D_2 x_2\end{pmatrix}.
\end{aligned}
$$

A.10 Proof of Proposition 7

It is straightforward to verify

$$
V_1^{(0)}(x) = W_1^{(0)}(x), \quad V_2^{(0)}(x) = W_2^{(0)}(x), \quad V_2^{(1)}(x) = W_2^{(1)}(x)
$$

since they are in a form of $\alpha_i D_i x - K_i$. We need to check terms appearing in $W_1^{(1)}(x)$, $W_1^{(2)}(x)$ and $W_2^{(2)}(x)$,

$$
\begin{aligned}
W_1^{(1)}(x) &= \alpha_1 D_1 F_1^{(1,1)}(x) - K_1 G_1^{(1,1)}(x) + \alpha_2 D_2 F_1^{(2,1)}(x) - K_2 G_1^{(2,1)}(x) \\
&= (\alpha_1 D_1 x_1 - K_1)\left(a_1\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + a_2\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}}\right) \\
&\quad + \alpha_2 D_2\left(B_1\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + B_2\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + p_1 x\right) - K_2\left(b_1\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + b_2\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + p_2\right), \\
W_1^{(2)}(x) &= \alpha_1 D_1 F_1^{(1,2)}(x) - K_1 G_1^{(1,2)}(x) + \alpha_2 D_2 F_1^{(2,2)}(x) - K_2 G_1^{(2,2)}(x) \\
&= -q_2(\alpha_1 D_1 x_1 - K_1)\left(l_1 a_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + l_2 a_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}\right) \\
&\quad - q_2\alpha_2 D_2\left(l_1 B_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + l_2 B_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}\right) + q_2 K_2\left(l_1 b_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + l_2 b_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}\right),
\end{aligned}
$$

$$
\begin{aligned}
W_2^{(2)}(x) &= \alpha_1 D_1 F_2^{(1,2)}(x) - K_1 G_2^{(1,2)}(x) + \alpha_2 D_2 F_2^{(2,2)}(x) - K_2 G_2^{(2,2)}(x) \\
&= -q_2(\alpha_1 D_1 x_1 - K_1)\left(a_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + a_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}\right) \\
&\quad - q_2\alpha_2 D_2\left(B_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + B_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}\right) + q_2 K_2\left(b_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + b_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}\right).
\end{aligned}
$$

By comparing with the results in Proposition 3, it is sufficient to show that

$$
\begin{aligned}
K_1\begin{pmatrix}\gamma_{11}(x_1) \\ \gamma_{12}(x_1)\end{pmatrix} &= (\alpha_1 D_1 x_1 - K_1)\begin{pmatrix}a_1 \\ a_2\end{pmatrix} + \alpha_2 D_2\begin{pmatrix}B_1 \\ B_2\end{pmatrix} - K_2\begin{pmatrix}b_1 \\ b_2\end{pmatrix}, && (53) \\
-\frac{K_2}{q_2}\begin{pmatrix}\gamma_{21}(x_2) \\ \gamma_{22}(x_2)\end{pmatrix} &= (\alpha_1 D_1 x_1 - K_1)\begin{pmatrix}a_3 \\ a_4\end{pmatrix} + \alpha_2 D_2\begin{pmatrix}B_3 \\ B_4\end{pmatrix} - K_2\begin{pmatrix}b_3 \\ b_4\end{pmatrix} && (54)
\end{aligned}
$$

to prove $V_i^{(n)}(x) = W_i^{(n)}(x)$. By (23) and (27), the LHS are

$$
\begin{aligned}
K_1 \begin{pmatrix} \gamma_{11}(x_1) \\ \gamma_{12}(x_1) \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ \beta_1^{(1)} & \beta_2^{(1)} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 D_1 x_1 - K_1 - p_1 \alpha_2 D_2 x_1 + p_2 K_2 \\ \alpha_1 D_1 x_1 - p_1 \alpha_2 D_2 x_1 \end{pmatrix}, \\
-\frac{K_2}{q_2} \begin{pmatrix} \gamma_{21}(x_2) \\ \gamma_{22}(x_2) \end{pmatrix} &= -\frac{1}{q_2} \begin{pmatrix} 1 & 1 \\ \beta_1^{(2)} & \beta_2^{(2)} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_2 D_2 - K_2 \\ \alpha_2 D_2 x_2 \end{pmatrix}.
\end{aligned}
$$

In the proof of Proposition 6. we know that

$$
\begin{aligned}
&\mathbf{W} \begin{pmatrix} a_1 \\ a_3 \end{pmatrix} = \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix}, \quad \mathbf{W} \begin{pmatrix} B_1 \\ B_3 \end{pmatrix} = -p_1 x_1 \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} + p_1 x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - l_2 x_2 \begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix}, \\
&\mathbf{W} \begin{pmatrix} b_1 \\ b_3 \end{pmatrix} = -p_2 \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - l_2 \begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix}.
\end{aligned}
$$

Thus it follows that

$$
\begin{aligned}
& \mathbf{W} \left[(\alpha_1 D_1 x_1 - K_1) \begin{pmatrix} a_1 \\ a_3 \end{pmatrix} + \alpha_2 D_2 \begin{pmatrix} B_1 \\ B_3 \end{pmatrix} - K_2 \begin{pmatrix} b_1 \\ b_3 \end{pmatrix} \right] \\
= \; & (\alpha_1 D_1 x_1 - K_1) \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} + \alpha_2 D_2 \left[-p_1 x_1 \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} + p_1 x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - l_2 x_2 \begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix} \right] \\
& -K_2 \left[-p_2 \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - l_2 \begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix} \right] \\
= \; & ((\alpha_1 D_1 - p_1 \alpha_2 D_2) x_1 - K_1 + p_2 K_2) \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} \\
& + p_1 \alpha_2 D_2 x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - p_2 K_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - l_2 (\alpha_2 D_2 x_2 - K_2) \begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix},
\end{aligned}
$$

which is further rewritten by Lemma 6 as

$$
\begin{aligned}
= \; & -\frac{\left(1 - \beta_2^{(1)}\right)(\alpha_1 D_1 - p_1 \alpha_2 D_2) x_1 + \beta_2^{(1)} (K_1 - p_2 K_2)}{\beta_1^{(1)} - \beta_2^{(1)}} \lambda^{\beta_1^{(1)}} \begin{pmatrix} 1 \\ \beta_1^{(1)} \end{pmatrix} \\
& + \frac{\left(1 - \beta_2^{(1)}\right)(\alpha_1 D_1 - p_1 \alpha_2 D_2) x_1 + \beta_2^{(1)} (K_1 - p_2 K_2)}{\beta_1^{(1)} - \beta_2^{(1)}} \lambda^{\beta_2^{(1)}} \begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} \\
& + l_1 \frac{\left(1 - \beta_2^{(2)}\right) \alpha_2 D_2 x_2 + \beta_2^{(2)} K_2}{\beta_1^{(2)} - \beta_2^{(2)}} \begin{pmatrix} 1 \\ \beta_1^{(2)} \end{pmatrix} + l_2 \frac{\left(1 - \beta_2^{(2)}\right) \alpha_2 D_2 x_2 + \beta_2^{(2)} K_2}{\beta_1^{(2)} - \beta_2^{(2)}} \begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix} \\
= \; & \begin{pmatrix} -\lambda^{\beta_1^{(1)}} + \lambda^{\beta_2^{(1)}} & -q_2 (l_1 - l_2) \\ -\beta_1^{(1)} \lambda^{\beta_1^{(1)}} + \beta_2^{(1)} \lambda^{\beta_2^{(1)}} & -q_2 \left(l_1 \beta_1^{(2)} - l_2 \beta_2^{(2)} \right) \end{pmatrix} \begin{pmatrix} \dfrac{\left(1 - \beta_2^{(1)}\right)(\alpha_1 D_1 - p_1 \alpha_2 D_2) x_1 + \beta_2^{(1)} (K_1 - p_2 K_2)}{\beta_1^{(1)} - \beta_2^{(1)}} \\ \dfrac{\left(1 - \beta_2^{(2)}\right) \alpha_2 D_2 x_2 + \beta_2^{(2)} K_2}{-q_2 \left(\beta_1^{(2)} - \beta_2^{(2)}\right)} \end{pmatrix} \\
= \; & \mathbf{W} \begin{pmatrix} K_1 \gamma_{11}(x_1) \\ -\dfrac{K_2}{q_2} \gamma_{21}(x_1) \end{pmatrix}.
\end{aligned}
$$

Similarly, one can show that

$$
\mathbf{W} \left[(\alpha_1 D_1 x_1 - K_1) \begin{pmatrix} a_2 \\ a_4 \end{pmatrix} + \alpha_2 D_2 \begin{pmatrix} B_2 \\ B_4 \end{pmatrix} - K_2 \begin{pmatrix} b_2 \\ b_4 \end{pmatrix} \right] = \mathbf{W} \begin{pmatrix} K_1 \gamma_{12}(x_1) \\ -\dfrac{K_2}{q_2} \gamma_{22}(x_1) \end{pmatrix}.
$$

Since **W** is regular, (53) and (54) are proved as desired.

We consider the smooth pasting conditions. By (53) we see that

$$\begin{aligned}
x\frac{d}{dx}W_1^{(1)}(x) &= (\alpha_1 D_1 x_1 - K_1)\left(a_1\beta_1^{(1)}\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + a_2\beta_2^{(1)}\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}}\right) \\
&\quad +\alpha_2 D_2\left(B_1\beta_1^{(1)}\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + B_2\beta_2^{(1)}\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + p_1 x\right) \\
&\quad -K_2\left(b_1\beta_1^{(1)}\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + b_2\beta_2^{(1)}\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}}\right) \\
&= \beta_1^{(1)}\gamma_{11}(x_1)\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + \beta_2^{(1)}\gamma_{12}(x_1)\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + p_1\alpha_2 D_2 x \\
&= x\frac{d}{dx}V_1^{(1)}(x).
\end{aligned}$$

One can show the same results for other $W_i^{(n)}$ in a parallel way.