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A CHAOS EXPANSION APPROACH FOR THE PRICING OF CONTINGENT CLAIMS

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ABSTRACT. In this paper, we propose an approximation method based on the Wiener-Ito chaos expansion for the pricing of European-style contingent claims. Our method is applicable to the general class of continuous Markov processes. The resulting approximation formula requires at most three-dimensional numerical integration. It will be shown through numerical examples that, unlike existing approximation methods, the accuracy of our approximation remains quite high even for the case of high volatility and long maturity.

Keywords: Wiener-Ito chaos expansion, Hermite polynomial, Black-Scholes model, successive substitution, diffusion process.

1. INTRODUCTION

Markov processes are used in a wide range of economics and finance to model underlying uncertainties, due to their expressive richness and analytical tractability. In the option pricing theory, underlying asset prices are often assumed to follow diffusion processes. However, analytical solutions for European options written on the asset prices are, in most cases, hard to obtain. On the other hand, efficient methods for calculating European options are required in practice for any model, since European options are usually the only options that are liquid enough to be used for the model calibration. Motivated by this gap, we study an approximation method based on the Wiener-Ito chaos expansion for the pricing of financial contingent claims.

A vast number of articles have addressed the valuation problem of contingent claims. As the models become more realistic, it becomes harder to obtain pricing formulas in closed form. As a result, numerical methods for solving partial differential equations or Monte Carlo methods are required to resolve the shortcoming. Unfortunately, however, these methods are in general computationally too extensive to be used in practice, because the entire optimization procedure is extremely time-consuming. Therefore, closed-form approximation formulas could be the only feasible solution for practitioners.

Fouque et al. (2000) apply the singular perturbation technique to the option pricing under a stochastic volatility model. They asymptotically expand the partial differential equation (PDE for short) derived from a stochastic differential equation (SDE for short) of the underlying asset price around the invariant distribution of volatility process. They also consider a fast mean-reverting stochastic volatility model and succeed to capture the short timescale volatility, a well-known phenomenon in practice (see Fouque et al., 2003a). The theoretical validity of the singular perturbation method is argued in Fouque et al. (2003b). Hagan et al. (2002) use the singular perturbation technique to obtain the prices of European options under the SABR model. By using matched asymptotic expansions, de Jong (2010) solves singularly perturbed problems and obtains approximated solutions in closed form for European options under various models.

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Kunitomo and Takahashi (1992) first develop the asymptotic theory, called the small disturbance asymptotic, for solving the valuation problem of average options when the underlying asset price follows a geometric Brownian motion. Yoshida (1992a) uses the results derived by Watanabe (1987) to obtain some useful results on the validity of the asymptotic expansions of some functional on time-homogeneous diffusion processes. Moreover, his result confirms that this approximation converges to the exact value when the volatility of the underlying asset goes to zero. Takahashi (1999) proposes approximated formulas in closed form based on the asymptotic expansion method for European options when the underlying asset follows a diffusion. Other important developments within this approach include Kunitomo and Takahashi (2001, 2003) and Takahashi and Takehara (2007). These articles use the small volatility asymptotic expansion based on the infinite dimensional analysis, called the Watanabe-Yoshida theory, on the Malliavin calculus.

Approximating the transition density or likelihood expansion for a diffusion process is other successful examples. Aït-Sahalia (2002, 2008) proposes a method to approximate the transition probability by means of a Hermite polynomial expansion and derives closed-form approximations for the log-likelihood functions of univariate and multivariate diffusions. Xiu (2011) has applied a similar strategy to derive a series approximation in closed form of European option prices under a variety of diffusion models by using either the Hermite polynomial approach or the undetermined coefficient method.

Recently, many of these approximate expansions have become popular in practice. However, these approximations usually get worse in the case of high volatility and long maturity, which causes a serious problem in practice, because derivatives with long maturities (beyond 10 years) become common in the OTC markets, and options for those maturities frequently exhibit a significant volatility smiles and skews. Handling these market skews and smiles are essential to equity, fixed income and foreign exchange desks, because they have large exposures for a wide range of strikes and maturities. In order to satisfy these trends, more accurate approximation schemes are inevitable.

The Edgeworth and/or Gram-Charlier expansions work quite effectively when the cumulants (or moments) of underlying process can be computed. The main idea of these expansions is to derive the characteristic function of the distribution whose probability density function (PDF for short) is approximated by the characteristic function of a known distribution (normal distribution in most cases), and to recover the PDF through the inverse Fourier transform. In the case of finite series, these expansions give the same result; however, since the arrangement of terms differs, there could be a difference between the accuracy of truncated expansions. Based on an Edgeworth expansion technique, Collin-Dufresne and Goldstein (2002) derive the probability distribution of a coupon bond's future price and propose an algorithm for the pricing of swaptions when the underlying term-structure dynamics are affine. By applying the Gram-Charlier expansion, Tanaka et al. (2010) provide an efficient method to approximate prices of several interest-rate related derivatives including swaptions, CMS, CMS options, and credit derivatives. However, in the case of diffusion processes, cumulants are rarely computable and, therefore, these expansions are not applicable directly in general.

In order to overcome these shortcomings, we propose a new methodology. The outline of our approach is as follows. First, we expand the underlying dynamics by Hermite polynomials based on the Wiener-Ito chaos expansion. Second, we approximate it by a truncated sum of iterated Ito stochastic integrals by means of successive substitution. Finally, we derive the characteristic function of the approximated underlying asset price and convert it to the PDF by inversion formula. The value of a European contingent claim is then derived in closed form by the approximated probability density.

We provide numerical examples to investigate the accuracy of our approximations. Through the comparison with previous works, we show that our approach provides far greater accuracy than the previously proposed schemes over a wide range of data sets. Especially, we emphasize that our approximation works quite well even in the case of high volatility and long maturity, unlike the previously proposed methods.

This paper is organized as follows. In the next section, we explain the backgrounds of the ideas of approximation method developed in this paper. Section 3 provides the proposed approximation, while the detailed development of our approximation is explained in Section 4. Based on the approximation scheme, we derive approximated formulas for the transition density function of the underlying diffusion process and for European call option prices in Section 5. Section 6 is devoted to numerical examples. Comparing with the exact formulas and Monte Carlo simulation results, it is observed that our approximation formulas exhibit a very high accuracy even for the case of high volatility and long maturity. Finally, Section 7 concludes this paper.

Throughout the paper, we consider the complete probability space $(\Omega, \mathcal{F}, \mathbb{Q}, \{\mathcal{F}_t\}_{t\geq 0})$ and assume that the filtration satisfies the usual conditions. The probability measure $\mathbb{Q}$ is a risk-neutral measure, because we are interested in the pricing of contingent claims. The expectation operator under $\mathbb{Q}$ is denoted by $\mathbb{E}$.

2. The Backgrounds

In this paper, we assume that the price of the underlying asset $\{S_t\}_{0\leq t\leq T}$ follows the stochastic differential equation (SDE for short)

$$\frac{\mathrm{d}S_t}{S_t} = r(t)\mathrm{d}t + \sigma(S_t, t)\mathrm{d}W_t, \tag{2.1}$$

where the short rate $r(t)$ is a deterministic function of time t, the volatility $\sigma(s,t)$ is a deterministic function of both asset price and time, and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$.

By applying Ito's formula, we obtain

$$\begin{aligned} S_t &= S_0 \exp\left[\int_0^t \left(r(u) - \frac{1}{2}\sigma^2(S_u, u)\right)\mathrm{d}u + \int_0^t \sigma(S_u, u)\mathrm{d}W_u\right] \\ &= F(0,t)\exp\left[\int_0^t \sigma(S_u,u)\mathrm{d}W_u - \frac{1}{2}\int_0^t \sigma^2(S_u,u)\mathrm{d}u\right], \end{aligned}$$

where $F(0,t) = S_0 \mathrm{e}^{\int_0^t r(u)\mathrm{d}u}$ is the forward price of the underlying asset with delivery date t. Denoting $\|g\|_t^2 = \int_0^t g^2(u)\mathrm{d}u$ and $J_t(g) = \int_0^t g(u)\mathrm{d}W_u$, we thus have

$$S_t = F(0,t)\exp\left[J_t(\sigma) - \frac{1}{2}\|\sigma\|_t^2\right]. \tag{2.2}$$

Therefore, it is essential to know (or to approximate) the distribution of the random variable of the form $\exp[J_t(\sigma) - \|\sigma\|_u^2/2]$, because the derivative price is given in terms of the expectation $\mathbb{E}[f(S_t)]$ for some payoff function $f(S)$.

To this end, we found the following observations useful to approximate the distribution of the random variable.

2.1. **Expansion in terms of Hermite Polynomials.** Let us denote by $h_n(x)$ the Hermite polynomial of order n defined by

$$h_n(x) = (-1)^n \mathrm{e}^{x^2/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2/2}, \quad n = 1, 2, \ldots, \tag{2.3}$$

with $h_0(x) = 1$. For example, we have $h_1(x) = x$, $h_2(x) = x^2 - 1$, $h_3(x) = x^3 - 3x$, etc.

The following expansion offers excellent convergence. The result may be known; however, for the sake of completeness, we provide a proof.

Lemma 2.1. *For any $x \in \mathcal{R}$ and $\lambda > 0$, we have*

$$\exp\left[tx - \frac{(\sqrt{\lambda}t)^2}{2}\right] = \sum_{n=0}^{\infty} \frac{(\sqrt{\lambda}t)^n}{n!} h_n\left(\frac{x}{\sqrt{\lambda}}\right). \tag{2.4}$$

Proof. Let $y = x/\sqrt{\lambda}$, so that

$$\exp\left[tx - \frac{(\sqrt{\lambda}t)^2}{2}\right] = \mathrm{e}^{y^2/2} \exp\left[-\frac{(\sqrt{\lambda}t - y)^2}{2}\right].$$

By Taylor's expansion about $t = 0$, we have

$$\begin{aligned}
\exp\left[-\frac{(\sqrt{\lambda}t - y)^2}{2}\right] &= \sum_{n=0}^{\infty} \left\{ \frac{\mathrm{d}^n}{\mathrm{d}t^n} \exp\left[-\frac{(\sqrt{\lambda}t - y)^2}{2}\right] \Big|_{t=0} \right\} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} (\sqrt{\lambda})^n \left\{ \frac{\mathrm{d}^n}{\mathrm{d}z^n} \mathrm{e}^{-z^2/2} \Big|_{z=-y} \right\} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}y^n} \mathrm{e}^{-y^2/2} \frac{(\sqrt{\lambda}t)^n}{n!}.
\end{aligned}$$

The result follows from (2.3). □

In order to check the accuracy of this expansion, we consider the log-normal random variable defined by

$$X(t; \theta_t) := \exp\left(\theta_t - \frac{1}{2}\Sigma_t\right), \tag{2.5}$$

where $\theta_t = \sigma W_t$ and $\Sigma_t = \sigma^2 t$. Note that this is the special case of $\exp[I_t(\sigma) - \|\sigma\|_t^2/2]$ when $\sigma(S, t) = \sigma$.

Now, take $t = 1$ and $\lambda = \Sigma_t$ in (2.4), and define

$$X_N^{\mathrm{H}}(t; x) := \sum_{n=0}^{N} \frac{\left(\sqrt{\Sigma_t}\right)^n}{n!} h_n\left(\frac{x}{\sqrt{\Sigma_t}}\right).$$

Also, for the comparison purpose, we consider

$$X_N^{\mathrm{M}}(t; x) := \mathrm{e}^{-\frac{1}{2}\Sigma_t} \sum_{n=0}^{N} \frac{x}{n!},$$

the Maclaurin expansion counterpart. The square errors resulted from the truncation are calculated as

$$\mathbb{E}\left[\left(X_3^{\mathrm{H}}(t; \theta_t) - X(t; \theta_t)\right)^2\right] \approx 0.03, \qquad \mathbb{E}\left[\left(X_3^{\mathrm{M}}(t; \theta_t) - X(t; \theta_t)\right)^2\right] \approx 0.11,$$

where we set $t = 10$ and $\sigma = 0.3$, a typical volatility level observed in the market. Hence, the Hermite expansion truncation provides an accurate approximation.

In order to check the accuracy more carefully, we compare the values of $X_N^{\mathrm{H}}(t, x)$ and $X_N^{\mathrm{M}}(t, x)$ for the interval $-3\sqrt{\Sigma_t} \le x \le 3\sqrt{\Sigma_t}$. Table 1 reports the values of $X(t, x)$, $X_N^{\mathrm{H}}(t, x)$ and $X_N^{\mathrm{M}}(t, x)$ for $N = 1, 2, 3$ and $t = 1, 5, 10$. Here, we fix $\sigma = 0.3$ for the volatility. Clearly, the speed of convergence in the Hermite expansion is much faster than that of the Maclaurin counterpart.

Table 1: A comparison between the Hermite and Maclaurin approximation

x	Exact	$N=1$ Hermite	$N=1$ Maclaurin	$N=2$ Hermite	$N=2$ Maclaurin	$N=3$ Hermite	$N=3$ Maclaurin
1 year ($t=1$)							
$3\sqrt{\Sigma_t}$	1.55	-0.45	-0.53	-0.09	-0.15	-0.01	-0.03
$2.33\sqrt{\Sigma_t}$	1.40	-0.22	-0.30	-0.02	-0.07	0.00	-0.01
$2\sqrt{\Sigma_t}$	1.33	-0.14	-0.21	-0.01	-0.04	0.00	-0.01
$\sqrt{\Sigma_t}$	1.15	0.01	-0.05	0.01	0.00	0.00	0.00
0	0.99	0.04	0.00	0.00	0.00	0.00	0.00
$-\sqrt{\Sigma_t}$	0.85	-0.01	-0.04	-0.01	0.00	0.00	0.00
$-2\sqrt{\Sigma_t}$	0.73	-0.12	-0.14	0.01	0.03	0.00	0.00
$-2.33\sqrt{\Sigma_t}$	0.70	-0.17	-0.19	0.03	0.05	0.00	-0.01
$-3\sqrt{\Sigma_t}$	0.63	-0.29	-0.29	0.07	0.09	-0.01	-0.02
5 years ($t=5$)							
$3\sqrt{\Sigma_t}$	2.59	-2.96	-3.57	-1.16	-1.95	-0.26	-0.87
$2.33\sqrt{\Sigma_t}$	2.07	-1.25	-1.76	-0.25	-0.79	0.03	-0.28
$2\sqrt{\Sigma_t}$	1.85	-0.71	-1.18	-0.04	-0.47	0.06	-0.14
$\sqrt{\Sigma_t}$	1.32	0.11	-0.23	0.11	-0.05	0.01	-0.01
0	0.95	0.20	0.00	-0.02	0.00	-0.02	0.00
$-\sqrt{\Sigma_t}$	0.68	-0.08	-0.15	-0.08	0.03	0.02	-0.01
$-2\sqrt{\Sigma_t}$	0.48	-0.55	-0.48	0.12	0.24	0.02	-0.08
$-2.33\sqrt{\Sigma_t}$	0.43	-0.73	-0.62	0.27	0.36	-0.02	-0.15
$-3\sqrt{\Sigma_t}$	0.35	-1.12	-0.92	0.68	0.70	-0.22	-0.38
10 years ($t=10$)							
$3\sqrt{\Sigma_t}$	3.71	-7.13	-8.53	-3.53	-5.94	-0.97	-3.50
$2.33\sqrt{\Sigma_t}$	2.70	-2.60	-3.77	-0.61	-2.21	0.19	-1.06
$2\sqrt{\Sigma_t}$	2.31	-1.35	-2.40	0.00	-1.26	0.28	-0.53
$1\sqrt{\Sigma_t}$	1.44	0.30	-0.40	0.30	-0.12	0.02	-0.03
0	0.89	0.36	0.00	-0.09	0.00	-0.09	0.00
$-\sqrt{\Sigma_t}$	0.56	-0.20	-0.21	-0.20	0.07	0.09	-0.02
$-2\sqrt{\Sigma_t}$	0.35	-0.99	-0.67	0.36	0.48	0.07	-0.25
$-2.33\sqrt{\Sigma_t}$	0.30	-1.28	-0.84	0.71	0.72	-0.09	-0.43
$-3\sqrt{\Sigma_t}$	0.22	-1.88	-1.21	1.72	1.37	-0.84	-1.08

The numbers indicate the difference between the exact value and approximation.

2.2. **Wiener-Ito Chaos Expansion.** First, note that the expansion (2.4) is valid also for random variable. Hence, putting $t=1$, $x=J_t(\sigma)$ and $\lambda=\|\sigma\|_t^2$, we obtain

$$\exp\left(J_t(\sigma)-\frac{1}{2}\|\sigma\|_t^2\right)=\sum_{n=0}^{\infty}\frac{\|\sigma\|_t^n}{n!}h_n\left(\frac{J_t(\sigma)}{\|\sigma\|_t}\right). \tag{2.6}$$

Second, Ito (1951) showed that, for $n\geq 1$,

$$\frac{\|\sigma\|_t^n}{n!}h_n\left(\frac{J_t(\sigma)}{\|\sigma\|_t}\right)=\int_0^t\int_0^{t_n}\cdots\int_0^{t_2}\sigma(t_1)\sigma(t_2)\cdots\sigma(t_n)\mathrm{d}W_{t_1}\cdots\mathrm{d}W_{t_n}, \tag{2.7}$$

where $\{W_t\}_{t\geq 0}$ is a one-dimensional Brownian motion. This expansion is a special case of the so-called Wiener-Ito Chaos expansion.

Applying these results to equation (2.2), we then obtain

$$\begin{aligned} \frac{S_t}{F(0,t)} &= \exp\left[J_t(\sigma) - \frac{1}{2}\|\sigma\|_t^2\right] \\ &= 1 + \sum_{n=1}^{\infty}\int_0^t\int_0^{t_n}\cdots\int_0^{t_2}\sigma(t_1)\sigma(t_2)\cdots\sigma(t_n)\mathrm{d}W_{t_1}\cdots\mathrm{d}W_{t_n}. \end{aligned} \tag{2.8}$$

As we shall show later, when the volatility $\sigma(t)$ is deterministic, the iterated integral can be evaluated very easily.

2.3. **Successive Substitution.** Consider a stochastic process $\{X_t; 0 \le t \le T\}$ that satisfies the stochastic integral equation (SDE)

$$X_t = X_0 \exp\left(\int_0^t a(s)\mathrm{d}s - \frac{1}{2}\int_0^t b^2(X_s, s)\mathrm{d}s + \int_0^t b(X_s, s)\mathrm{d}W_s\right), \tag{2.9}$$

where $a(t)$ and $b(x,t)$ satisfy some regularity conditions.

In certain cases, the solution $\{X_t\}$ can be constructed by successive substitution. Namely, let $X_t^{(0)} = X_0 \mathrm{e}^{\int_0^t a(s)\mathrm{d}s}$, and define $X_t^{(k+1)}$ by

$$X_t^{(k+1)} = X_0 \exp\left(\int_0^t a(s)\mathrm{d}s - \frac{1}{2}\int_0^t b^2(X_s^{(k)}, s)\mathrm{d}s + \int_0^t b(X_s^{(k)}, s)\mathrm{d}W_s\right) \tag{2.10}$$

successively. Then, it is expected that $X_t^{(k)}$ converges to the solution X_t almost surely.

A sufficient condition is known for the convergence, which we state below for the sake of completeness. The proof is similar to the one for the existence of strong solution in the SDE (2.1), and is omitted. See, e.g., Chapter 5 of Øksendal (2000) for the detailed proof.

Proposition 2.1. *Let $T > 0$, and suppose that $a(t)$ and $b(t,x)$ satisfy*

$$|b(t,x)^2| + |b(t,x)| \le C(1 + |\log x|), \quad x \in \mathcal{R},\ t \in [0,T],$$

and

$$|b(t,x)^2 - b(t,y)^2| + |b(t,x) - b(t,y)| \le D\left|\log\left(\frac{x}{y}\right)\right|, \quad x, y \in \mathcal{R},\ t \in [0,T],$$

for some constant C and D. Then, $X_t^{(k)}$ converges to the solution X_t almost surely.

Remark 2.1. *It should be noticed that the condition in the above proposition is often too strong for practical uses. Hence, in the following development, we shall assume that the successive substitution produces the solution without checking the condition.*

In the following sections, based on these observations, we intend to develop an approximation method for the pricing of European contingent claims.

3. The Proposed Approximation

In this section, we propose an approximation method based on the observations explained in the previous section.

Suppose that the SDE (2.1) admits a solution

$$S_t = F(0,t)\exp\left[\int_0^t \sigma(S_u, u)\mathrm{d}W_u - \frac{1}{2}\int_0^t \sigma^2(S_u, u)\mathrm{d}u\right]. \tag{3.1}$$

Let $S_t^{(0)} = F(0,t)$ and define $S_t^{(m)}$ successively according to (2.10), i.e.,

$$\begin{aligned} S_t^{(m+1)} &= F(0,t)\exp\left[\int_0^t \sigma_m(u)\mathrm{d}W_u - \frac{1}{2}\int_0^t \sigma_m^2(u)\mathrm{d}u\right] \\ &= F(0,t)\exp\left[J_t(\sigma_m) - \frac{1}{2}\|\sigma_m\|_t^2\right], \end{aligned} \tag{3.2}$$

where $\sigma_m(t) = \sigma(S_t^{(m)}, t)$. It is assumed throughout the rest of this paper that $S_t^{(m)}$ converges to S_t as $m \to \infty$. It follows that

$$S_t = S_t^{(1)} + \sum_{m=1}^{\infty}\{S_t^{(m+1)} - S_t^{(m)}\}. \tag{3.3}$$

On the other hand, from (2.6) and (3.2), we have

$$\frac{S_t^{(m+1)}}{F(0,t)} = 1 + \sum_{n=1}^{\infty} \frac{\|\sigma_m\|_t^n}{n!} h_n\left(\frac{J_t(\sigma_m)}{\|\sigma_m\|_t}\right).$$

Hence, defining

$$I_{m,n}(t) = \frac{1}{n!}\left\{\|\sigma_m\|_t^n h_n\left(\frac{J_t(\sigma_m)}{\|\sigma_m\|_t}\right) - \|\sigma_{m-1}\|_t^n h_n\left(\frac{J_t(\sigma_{m-1})}{\|\sigma_{m-1}\|_t}\right)\right\}, \tag{3.4}$$

we obtain from (3.3) that

$$S_t = S_t^{(1)} + F(0,t)\sum_{m,n=1}^{\infty} I_{m,n}(t). \tag{3.5}$$

Our approximation is to truncate the infinite sum at $m+n \le 3$. As we shall see later through numerical experiments, this truncation produces a highly accurate approximation for S_t.

To be more specific, consider an iterated integral

$$I := \sum_{n=1}^{\infty}\int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \sigma_1(t_1)\sigma_2(t_2)\cdots\sigma_n(t_n)\mathrm{d}W_{t_1}\cdots\mathrm{d}W_{t_n}.$$

When the volatilities $\sigma_n(t)$ are deterministic functions, the iterated integral converges very quickly, so that we can truncate it at $n = 3$. In other words, we can neglect the terms of order n, $n \ge 4$.

Before proceeding, note from (2.8) that

$$\frac{S_t^{(1)}}{F(0,t)} = 1 + \sum_{n=1}^{\infty}\int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \sigma_0(t_1)\sigma_0(t_2)\cdots\sigma_0(t_n)\mathrm{d}W_{t_1}\cdots\mathrm{d}W_{t_n},$$

where $\sigma_0(t) = \sigma(F(0,t), t)$, which is a deterministic function. Hence, we define

$$\begin{aligned} \widetilde{S}_t^{(1)} &= F(0,t)\left[1 + \int_0^t \sigma_0(t_1)\mathrm{d}W_{t_1} + \int_0^t\int_0^{t_2}\sigma_0(t_1)\sigma_0(t_2)\mathrm{d}W_{t_1}\mathrm{d}W_{t_2}\right. \\ &\qquad \left. + \int_0^t\int_0^{t_3}\int_0^{t_2}\sigma_0(t_1)\sigma_0(t_2)\sigma_0(t_3)\mathrm{d}W_{t_1}\mathrm{d}W_{t_2}\mathrm{d}W_{t_3}\right], \end{aligned} \tag{3.6}$$

as an approximation for $S_t^{(1)}$.

Summarizing, we approximate the quantity S_t by

$$S_t = \widetilde{S}_t^{(1)} + F(0,t)\sum_{m,n\ge 1;\ m+n\le 3} I_{m,n}(t), \tag{3.7}$$

where $\widetilde{S}_t^{(1)}$ is given by (3.6) and $I_{m,n}(t)$ by (3.4). In the next section, we approximate each $I_{m,n}(t)$ by an iterated integral with deterministic volatilities.

To this end, it is essential to reduce the iteration subscript m by some means. In this paper, we employ Taylor's expansion around $S_t^{(m-1)}$ for this purpose. Recall that $J_t(\sigma_m) = \int_0^t \sigma_m(u)\mathrm{d}W_u = \int_0^t \sigma(S_u^{(m)}, u)\mathrm{d}W_u$. It follows that

$$\begin{aligned} J_t(\sigma_m) \;\approx\; & J_t(\sigma_{m-1}) + \int_0^t \sigma'_{m-1}(u)\{S_u^{(m)} - S_u^{(m-1)}\}\mathrm{d}W_u \\ & + \frac{1}{2}\int_0^t \sigma''_{m-1}(u)\{S_u^{(m)} - S_u^{(m-1)}\}^2\mathrm{d}W_u \end{aligned} \tag{3.8}$$

and

$$J_t^2(\sigma_m) \approx J_t^2(\sigma_{m-1}) + 2J_t(\sigma_{m-1})\int_0^t \sigma'_{m-1}(u)\{S_u^{(m)} - S_u^{(m-1)}\}\mathrm{d}W_u, \tag{3.9}$$

where we denote

$$\sigma'_m(t) := \partial_x\sigma(x,t)\big|_{x=S_t^{(m)}}, \qquad \sigma''_m(t) := \partial_{xx}\sigma(x,t)\big|_{x=S_t^{(m)}}$$

for the sake of notational simplicity.

4. APPROXIMATION OF $I_{m,n}(t)$

In this section, we approximate each $I_{m,n}(t)$ by using the approximations (3.8) and (3.9). Recall that

$$S_t^{(m+1)} - S_t^{(m)} = F(0,t)\sum_{n=1}^{\infty} I_{m,n}(t) \;\approx\; F(0,t)\sum_{n\le 3-m} I_{m,n}(t) \tag{4.1}$$

by definition.

4.1. Approximation of $I_{1,1}(t)$.

By definition, $I_{1,1}(t) = J_t(\sigma_1) - J_t(\sigma_0)$ and so, from (3.8), we have

$$I_{1,1}(t) \;\approx\; \int_0^t \sigma'_0(u)\{S_u^{(1)} - S_u^{(0)}\}\mathrm{d}W_u + \frac{1}{2}\int_0^t \sigma''_0(u)\{S_u^{(1)} - S_u^{(0)}\}^2\mathrm{d}W_u.$$

Since $S_t^{(0)} = F(0,t)$ and $S_t^{(1)}$ is approximated by (3.6), by ignoring the terms of higher orders, we obtain

$$\begin{aligned} I_{1,1}(t) \;\approx\; & \int_0^t \sigma'_0(s)F(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)\mathrm{d}W_s \\ & + \int_0^t \sigma'_0(s)F(0,s)\left(\int_0^s \sigma_0(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\ & + \frac{1}{2}\int_0^t \sigma''_0(s)F^2(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)^2\mathrm{d}W_s. \end{aligned} \tag{4.2}$$

Further, by Ito's formula, we get

$$\begin{aligned} & \int_0^t \sigma''_0(s)F^2(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)^2\mathrm{d}W_s \\ = \; & 2\int_0^t \sigma''_0(s)F^2(0,s)\left(\int_0^s \sigma_0(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\ & + \int_0^t \sigma''_0(s)F^2(0,s)\left(\int_0^s \sigma_0^2(u)\mathrm{d}u\right)\mathrm{d}W_s. \end{aligned} \tag{4.3}$$

Finally, substitution of (4.3) into (4.2) yields

$$\begin{aligned}
I_{1,1}(t) &\approx \int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)\mathrm{d}W_s \\
&\quad + \int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\
&\quad + \int_0^t \sigma_0''(s)F^2(0,s)\left(\int_0^s \sigma_0(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\
&\quad + \frac{1}{2}\int_0^t \sigma_0''(s)F^2(0,s)\left(\int_0^s \sigma_0^2(u)\mathrm{d}u\right)\mathrm{d}W_s.
\end{aligned} \tag{4.4}$$

4.2. **Approximation of $I_{1,2}(t)$.** By the definition of Hermite polynomials, we have

$$\begin{aligned}
I_{1,2}(t) &= \frac{1}{2}\left\{(J_t^2(\sigma_1) - J_t^2(\sigma_0)) - (\|\sigma_1\|_t^2 - \|\sigma_0\|_t^2)\right\} \\
&\approx J_t(\sigma_0)\int_0^t \sigma_0'(u)\{S_u^{(1)} - S_u^{(0)}\}\mathrm{d}W_u - \frac{1}{2}(\|\sigma_1\|_t^2 - \|\sigma_0\|_t^2),
\end{aligned}$$

where we have used (3.9) for the approximation. Hence, since $J_t(\sigma_0) = \int_0^t \sigma_0(u)\mathrm{d}W_u$, $\sigma_0(t) = \sigma(F(0,t),t)$, by ignoring the terms of higher orders, we obtain

$$\begin{aligned}
I_{1,2}(t) &\approx \left(\int_0^t \sigma_0(s)\mathrm{d}W_s\right)\left(\int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)\mathrm{d}W_s\right) \\
&\quad - \frac{1}{2}\left(\|\sigma_1\|_t^2 - \|\sigma_0\|_t^2\right),
\end{aligned} \tag{4.5}$$

where we have applied (3.6) for the further approximation.

Now, by Ito's formula, the first term in (4.5) is rewritten as

$$\begin{aligned}
&\left(\int_0^t \sigma_0(s)\mathrm{d}W_s\right)\left(\int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)\mathrm{d}W_s\right) \\
&= \int_0^t \sigma_0(s)\left(\int_0^s \sigma_0'(u)F(0,u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\
&\quad + \int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)^2\mathrm{d}W_s \\
&\quad + \int_0^t \sigma_0(s)\sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)\mathrm{d}s.
\end{aligned}$$

Similarly, for the second and third terms in the above expression, we get

$$\begin{aligned}
&\int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)^2\mathrm{d}W_s \\
&= 2\int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\
&\quad + \int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0^2(u)\mathrm{d}u\right)\mathrm{d}W_s
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \sigma_0(s)\sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)\mathrm{d}s \\
=\ &\left(\int_0^t \sigma_0(s)\sigma_0'(s)F(0,s)\mathrm{d}s\right)\left(\int_0^t \sigma_0(s)\mathrm{d}W_s\right) \\
&\quad - \int_0^t \sigma_0(s)\left(\int_0^s \sigma_0(u)\sigma_0'(u)F(0,u)\mathrm{d}u\right)\mathrm{d}W_s,
\end{aligned}$$

respectively.

On the other hand, the second term in (4.5) is approximated as

$$\|\sigma_1\|_t^2 - \|\sigma_0\|_t^2 = \int_0^t (\sigma_1^2(s) - \sigma_0^2(s))\mathrm{d}s \ \approx\ 2\int_0^t \sigma_0(u)\sigma_0'(u)\{S_u^{(1)} - S_u^{(0)}\}\mathrm{d}s,$$

by Taylor's expansion around $S_t^{(0)}$. Applying the approximation (3.6) and then using Ito's lemma, we obtain

$$\begin{aligned}
\|\sigma_1\|_t^2 - \|\sigma_0\|_t^2 &\approx 2\int_0^t \sigma_0(s)\sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)\mathrm{d}s \\
&= 2\left(\int_0^t \sigma_0(s)\sigma_0'(s)F(0,s)\mathrm{d}s\right)\left(\int_0^t \sigma_0(s)\mathrm{d}W_s\right) \\
&\quad - 2\int_0^t \sigma_0(s)\left(\int_0^s \sigma_0(u)\sigma_0'(u)F(0,u)\mathrm{d}u\right)\mathrm{d}W_s.
\end{aligned}$$

Finally, we put these results together to obtain the approximation for $I_{1,2}(t)$ as

$$\begin{aligned}
I_{1,2}(t) &\approx \int_0^t \sigma_0(s)\left(\int_0^s \sigma_0'(u)F(0,u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\
&\quad + 2\int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\
&\quad + \int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0^2(u)\mathrm{d}u\right)\mathrm{d}W_s.
\end{aligned} \tag{4.6}$$

4.3. **Approximation of $I_{2,1}(t)$.** By definition, $I_{2,1}(t) = J_t(\sigma_2) - J_t(\sigma_1)$ and so, from (3.8), we have

$$I_{2,1}(t) \ \approx\ \int_0^t \sigma_1'(u)(S_u^{(2)} - S_u^{(1)})\mathrm{d}W_s + \frac{1}{2}\int_0^t \sigma_1''(u)(S_u^{(2)} - S_u^{(1)})^2\mathrm{d}W_s.$$

Since $S_u^{(2)} - S_u^{(1)} = F(0,t)I_{1,1}(t)$ due to (4.1), and since $I_{1,1}(t) = J_t(\sigma_1) - J_t(\sigma_0)$, we have

$$I_{2,1}(t) \ \approx\ \int_0^t \sigma_1'(s)F(0,s)I_{1,1}(s)\mathrm{d}W_s + \frac{1}{2}\int_0^t \sigma_1''(s)F^2(0,s)I_{1,1}^2(s)\mathrm{d}W_s.$$

Hence, from (4.2), by ignoring higher terms, we obtain

$$\begin{aligned}
I_{2,1}(t) &\approx \int_0^t \sigma_1'(s)F(0,s)\left(\int_0^s \sigma_0'(u)F(0,u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_s\right)\mathrm{d}W_s \\
&\quad + \frac{1}{2}\int_0^t \sigma_1''(s)F(0,s)\left(\int_0^s \sigma_0'(u)F(0,u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_s\right)^2\mathrm{d}W_s.
\end{aligned}$$

Now, we apply Taylor's expansion to $\sigma_1'(t) = \partial_x \sigma(S_t^{(1)}, t)$ and $\sigma_1''(t)$ around $S_t^{(0)}$. It follows by ignoring higher terms again that

$$\begin{aligned} I_{2,1}(t) &\approx \int_0^t \left\{\sigma_0'(u) + \sigma_0''(u)\{S_u^{(1)} - S_u^{(0)}\}\right\} F(0,u) \\ &\qquad \times \left(\int_0^u \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s. \end{aligned}$$

By applying (3.6) and ignoring higher terms again, we finally get

$$I_{2,1}(t) \approx \int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0'(u)F(0,u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s. \tag{4.7}$$

4.4. **Approximation of $I_{m,n}(t)$, $m+n \geq 4$.** Here, we prove the case that $I_{2,2}(t) \approx 0$ only. The other terms can be proved in a similar manner. Note that

$$I_{2,2}(t) = \frac{1}{2}\left\{(J_t^2(\sigma_2) - J_t^2(\sigma_1)) - (\|\sigma_2\|_t^2 - \|\sigma_1\|_t^2)\right\}.$$

From (3.9) and the approximation method used for $I_{2,1}$, we obtain

$$J_t^2(\sigma_2) \approx J_t^2(\sigma_1) + 2J_t(\sigma_1)\int_0^t \sigma_0'(s)F(0,s)\left(\int_0^s \sigma_0'(u)F(0,u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s,$$

the second term being zero because it involves higher terms only. Similarly, we have

$$\|\sigma_2\|_t^2 - \|\sigma_1\|_t^2 = E[J_t^2(\sigma_2) - J_t^2(\sigma_1)] \approx 0,$$

proving the claim.

5. Approximation Formulas

The proposed approximations developed so far are put together to conclude the following.

Theorem 5.1. *Let $X_t := \frac{S_t}{F(0,t)} - 1$. Then,*

$$\begin{aligned} X_t &\approx \int_0^t p_1(s)\mathrm{d}W_s + \int_0^t p_2(s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)\mathrm{d}W_s \\ &\quad + \int_0^t p_3(s)\left(\int_0^s \sigma_0(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\ &\quad + \int_0^t p_4(s)\left(\int_0^s p_5(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} p_1(s) &:= \left\{\sigma_0(s) + F(0,s)\sigma_0'(s)\left(\int_0^s \sigma_0^2(u)\mathrm{d}u\right) + \frac{1}{2}F^2(0,s)\sigma_0''(s)\left(\int_0^s \sigma_0^2(u)\mathrm{d}u\right)\right\}, \\ p_2(s) &:= \sigma_0(s) + F(0,s)\sigma_0'(s), \\ p_3(s) &:= \sigma_0(s) + 3F(0,s)\sigma_0'(s) + F^2(0,s)\sigma_0''(s), \\ p_4(s) &:= \sigma_0(s) + F(0,s)\sigma_0'(s), \\ p_5(s) &:= F(0,s)\sigma_0'(s). \end{aligned}$$

Note that $p_k(t)$ are all deterministic functions.

In order to calculate the probability distribution of X_t, we intend to derive an approximated characteristic function of X_t, which can then be inverted back to derive an approximation of the probability distribution of X_t. This idea has been widely used for deriving approximated distributions.

Before proceeding, we define the following variables for the sake of notational simplicity. Let

$$
\begin{aligned}
a_1(t) &= \int_0^t p_1(s)\mathrm{d}W_s, \\
a_2(t) &= \int_0^t p_2(s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)\mathrm{d}W_s,
\end{aligned}
$$

and

$$
\begin{aligned}
a_3(t) &= \int_0^t p_3(s)\left(\int_0^s \sigma_0(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\
&\quad + \int_0^t p_4(s)\left(\int_0^s p_5(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s.
\end{aligned}
$$

Then, from (5.1), we have

$$X_t \approx a_1(t) + a_2(t) + a_3(t).$$

Note that $a_1(t)$ follows a normal distribution with zero mean and variance $\Sigma_t := \int_0^t p_1^2(s)\mathrm{d}s$. It is well known that the moments of $a_j(t)$ conditional on the normal variate $a_1(t)$ can be obtained explicitly. We shall make use of this result for our approximation.

Let the characteristic function of X_t be $\Psi(\xi) := \mathbb{E}[\mathrm{e}^{i\xi X_t}]$. We approximate it as

$$
\begin{aligned}
\Psi(\xi) &\approx \mathbb{E}\left[\mathrm{e}^{\{i\xi(a_1(t)+a_2(t)+a_3(t))\}}\right] \\
&\approx \mathbb{E}\left[\mathrm{e}^{i\xi a_1(t)}\left(1 + i\xi a_2(t) + i\xi a_3(t) - \frac{1}{2}\xi^2 a_2(t)^2\right)\right].
\end{aligned}
$$

See, e.g., Takahashi (1999) and Kunitomo and Takahashi (2001). Taking the conditional expectation on $a_1(t)$, we then have

$$
\begin{aligned}
\Psi(\xi) &\approx \mathbb{E}[\mathrm{e}^{i\xi a_1(t)}] + i\xi\mathbb{E}\left[\mathrm{e}^{i\xi a_1(t)}\mathbb{E}[a_2(t)\,|\,a_1(t)]\right] \\
&\quad + i\xi\mathbb{E}\left[e^{i\xi a_1(t)}\mathbb{E}[a_3(t)\,|\,a_1(t)]\right] - \frac{1}{2}\xi^2\mathbb{E}\left[e^{i\xi a_1(t)}\mathbb{E}[a_2(t)^2\,|\,a_1(t)]\right].
\end{aligned} \tag{5.2}
$$

The conditional expectations can be evaluated explicitly, by just following the standard arguments.

Namely, by consulting the well-known results (see Appendix A), we have the following. From (A.1), (A.2) and (A.3), respectively, we obtain

$$\mathbb{E}[a_2(t)|a_1(t)=x] = q_1(t)\left(\frac{x^2}{\Sigma_t^2} - \frac{1}{\Sigma_t}\right), \tag{5.3}$$

$$\mathbb{E}[a_3(t)|a_1(t)=x] = q_2(t)\left(\frac{x^3}{\Sigma_t^3} - \frac{3x}{\Sigma_t^2}\right), \tag{5.4}$$

$$\mathbb{E}[a_2^2(t)|a_1(t)=x] = q_3(t)\left(\frac{x^4}{\Sigma_t^4} - \frac{6x^2}{\Sigma_t^3} + \frac{3}{\Sigma_t^2}\right) + q_4(t)\left(\frac{x^2}{\Sigma_t^2} - \frac{1}{\Sigma_t}\right) + q_5(t), \tag{5.5}$$

where

$$\begin{aligned}
\Sigma_t &= \int_0^t p_1^2(s)\mathrm{d}s,\\
q_1(t) &= \int_0^t p_1(s)p_2(s)\left(\int_0^s \sigma_0(u)p_1(u)\mathrm{d}u\right)\mathrm{d}s,\\
q_2(t) &= \int_0^t p_1(s)p_3(s)\left(\int_0^s \sigma_0(u)p_1(u)\left(\int_0^u \sigma_0(r)p_1(r)\mathrm{d}r\right)\mathrm{d}u\right)\mathrm{d}s\\
&\quad + \int_0^t p_1(s)p_4(s)\left(\int_0^s p_1(u)p_5(u)\left(\int_0^u \sigma_0(r)p_1(r)\mathrm{d}r\right)\mathrm{d}u\right)\mathrm{d}s,\\
q_3(t) &= q_1^2(t),\\
q_4(t) &= 2\int_0^t p_1(s)p_2(s)\left(\int_0^s p_1(u)p_2(u)\left(\int_0^u \sigma_0^2(r)\mathrm{d}r\right)\mathrm{d}u\right)\mathrm{d}s\\
&\quad +2\int_0^t p_1(s)p_2(s)\left(\int_0^s \sigma_0(u)p_2(u)\left(\int_0^u \sigma_0(r)p_1(r)\mathrm{d}r\right)\mathrm{d}u\right)\mathrm{d}s\\
&\quad + \int_0^t p_2^2(s)\left(\int_0^s \sigma_0(u)p_1(u)\mathrm{d}u\right)^2\mathrm{d}s,\\
q_5(t) &= \int_0^t p_2^2(s)\left(\int_0^s \sigma_0^2(u)\mathrm{d}u\right)\mathrm{d}s.
\end{aligned}$$

Recall that $a_1(t)$ follows a normal distribution with zero mean and variance Σ_t. Hence, we can apply the following well known inversion formula.

Lemma 5.1. *Suppose that X follows a normal distribution with zero mean and variance Σ. Then, for any polynomial functions $f(x)$ and $g(x)$, we have*

$$\frac{1}{2\pi}\int_{\mathcal{R}} \mathrm{e}^{-iky}g(-ik)\mathbb{E}\big[f(x)\mathrm{e}^{ikx}\big]\mathrm{d}k = g\left(\frac{\partial}{\partial y}\right)f(y)n(y;0,\Sigma),$$

where $n(x;a,b)$ denotes the normal density function with mean a and variance b.

The above formula is easily obtained by differentiating both sides of

$$\frac{1}{2\pi}\int_{\mathcal{R}} \mathrm{e}^{-iky}\mathbb{E}\big[h(x)\mathrm{e}^{ikx}\big]\mathrm{d}k = h(y)n(y;0,\Sigma)$$

with respect to y. See, e.g., Takahashi (1999).

Let us denote the density function of X_t by $f_{X_t}(x)$. By applying Lemma 5.1 to each term of the characteristic function (5.2), we obtain the approximation of the density function as

$$\begin{aligned}
f_{X_t}(x) &= n(x;0,\Sigma_t) - \frac{\partial}{\partial x}\left\{\mathbb{E}[a_2(t)|a_1(t)=x]n(x;0,\Sigma_t)\right\}\\
&\quad - \frac{\partial}{\partial x}\left\{\mathbb{E}[a_3(t)|a_1(t)=x]n(x;0,\Sigma_t)\right\}\\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{\mathbb{E}[a_2(t)^2|a_1(t)=x]n(x;0,\Sigma_t)\right\} + \cdots .
\end{aligned} \tag{5.6}$$

Now, by substituting (5.3)–(5.5) into (5.6), we obtain the following result.

Theorem 5.2. *The probability density function of X_t is approximated as*

$$\begin{aligned} f_{X_t}(x) \approx & \frac{1}{2\Sigma_t^6} n\left(x;0,\Sigma_t\right)\Big[q_3(t)\left(x^6-15x^4\Sigma_t+45x^2\Sigma_t^2-15\Sigma_t^3\right) \\ & \quad + \Sigma_t^2\left(2q_2(t)+q_4(t)\right)\left(x^4-6x^2\Sigma_t+3\Sigma_t^2\right) \\ & \quad + \Sigma_t^3\left\{2q_1(t)\left(x^3-3x\Sigma_t\right)+q_5(t)\left(x^2\Sigma_t-\Sigma_t^2\right)+2\Sigma_t^3\right\}\Big], \end{aligned} \tag{5.7}$$

where $n(x;a,b)$ denotes the normal density function with mean a and variance b.

We remark that the polynomials involved in (5.7) are Hermite polynomials $h_n(x)$, and the density function $f_{X_t}(x)$ can then be alternatively expressed as

$$\begin{aligned} f_{X_t}(x) = & \frac{1}{2} n\left(x;0,\Sigma_t\right)\left[\frac{q_3(t)}{\Sigma_t^3}h_6\left(\frac{x}{\sqrt{\Sigma_t}}\right)+\frac{(2q_2(t)+q_4(t))}{\Sigma_t^2}h_4\left(\frac{x}{\sqrt{\Sigma_t}}\right)\right. \\ & \left. +\frac{2q_1(t)}{\left(\sqrt{\Sigma_t}\right)^3}h_3\left(\frac{x}{\sqrt{\Sigma_t}}\right)+\frac{q_5(t)}{\Sigma_t}h_2\left(\frac{x}{\sqrt{\Sigma_t}}\right)+2\right]. \end{aligned}$$

Also, the density function of S_t is given by

$$f_{S_t}(x) = \frac{f_{X_t}\left(\frac{x}{F(0,t)}-1\right)}{F(0,t)}. \tag{5.8}$$

Finally, we are in a position to state an approximation formula for a European call option with strike K and maturity t written on the asset S_t. Recall that the value of the European call option is given by

$$C(t) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t r(s)\mathrm{d}s}\left(S_t-K\right)^+\right] = F(0,t)\mathbb{E}\left[\mathrm{e}^{-\int_0^t r(s)\mathrm{d}s}\left(X_t+\widetilde{K}\right)^+\right],$$

where $\widetilde{K} := 1-\frac{K}{F(0,t)}$. With the density function $f_{X_t}(x)$ at hand, it follows that

$$C(t) = S(0)\int_{-\widetilde{K}}^{\infty}\left(x+\widetilde{K}\right)f_{X_t}(x)\mathrm{d}x.$$

Calculating the integral by using the approximated density function, we conclude the following.

Theorem 5.3. *The value of a European call option with maturity t and strike K is approximated as*

$$\begin{aligned} C(t) \approx & \frac{S_0 n(\widetilde{K};0,\Sigma)}{2\sqrt{2}\Sigma^4}\Big[\sqrt{2}q_3(t)(\widetilde{K}^4-6\widetilde{K}^2\Sigma+3\Sigma^2) \\ & \quad + \Sigma^2\sqrt{2}\left(q_4(t)+2q_2(t)\right)\left(\widetilde{K}^2-\Sigma\right) \\ & \quad + \Sigma^3\left\{-2\sqrt{2}q_1(t)\widetilde{K}+\sqrt{2}q_5(t)\Sigma+2\sqrt{2}\Sigma^2\right\}\Big] \\ & + S_0\widetilde{K}\left(1-\Phi(-\widetilde{K}/\sqrt{\Sigma})\right), \end{aligned} \tag{5.9}$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution.

6. NUMERICAL EXAMPLES

In this section, the accuracy of our approximation is studied through numerical examples. For the so-called constant elasticity of variance (CEV) model, we calculate option prices using our method and compare them with the exact solution. We also examine a model with no closed-form solution by employing Monte Carlo simulations. As a comparison purpose, we calculate the option prices using the other two approximations previously proposed in the literature.

The first candidate is based on the asymptotic theory, called the small disturbance asymptotic (denoted SDA, hereafter). For a diffusion process, Takahashi (1999) asymptotically expands the price of underlying asset up to the third order of volatility around zero, and derives approximated solutions in closed form for European options. this approximation method is popular in practice, because it is very accurate for the case of low volatility and short maturity.

The second candidate is based on the singular perturbation method (denoted SPM, hereafter). By using matched asymptotic expansions, de Jong (2010) solves singularly perturbed problems and obtains approximated solutions in closed form for European options under various models.

6.1. **The CEV model.**

Suppose that the volatility in the SDE (2.1) is specified as

$$\sigma(S,t) := \nu(t)S^{\beta(t)-1}, \qquad t \geq 0,$$

where $\nu(t)$ and $\beta(t)$ are deterministic functions of time. If $\beta(t) = 1$ and $\nu(t) = \mu$, then the asset price S_t follows a geometric Brownian motion as in the Black-Scholes model (1973). On the other hand, if $\beta(t) = 0$, then the asset price is normally distributed. The function $\nu(t)$ is called a relative volatility, while $\beta(s)$ is a time-dependent CEV parameter.

In the following, we consider the two settings; the Black-Scholes setting (i.e., $\beta(t) = 1$) and the square-root setting (i.e., $\beta(t) = 0.5$). Note that closed-form solutions of option prices are known for these cases.

In either case, we perform numerical experiments for low volatility and high volatility cases, and short maturity ($T = 1$ year), medium maturity ($T = 5$ years) and long maturity ($T = 10$ years) cases. Other parameters are chosen as $S_0 = 80.00$ and $r = 3.0\%$. The strikes $K_i(T)$, $i = 1, 2, \cdots, 5$, are selected by using the formula

$$K_i(T) = F(0,T)\exp(0.1 \times \sqrt{T} \times \delta_i), \tag{6.1}$$

where $\delta_i = -1.0,\ -0.5,\ 0,\ 0.5,\ 1.0$.

In order to check the accuracy, we consider the relative error (labeled by RE) defined by

$$\text{RE} = \frac{\text{Approximate Value} - \text{Exact Value}}{\text{Exact Value}}$$

throughout the numerical experiments.

6.1.1. *The Black-Scholes setting.*

We first consider the Black-Scholes case, i.e., $\beta(t) = 1$ for all t. In this case, the underlying asset price is log-normally distributed, and the exact option prices are known as the Black-Scholes formula (1973).

Table 2 shows option prices for the low volatility case ($\nu = 0.15$), whereas Table 3 for the high volatility case ($\nu = 0.3$). In the tables, BS means the exact prices calculated by the Black-Scholes formula, and Ours indicates the approximated prices calculated by the formula given in Theorem 5.3. The values in column SPM are based on Equation (5.9.15) in de Jong (2010), while those in column SDA are calculated using Equation (2.22) in Takahashi (1999). The relative errors are appended in each box.

From the two tables, we observe that, in the entire range of strikes and maturities, the relative errors of our approximation method are quite small. The error becomes slightly large for long maturity and far in-the- and out-of-the money strikes. However, compared with the other methods, the errors are significantly smaller. The largest magnitude of the relative error in our

Table 2: European option prices in the Black–Scholes setting (Low volatility)

Strike	BS	Ours	RE (%)	SDA	RE (%)	SPM	RE (%)
T=1 Year							
91.11	1.90	1.90	0.01	1.88	-1.09	1.94	1.74
86.66	3.12	3.12	0.00	3.10	-0.81	3.14	0.42
82.44	4.78	4.78	-0.00	4.76	-0.56	4.78	-0.01
78.42	6.87	6.87	-0.00	6.85	-0.37	6.88	0.04
74.59	9.34	9.34	-0.00	9.31	-0.24	9.36	0.24
T=5 Years							
116.24	4.50	4.51	0.17	4.28	-4.89	5.21	15.77
103.94	7.17	7.17	0.03	6.89	-3.91	7.67	6.95
92.95	10.65	10.65	-0.00	10.35	-2.83	10.95	2.79
83.11	14.87	14.87	-0.02	14.58	-1.94	15.11	1.57
74.32	19.63	19.62	-0.03	19.37	-1.31	19.99	1.84
T=10 Years							
148.15	6.61	6.65	0.51	6.02	-8.96	9.55	44.44
126.49	10.31	10.32	0.09	9.52	-7.65	12.60	22.22
107.99	15.00	15.00	-0.01	14.15	-5.69	16.61	10.74
92.2	20.50	20.49	-0.05	19.68	-4.01	21.70	5.87
78.71	26.51	26.49	-0.08	25.76	-2.81	27.80	4.88

The parameters are chosen as $r = 3.0\%$, $S(0) = 80.0$, $\beta = 1.0$ and $\nu = 0.15$.

Table 3: European option prices in the Black–Scholes setting (High volatility)

Strike	BS	Ours	RE (%)	SDA	RE (%)	SPM	RE (%)
T=1 Year							
91.11	6.38	6.38	0.02	6.18	-3.15	6.56	2.83
86.66	7.87	7.87	0.01	7.66	-2.69	8.03	2.08
82.44	9.54	9.54	-0.00	9.32	-2.26	9.69	1.59
78.42	11.38	11.38	-0.01	11.17	-1.88	11.53	1.29
74.59	13.39	13.39	-0.01	13.18	-1.54	13.54	1.13
T=5 Years							
116.24	14.87	14.91	0.24	12.66	-14.86	17.53	17.90
103.94	17.82	17.84	0.09	15.47	-13.22	20.15	13.06
92.95	21.01	21.01	-0.03	18.60	-11.49	23.06	9.75
83.11	24.40	24.37	-0.11	21.99	-9.86	26.26	7.62
74.32	27.92	27.88	-0.16	25.57	-8.41	29.69	6.34
T=10 Years							
148.15	21.48	21.61	0.63	15.30	-28.74	30.94	44.09
126.49	25.23	25.28	0.20	18.58	-26.36	33.27	31.84
107.99	29.18	29.14	-0.13	22.33	-23.47	36.00	23.38
92.2	33.24	33.13	-0.35	26.39	-20.60	39.14	17.75
78.71	37.34	37.16	-0.48	30.62	-17.99	42.65	14.22

The parameters are chosen as $r = 3.0\%$, $S(0) = 80.0$, $\beta = 1.0$ and $\nu = 0.3$.

method is 0.51% in Table 2; however, considering the actual bid-ask spreads, this error would be smaller than the spreads.

6.1.2. *The Square-Root setting.* We next consider the square-root case, i.e., $\beta(t) = 0.5$ for all t. In this case, the underlying asset price follows a chi-square distribution, and exact option prices are known (see, e.g., Schroder, 1989).

Table 4: European option prices in the square-root setting (Low volatility)

Strike	Exact	Ours	RE (%)	SDA	RE (%)	SPM	RE (%)
T=1 Year							
91.11	1.74	1.74	-0.00	1.74	-0.29	1.80	3.46
86.66	2.99	2.99	0.00	2.99	-0.20	3.02	0.99
82.44	4.71	4.71	0.00	4.70	-0.14	4.71	-0.01
78.42	6.86	6.86	-0.00	6.85	-0.09	6.86	0.05
74.59	9.37	9.37	0.00	9.36	-0.05	9.41	0.42
T=5 Years							
116.24	3.61	3.61	-0.03	3.56	-1.40	4.61	27.74
103.94	6.43	6.43	0.01	6.37	-0.97	7.21	12.13
92.95	10.19	10.19	0.01	10.13	-0.64	10.58	3.84
83.11	14.71	14.71	0.00	14.65	-0.41	14.91	1.36
74.32	19.71	19.71	0.01	19.66	-0.26	20.13	2.13
T=10 Years							
148.15	4.64	4.63	-0.09	4.51	-2.66	8.36	80.29
126.49	8.61	8.62	0.03	8.46	-1.83	11.84	37.53
107.99	13.87	13.87	0.02	13.70	-1.19	15.97	15.19
92.2	19.99	19.99	0.00	19.84	-0.76	21.19	5.98
78.71	26.49	26.50	0.02	26.37	-0.48	27.74	4.70

The parameters are set as $r = 3.0\%$, $S(0) = 80.0$, $\beta = 0.5$ and $\nu = 1.33$.

Table 5: European option prices in the square-root setting (High volatility)

Strike	Exact	Ours	RE (%)	SDA	RE (%)	SPM	RE (%)
T=1 Year							
91.11	6.01	6.01	0.01	5.96	-0.81	6.19	2.97
86.66	7.60	7.60	0.01	7.55	-0.67	7.76	2.07
82.44	9.40	9.40	0.00	9.34	-0.54	9.54	1.49
78.42	11.36	11.36	0.00	11.31	-0.44	11.50	1.22
74.59	13.47	13.47	-0.00	13.42	-0.35	13.62	1.14
T=5 Years							
116.24	12.86	12.88	0.12	12.36	-3.89	15.32	19.07
103.94	16.39	16.41	0.12	15.87	-3.17	18.59	13.43
92.95	20.18	20.20	0.07	19.66	-2.57	22.09	9.43
83.11	24.14	24.15	0.03	23.64	-2.09	25.87	7.15
74.32	28.16	28.17	0.02	27.69	-1.69	29.91	6.21
T=10 Years							
148.15	17.12	17.18	0.33	15.85	-7.39	25.36	48.15
126.49	22.05	22.14	0.39	20.74	-5.95	29.43	33.44
107.99	27.20	27.28	0.26	25.90	-4.80	33.50	23.14
92.2	32.38	32.43	0.15	31.13	-3.87	37.82	16.79
78.71	37.42	37.47	0.12	36.25	-3.13	42.51	13.61

The parameters are set as $r = 3.0\%$, $S(0) = 80.0$,$\beta = 0.5$ and $\nu = 2.66$.

Table 4 shows option prices for the low volatility case ($\nu = 1.33$), whereas Table 5 for the high volatility case ($\nu = 2.66$). The volatilities ν are so determined that the variances of log-returns at each expiry are equivalent to those of the Black–Scholes setting whose volatilities are around 30%. In these tables, the labels are the same as the Black–Scholes setting, except that the values in column SPM are based on Equation (6.4.7) in de Jong (2010).

Table 6: European option prices in the ERV model (low volatility)

Strike	MC	Ours	RE (%)	SDA	RE (%)
T=1 Year					
91.11	1.81	1.80	-0.67	1.70	-6.04
86.66	3.05	3.05	0.00	2.99	-1.87
82.44	4.74	4.75	0.17	4.75	0.19
78.42	6.88	6.89	0.16	6.95	0.98
74.59	9.38	9.38	0.10	9.48	1.11
T=5 Years					
116.24	3.81	3.77	-0.98	3.29	-13.73
103.94	6.60	6.60	-0.11	6.30	-4.54
92.95	10.31	10.32	0.15	10.33	0.17
83.11	14.77	14.79	0.14	15.07	2.06
74.32	19.71	19.73	0.10	20.20	2.49
T=10 Years					
148.15	4.95	4.87	-1.48	3.92	-20.73
126.49	8.90	8.88	-0.19	8.28	-6.97
107.99	14.06	14.08	0.16	14.08	0.11
92.2	20.08	20.11	0.15	20.67	2.96
78.71	26.49	26.52	0.13	27.46	3.67

The parameters are set as $r = 3.0\%$, $\mu = S(0) = 80.0$, $\kappa = 0.01$ and $\nu = 0.075$.

It can be seen that, in the entire range of strikes and maturities, the errors of our approximation method are surprisingly small. The SDA may be comparable with ours for the short maturity case.

6.2. **The exponentially-retracting volatility model.** Suppose that the volatility in the SDE (2.1) is specified as

$$\sigma(S_t, t) := \nu(t)\left(1 + \mathrm{e}^{\kappa(t)(\mu(t)-S_t)}\right),$$

where $\nu(t)$, $\kappa(t)$ and $\mu(t)$ are positive deterministic functions of time. If the asset price S_t increases to infinity, the volatility becomes constant. We refer to this model as the exponentially-retracting volatility (ERV) model. Let us call the function $\nu(t)$ a relative volatility and $\kappa(t)$ a retraction parameter. The function $\mu(t)$ is set to be the initial asset price S_0 in this example. Note that closed-form solutions and SPM approximations for option prices are not available in this case. Hence, we compare our approximation results with Monte Carlo simulation.

As for the previous cases, tables 6 and 7 report the low volatility and high volatility cases, and the short maturity ($T = 1$ year), medium maturity ($T = 5$ years) and long maturity ($T = 10$ years) cases. The volatilities ν are set so that the variances of log-returns at each expiry are equivalent to those of the log-normal process whose volatilities are around 15%. Other parameters are chosen as $S_0 = 80.00$, $r = 3.0\%$, $\kappa(t) = 0.01$ and $\mu(t) = S_0$. The strikes $K_i(T)$, $i = 1, 2, \cdots, 5$, are selected by the formula (6.1). In these tables, the label MC means the prices calculated by Monte Carlo simulation. The other labels are the same as the previous example.

We can see from the two tables that the relative errors of our approximation are well within the observed bid-ask spreads in the actual market even in this complex setting.

7. Conclusion

In this paper, we propose an approximation method based on the Wiener-Ito chaos expansion for the pricing of European-style options. Our method is applicable to the general class of

Table 7: European option prices in the ERV model (High volatility)

Strike	MC	Ours	RE (%)	SDA	RE (%)
T=1 Year					
91.11	4.93	4.92	-0.16	4.72	-4.14
86.66	6.45	6.45	0.08	6.35	-1.48
82.44	8.21	8.22	0.14	8.22	0.19
78.42	10.18	10.20	0.14	10.30	1.18
74.59	12.34	12.35	0.13	12.55	1.70
T=5 Years					
116.24	10.60	10.56	-0.37	9.54	-10.02
103.94	13.96	13.97	0.05	13.42	-3.88
92.95	17.68	17.70	0.13	17.70	0.09
83.11	21.65	21.67	0.08	22.19	2.48
74.32	25.76	25.77	0.07	26.74	3.82
T=10 Years					
148.15	14.13	14.02	-0.77	11.99	-15.14
126.49	18.85	18.86	0.08	17.75	-5.80
107.99	23.91	23.95	0.17	23.96	0.21
92.2	29.12	29.14	0.06	30.24	3.83
78.71	34.28	34.29	0.03	36.29	5.88

The parameters are set as $r = 3.0\%$, $\mu(t) = S_0 = 80.0$, $\kappa = 0.01$ and $\nu(t) = 0.13$.

continuous Markov processes. The resulting approximation formula requires at most three-dimensional numerical integration; whence it is not computer-intensive for the valuation. It is shown through numerical examples that, unlike existing approximation methods, the accuracy of our approximation remains quite high even for the case of high volatility and long maturity.

The outline of our approach is as follows. First, we expand the underlying dynamics by Hermite polynomials based on the Wiener-Ito chaos expansion. Second, we approximate it by a truncated sum of iterated Ito stochastic integrals by means of successive substitution. Finally, we derive the characteristic function of the approximated underlying asset price and convert it to the probability density function by inversion formula. The value of a European contingent claim is then derived in closed form by the approximated probability density.

As future works, we extend our method to multi-dimensional diffusions such as stochastic interest-rate models and stochastic volatility models and consider applications for the valuation problem of other financial contingent claims.

Appendix A. Formulas for Conditional Expectations

Let W_t^i, $i = 1, \dots, 4$, be one-dimensional, independent standard Brownian motions, and let $y_i(x)$ be some deterministic functions. Moreover, let $\Sigma := \int_0^T y_1^2(t)\mathrm{d}t$, and denote $J_T(y_1) = \int_0^T y_1(t)\mathrm{d}W_t^1$.

Then, the following formulas are well known:

$$E\left[\int_0^T y_3(t)\left(\int_0^t y_2(s)\mathrm{d}W_s^2\right)\mathrm{d}W_t^3 \middle| J_T(y_1) = x\right] = v_1\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right), \tag{A.1}$$

where

$$v_1 = \int_0^T y_3(t)y_1(t)\left(\int_0^t y_2(s)y_1(s)\mathrm{d}s\right)\mathrm{d}t.$$

$$E\left[\int_0^T y_4(t)\left(\int_0^t y_3(s)\left(\int_0^s y_2(u)\mathrm{d}W_u^2\right)\mathrm{d}W_s^3\right)\mathrm{d}W_t^4\middle|J_T(y_1)=x\right]$$

$$= v_2\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right), \tag{A.2}$$

where

$$v_2=\int_0^T y_4(t)y_1(t)\left(\int_0^t y_3(s)y_1(s)\left(\int_0^s y_2(u)y_1(u)\mathrm{d}u\right)\mathrm{d}s\right)\mathrm{d}t.$$

Furthermore,

$$E\left[\left(\int_0^T y_3(t)\left(\int_0^t y_2(s)\mathrm{d}W_s^2\right)\mathrm{d}W_t^3\right)^2\middle|J_T(y_1)=x\right]$$

$$= v_3\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}-\frac{3}{\Sigma^2}\right)+v_4\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)+v_5, \tag{A.3}$$

where

$$\begin{aligned}
v_3 &= \left(\int_0^T y_3(t)y_1(t)\left(\int_0^t y_2(t)y_1(t)\mathrm{d}s\right)\mathrm{d}t\right)^2,\\
v_4 &= 2\int_0^T y_3(t)y_1(t)\left(\int_0^t y_3(s)y_1(s)\left(\int_0^s y_2^2(u)\mathrm{d}u\right)\mathrm{d}s\right)\mathrm{d}t\\
&\quad +2\int_0^T y_3(t)y_1(t)\left(\int_0^t y_2(s)y_3(s)\left(\int_0^s y_2(u)y_1(u)\mathrm{d}u\right)\mathrm{d}s\right)\mathrm{d}t\\
&\quad +\left\{\int_0^T y_3^2(t)\left(\int_0^t y_2(s)y_1(s)\mathrm{d}s\right)^2\mathrm{d}t\right\},\\
v_5 &= \int_0^T y_3^2(t)\left(\int_0^t y_2^2(u)\mathrm{d}u\right)\mathrm{d}t.
\end{aligned}$$

Formulas (A.1), (A.2) and (A.3) are one-dimensional versions of Lemma 2.1 in Takahashi (1999). See also Yoshida (1992b) for detailed discussions on the conditional expectations.

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