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CREDIT-EQUITY MODELING UNDER A LATENT L´EVY FIRM PROCESS

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ABSTRACT. We propose a unified credit-equity modeling by extending the latent structural model originally proposed by Kijima et al. (2009) so as to include jumps and regime switching. As in the original latent model, we treat the actual firm value to be unobservable and one can extract information from the correlated marker process, the equity value, which is actively traded in the market. Default occurs when the actual firm value reaches a default threshold at the first time before the maturity of debt. The purpose of such extension is to capture more realistic credit spreads under different economic environments. We illustrate the application of the model through the pricing of credit default swaps and equity options. Our model can evaluate corporate securities and their derivatives in a unified framework.

1. INTRODUCTION

Vast amount of studies have been conducted to model credit risk of a corporate firm. Among them, modeling the credit risk under the structural approach has an economic appeal since it provides an intuitive linkage between the firm value and the values of corporate securities such as debt and equity.

However, until recently, the equity and credit modeling seems to be two separate themes in the finance literature. That is, equity processes are modeled through the eyes of the investors, while structural models reflect the perceptions of players inside firms. Consequently, many prominent equity models have not taken the creditworthiness of the firm value into account. However, recent credit crisis shows the intimidate relationship between the credit and equity markets.[1] Hence, new attempts are required to construct the credit-equity modeling in a unified manner.

Consider a corporate firm that issues a debt and an equity. Let D and S be the debt and equity values, respectively, and let V be the firm value. According to the basic accounting assumption, we have the relationship $V = D + S$. The default occurs when V reaches a default threshold either at the maturity or at the first time before maturity. This is the basic setting of the structural approach.

The difficulty to construct the credit-equity modeling in a unified manner stems from the fact that the debt and equity possess different properties. That is, while debt has finite maturity and face value, equity has neither maturity nor face value. To overcome this difficulty in the framework of structural models, the Merton model (1974) and its variants (see, for example, Merton (1976), Zhou (1997, 2001), and Kijima and Suzuki (2001)) assume that the firm is liquidated at the debt maturity and the equity value is evaluated as a call option with the same maturity written on the firm value. On the other hand, the Leland model (1994) and its variants (see, for example, Leland and Toft (1996), Hilberink and Rogers (2001), and Chen and Kou (2009)) consider a perpetual bond and the equity value is obtained by the balance theory.

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[1] For example, during the credit crisis, both CDS (credit default swap) premiums and equity volatilities are at their historical high.

New attempt to overcome the difficulty appears in CreditGrades proposed by Finger et al. (2002), where the present value of debt, D, is assumed to be the discounted face value of debt and the equity value S is modeled by a geometric Brownian motion. Through the basic accounting relationship, the firm value V is determined by $V = D + S$ and default is the first passage time of V to the face value.

Due to its simple formulation, CreditGrades quickly receives widespread popularity among practitioners and many studied have been conducted to exemplify its power in the credit-equity modeling. For example, Byström (2006) shows that the equity process has a leading effect to the credit spread and there is a positive relationship between the empirical credit spread and the theoretical credit spread computed by the CreditGrades model.

As is well known in the finance literature, if the firm value is assumed to follow a diffusion process, the model always generates unrealistic zero short-term credit spreads. To correct this shortcoming without losing tractability, the original CreditGrades model has been extended by assuming the default barrier to be stochastic, independent of the firm value process. Recently, Sepp (2006) extends the CreditGrades framework to include the case when the firm value process follows either the double-exponential jump-diffusion model or stochastic volatility model. Ozeki et al. (2011) make use of the Weiner-Hopf factorization scheme to study credit and equity problems when the firm dynamics is driven by a spectrally negative Lévy process.

In this paper, we propose another unified credit-equity model by extending the latent structural model originally proposed by Kijima et al. (2009), where we treat the *actual* firm value to be unobservable and one can extract information from the marker process that is observable in the market. Unlike the latent structural model in Kijima et al. (2009), however, the marker process in this paper is chosen to be the firm's equity. In the original latent structural model in Kijima et al. (2009), the marker process V represents the *tangible* asset of the firm correlated to the *actual* firm value. For a given maturity T, the value of the debt is given in terms of V and A, with the default feature embedded in A. Equity value is obtained as a residual value after the payment has been made to the debt holder in time of default or at maturity, whichever comes first. Hence, by defining the firm's tangible asset as the (correlated) marker process, equity comes as a by-product with same maturity as the corporate bond. As commented in Kijima et al. (2009), the fact that equity has maturity is rather unrealistic. Moreover, the complicated form of equity in the original latent structural model makes it difficult to price equity options, even under the case of Brownian motion. Motivated by the fact that the liquidity of the equity markets supersedes that of the corporate bond markets, we shall use the equity data of the firm as means to extract firm's credit information in this paper. We then proceed to price the CDS and equity options with default feature.

From the financial standpoint, pricing the CDS and the equity options using the equity data alone enables us to study the relationship between CDS and equity under a joint framework. More specifically, embedding the firm value into the equity value allows us to introduce the firm's creditworthiness into the equity process. As we shall see in the subsequent discussion, the current framework allows us to explain the effect of credit spread on the option's implied volatility.

On the modeling perspective, this paper extends the original latent process model on two grounds. The first extension is to model the actual firm value and equity processes by jump-diffusion models. The reason to include jumps is to reproduce realistic short-term credit spreads. Jumps are considered in both the actual firm value and equity processes, since surprise shocks can occur both internally (i.e., firm restructuring) and externally (i.e., market reaction to the equity issued by the firm). The second extension is to include the regime-switching dynamics into the economy. Economic motivation of regime-switching is to capture the macroeconomic effect.[2] During the credit crisis or economic downturn, the volatilities of firm value and equity are skyrocketing. When the economic environment returns back to normal, the credit becomes calm and the volatilities drop significantly. In particular, regime-switching jump-intensity is served to describe the arrival frequency of the positive and negative news under different economic landscape. As we shall soon see in the case of the CDS pricing, the persistence of the firm staying in a particular credit environment can be well captured by the regime-switching frequency. In fact, Byström (2006) demonstrates the autocorrelative behavior of the credit spread, an evidence of the credit clustering.

When studying the latent process with both jump-diffusion and regime-switching, it is imperative to investigate which effect, jump-diffusion or regime-switching, has greater impact on the price of the securities across a time horizon. Intuitively, Lévy jump should have more dominant effect than the regime-switching on a very short time interval, whereas the situation is reversed as the time interval expands. We shall verify this intuition analytically and numerically when pricing CDSs and equity options under the current framework.

The rest of the paper is organized as follows. Section 2 gives the setup of our model with the aforementioned structures. In particular, a regime-switching double-exponential jump-diffusion model is tractable for our purpose, and the key results obtained in Kijima and Siu (2011) are summarized in Section 3. Section 4 provides the pricing of credit default swaps in our setting, while Section 5 considers equity derivatives with default feature. To enhance the versatility of our model, we also discuss the extension to the case of randomized default barrier. Section 6 provides a comprehensive numerical analysis on the effects of model's parameters on the CDSs and equity options. Section 7 concludes the paper. Appendix A contains proofs of some results mentioned in the paper.

2. Model Setup

In this section, we discuss the structure of our extended latent firm model and its assumptions. As in Kijima et al. (2009), we assume that the capital market is frictionless and there exists no information asymmetry. Moreover, we also assume that money can be borrowed from the money market at a constant, riskfree interest rate r. Unless otherwise stated, we shall work only on the probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$ and always assume that the risk-neutral probability measure $\mathbb{P}$ exists. Moreover, filtration $\mathcal{F}$ is generated by the stochastic processes considered in this paper.

As commonly observed, asset prices fluctuate under different economic or credit environment. Intuitively, the actual firm value and the equity value are driven by two factors, idiosyncratic and systematic factors. Systematic factors refer to the macroeconomic influence on the asset prices. Examples of macroeconomic indicators include Gross Domestic Products (GDP), inflationary/deflationary pressure, and sovereign risks. These systematic factors in turn affect the prices and volatilities of equity prices, as well as the actual firm values, as it is equally susceptible to the credit environment. We shall use the Markov chain as the driving factor of switching from one economic regime to another. Each regime corresponds to different parameters in the latent firm model, indicating that equity and actual firm values behave differently under different economic or credit background.

Let $\{J_t : t \geq 0\}$ be a Markov chain with state space E. For simplicity, we assume that E is finite and contains d elements, i.e. $E = \{1, 2, \ldots, d\}$. Let $\mathbf{Q}$ be the intensity matrix of J_t with respect to the Lebesgue measure, i.e.

$$\mathbf{Q} = \{q_{ij}\}_{i,j\in E}\,,$$

[2]Siu et al. (2008) discuss in detail on the pricing of CDSs under the regime-switching Brownian motion within the structural framework.

where

$$q_{ii} = -\sum_{i \neq j} q_{ij}.$$

Assume that $J_0 = i$ and, defining a holding time ζ as a positive random variable

$$\zeta = \inf\{t : J_t \neq i\},$$

the standard Markov chain theory shows that, for any time $t \geq 0$,

$$\mathbb{P}(\zeta > t) = \exp(-q_{ii}t). \tag{2.1}$$

That is, ζ is exponentially distributed with mean $\mathbb{E}_i[\zeta] = \frac{1}{q_{ii}}$. This implies that the higher the value of q_{ii} the faster the Markov chain J_t leaves state i. As discussed in Fouque et al. (2000), if we model the volatility function of an equity process in terms of a Markov chain J_t, the parameter q_{ii} can be seen as the parameter governing the degree of volatility persistence. Together with an additional assumption of J_t being ergodic, the long term behavior of the volatility process can then be fully captured by the invariant distribution of J_t. See Fouque et al. (2000) for details on the common stochastic volatility models used in finance.

In what follows, we shall use the Markov chain J_t to describe both the volatility and the Lévy measure of the firm and equity processes. The presence of J_t can then relax the independent increment assumption embedded in every Lévy process, thereby bringing forth the level of volatility persistence that is well-documented by the empirical studies mentioned in the introduction.

With the regime-switching dynamics in place, we are now in a position to discuss the structures of the actual firm value and equity processes.

2.1. **Firm value dynamics.** First, we consider the actual firm value, which is latent or unobservable in the market. Let X_t be a regime-switching Lévy process with the following canonical representation:

$$X_t = \int_0^t b^X(J_s)\mathrm{d}s + \int_0^t \sigma^X(J_s)\mathrm{d}W_s^X + \int_0^t \int_{\mathbb{R}} y(\mu^X(J_s) - \nu^X(J_s))(\mathrm{d}y)\mathrm{d}s, \tag{2.2}$$

where, under the regime $J_t = j$, $b^X(J_t) \equiv b_j^X$ denotes the drift, $\sigma^X(J_t) \equiv \sigma_j^X$ the volatility, and $\mu^X(J_t) \equiv \mu_j^X$ represents the random jump measure with compensator $\nu^X(J_t) \equiv \nu_j^X$. The process W_t^X represents the standard Brownian motion and all the random processes are mutually independent.

To compute the moment generating function $\mathbb{E}_i[\exp(uX_t)]$ for $u \in \mathbb{R}$, we need to impose one restriction on X_t: X_t has the second finite moment under each regime, i.e., for every $t \geq 0$,

$$\max_j \int_{\mathbb{R}} (1 \wedge y^2)\nu_j^X(\mathrm{d}y) < \infty.$$

Then, for any $u \in \mathbb{R}$, the moment generating function $\mathbb{E}_i[\exp(uX_t)]$ is finite and takes the form as follows (see Asmussen (2000)):

$$\mathbb{E}_i[\exp(uX_t)] \equiv \exp\left(\mathbf{K}^X[u]t\right), \tag{2.3}$$

$$\mathbf{K}^X[u] \equiv \{\kappa_j^X(u)\}_{\text{diag}} + \mathbf{Q}, \tag{2.4}$$

where

$$\kappa_j^X(u) = b_j^X u + \frac{1}{2}(\sigma_j^X u)^2 + \int_{\mathbb{R}} (\mathrm{e}^{uy} - 1 - y1_{\{|y| \leq 1\}})\nu_j^X(\mathrm{d}y).$$

Now, the *actual* firm value at time t, denoted by A_t, is assumed to be given by

$$A_t = \exp(X_t), \qquad t \geq 0, \tag{2.5}$$

where $X_0 \equiv \log(A_0)$ denotes the value of the actual log-firm value at $t = 0$. It is assumed that A_t itself is latent, i.e. unobservable to the investors, and thus nontradable in the market. Observe that, if there is only one regime and there are no jumps, we return to the Brownian motion framework as in Kijima et al. (2009). The inclusion of jumps is to capture the effects of surprise shocks to the firm. Note that regime-switching can also affect both the volatility and the random jump measure. Volatility driven by the Markov chain results in the volatility persistence as delineated in Fouque et al. (2000). The Markov-modulated jump process is to capture the impact of the shocks to the firm under different macroeconomic environments. For example, sudden shock, both systematic and idiosyncratic, during the time of credit crisis should have bigger (negative) impact to the firm than that, say, during the time of economic prosperity.

In contrast to the ordinary structural models, we assume that default occurs when the actual firm value exceeds a default threshold. That is, define

$$\tau = \inf\{t \geq 0 : A_t \leq \Gamma\} = \inf\{t \geq 0 : X_t \leq L\} \tag{2.6}$$

for some $\Gamma = \mathrm{e}^L$. Under this definition of default time τ, default is firm-specific.

2.2. **Equity process.** Let S_t be the equity value of the firm at time t. Contrast to the actual firm value, the equity value is observable and assumed to be actively traded in the market.

Let $Y_t = \log S_t$, and assume that, for each regime,

$$Y_t = \rho X_t + Z_t$$

for some constant $\rho \in [0, 1]$, where Z_t can be seen as the impact factor that is non-firm specific. Thus, the parameter ρ describes the importance of the actual firm value on equity. Higher the value of ρ, the greater influence the actual firm value has on the equity. As we shall soon see, this simple structure on equity allows us to draw the information of the firm quality into the equity valuation and thus embed credit feature into the corresponding equity derivative products.

In what follows, we assume that Z_t is also modeled by a regime-switching Lévy process, independent of other processes. That is,

$$S_t = \exp(\rho X_t + Z_t), \tag{2.7}$$

also, as in (2.2), Z_t has the following canonical representation:

$$Z_t = \int_0^t b^Z(J_s)\mathrm{d}s + \int_0^t \sigma^Z(J_s)\mathrm{d}W_s^Z + \int_0^t \int_{\mathbb{R}} y(\mu^Z(J_s) - \nu^Z(J_s))(\mathrm{d}y)\mathrm{d}s,$$

where $b^Z(J_t) \equiv b_j^Z$ denotes the drift, $\sigma^Z(J_t) \equiv \sigma_j^Z$ the volatility, and $\mu^Z(J_t) \equiv \mu_j^Z$ represents the random jump measure with compensator $\nu^Z(J_t) \equiv \nu_j^Z$. Additionally, we set $Z_0 \equiv \log(S_0/A_0^\rho)$.

Similar to the case of X_t, we need an additional assumption for the moment generating function $\mathbb{E}_i[\exp(uZ_t)]$, $u \in \mathbb{R}$, to be finite. That is, for every $t \geq 0$, we assume

$$\max_j \int_{\mathbb{R}} (1 \wedge y^2)\nu_j^Z(\mathrm{d}y) < \infty.$$

The moment generating function $\mathbb{E}_i[\exp(uZ_t)]$ is given as follows:

$$\mathbb{E}_i[\exp(uZ_t)] \equiv \exp\left(\mathbf{K}^Z[u]t\right), \tag{2.8}$$

$$\mathbf{K}^Z[u] \equiv \{\kappa_j^Z(u)\}_{\text{diag}} + \mathbf{Q}, \tag{2.9}$$

where

$$\kappa_j^Z(u) = b_j^Z u + \frac{1}{2}(\sigma_j^Z u)^2 + \int_{\mathbb{R}} (\mathrm{e}^{uy} - 1 - y1_{\{|y|\leq 1\}})\nu_j^Z(\mathrm{d}y).$$

Note from (2.4) and (2.9) that the regime switching factor J_t affects both X_t and Z_t simultaneously.

Similar to the discussion on the actual firm value process, the equity process has jumps from X_t and Z_t given a regime J_t. While jumps in X_t refer to the jumps brought by sudden changes of the firm value, jumps in Z_t refer to sudden shocks in the market, unrelated to the firm value. The latter shocks may be due to the sudden changes in the market perception on the equity market.

Parallel to the original Black-Cox (1976) model, default of the firm would trigger a drop in the equity value. However, unlike typical structural models, default of the firm may not necessarily imply an immediate default in equity, as is often observed in actual markets. This is a significant departure from the traditional structural framework. The jump effect on equity values in time of credit event would become particularly pronounced when evaluating the equity products under the latent firm model, as we will soon see in later sections.

From the construction of the equity value, it is clear that the pricing of any equity derivative under the current framework requires a prior knowledge on the joint distribution of X_t and Z_t at any time t. Although X_t and Z_t are conditionally independent given J_t, X_t and Z_t are in general *not* independent, as they share the same regime-switching process J_t. To ease computation, we shall adopt the change-of-measure technique for the case of regime-switching Lévy processes. Hence, before proceeding further, let us first investigate the necessary tools to justify the change-of-measure technique.

Lemma 2.1. Suppose that the function $g : [0, T] \times E \to \mathbb{R}$ is bounded and $g(\cdot, j)$ is continuous for each $j \in E$. Define, for $\zeta \in \mathbb{R}$,

$$V_t = \exp\left(g(t, J_t)t + \zeta Z_t\right).$$

Then, V_t is a $\mathbb{P}$-martingale with respect to $\mathcal{F}_t$ if and only if

$$g(t, J_t) = -\kappa^Z_{(J_t)}(\zeta). \tag{2.10}$$

Proof. See Appendix A.1

Define an equivalent probability measure $\tilde{\mathbb{P}}$ by its Radon-Nikodym derivative as follows:

$$\left.\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} \equiv V_t, \tag{2.11}$$

where $\mathcal{F}_t \equiv \{\sigma(Z_s),\ 0 \le s \le t\}$.

Denoting $\tilde{\mathbb{E}}_i[.]$ as an expectation operator under $\tilde{\mathbb{P}}$, we have the following results.

Corollary 2.1. Let $\zeta, \gamma \in \mathbb{R}$. Then,

$$\tilde{\mathbb{E}}_i\left[\exp\left(\kappa^Z_{(J_t)}(\zeta)t + \gamma X_t\right)\right] = \exp\left(\left(\mathbf{Q} + \{\kappa^Z_j(\zeta) + \kappa^X_j(\gamma)\}_{\text{diag}}\right)t\right) \tag{2.12}$$

and

$$\tilde{\mathbb{E}}_i\left[\exp\left(\gamma X_t\right)\right] = \exp\left(\left(\mathbf{Q} + \{\kappa^X_j(\gamma)\}_{\text{diag}}\right)t\right). \tag{2.13}$$

Proof. See Appendix A.2

Remark 2.1. Observe that, from Corollary 2.1, the distribution of X_t is unchanged under $\tilde{\mathbb{P}}$. This is in agreement with our intuition that X_t and Z_t are conditionally independent processes given J_t. As we will soon see, this change-of-measure formula greatly reduces the complexity when pricing equity options in our framework.

From equation (2.12), we can now easily derive the drift condition on the equity process so that $\mathrm{e}^{-rt}S_t$ is a martingale, i.e.

$$\mathbb{E}_i\left[\mathrm{e}^{-rt}\frac{S_t}{S_0}\right] = 1.$$

Let $\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)^\top \in \mathbb{R}^d$ is the unit vector with 1 in the i-th component and $\mathbf{1}_d = (1, \ldots, 1)^\top \in \mathbb{R}^d$.

By the definition of S_t, this is equivalent to finding the drift of S_t such that

$$\mathbf{e}_i^\top \mathbb{E}_i \left[\mathrm{e}^{-rt} \exp(\rho X_t + Z_t) \right] \mathbf{1}_d = 1,$$

for all $i \in E$. By invoking Corollary 2.1, we have

$$\mathbf{e}_i^\top \mathbb{E}_i \left[\mathrm{e}^{-rt} \exp(\rho X_t + Z_t) \right] \mathbf{1}_d = \mathbf{e}_i^\top \exp((\mathbf{Q} + \{\kappa_j^X(\rho) + \kappa_j^Z(1) - r\}_{\text{diag}})t)\mathbf{1}_d. \tag{2.14}$$

Following the proof of Lemma 2.1, it is clear that

$$\mathbf{e}_i^\top \exp((\mathbf{Q} + \{\kappa_j^X(\rho) + \kappa_j^Z(1) - r\}_{\text{diag}})t)\mathbf{1}_d = 1$$

if and only if

$$\kappa_j^X(\rho) + \kappa_j^Z(1) - r = 0.$$

We then have the following sufficient condition on S_t to preclude arbitrage opportunity:

$$\begin{aligned} \rho b_j^X + b_j^Z &= r - \frac{1}{2}(\rho\sigma_j^X)^2 - \frac{1}{2}(\sigma_j^Z)^2 \\ &\quad - \int_{\mathbb{R}} (\mathrm{e}^{\rho y} - 1 - \rho y 1_{\{|y| \le 1\}}) \nu_j^X(\mathrm{d}y) - \int_{\mathbb{R}} (\mathrm{e}^{y} - 1 - y 1_{\{|y| \le 1\}}) \nu_j^Z(\mathrm{d}y), \ j \in E. \end{aligned}$$

2.3. **Calibration procedure.** Before applying the latent model to the pricing of CDSs and equity options, we shall describe briefly the calibration procedures against the actual data in this subsection. Since we treat the actual firm value to be unobservable, our only source of information on the firm value is the equity process, via the relation (2.7), or equity derivatives.[3] Thus, before using the latent model to price the firm-related products, we must first calibrate the latent model against the equity data. After the successful calibration, we can then proceed to price firm-related products with the calibrated model.

Specifically, denote by N the number of observations and by $\mathbf{X} = (X_i, i = 1, ..., N)$ the vector of the equity (or its derivatives) values of length N, sampled from the equity market. Let Θ be the vector of the equity model parameters, and denote by $\widehat{\mathbf{X}}(\Theta) = (\widehat{X}_i(\Theta), i = 1, ..., N)$ the vector of the equity values of the model counterparts. Calibration procedure amounts to finding the optimal Θ^* such that the squared sum of the difference between $\mathbf{X}$ and $\widehat{\mathbf{X}}(\Theta)$ is minimized. In other words,

$$\Theta^* = \arg\min_{\Theta} \frac{1}{N} \sum_{i=1}^{N} (\widehat{X}_i(\Theta) - X_i)^2. \tag{2.15}$$

However, since the main scope of this paper is to study the economic underpinning behind the model's parameters, we have chosen the parameters *a priori* with the intention to illustrate specific effects. The actual performance of the calibrated latent model and the implications thereof are left for future research.

3. THE REGIME-SWITCHING DOUBLE-EXPONENTIAL JUMP MODEL

For practical use, it is important to derive analytic or semi-analytic solutions of derivative prices for the purpose of efficient computation. To this end, we shall assume that, for each regime $J_t = j$, $X_t^{(j)}$ follows the double-exponential jump-diffusion model, first developed by Kou (2002). The benefits of the Kou model are two-folds. First, the double-exponential jump-diffusion model allows two-sided jumps. This means that both positive and negative news can affect the actual firm and equity values. Second, we can derive the Laplace transform of the first passage time to the default threshold, which can be inverted numerically with ease. Thus, the assumption of the Kou process under each regime will maintain the necessary tractability for efficient computation of first-passage-time distributions; see Kou and Wang (2003). In the

[3] In Section 6, we consider the pricing of *defaultable* equity options.

following, for the sake of simplicity, we assume that E contains only 2 elements, i.e. $E = \{1,2\}$.

In this setting, from (2.2), the log-firm value process X_t is defined by

$$X_t = \int_0^t b^X(J_s)\mathrm{d}s + \int_0^t \sigma^X(J_s)\mathrm{d}W_s^X + \sum_{j\in E}\int_0^t 1_{\{J_s=j\}}\mathrm{d}N_s^X(j). \tag{3.1}$$

Given $J_t = j \in E$, b_j^X and σ_j^X are constants, and $\{N_t^X(j) : t \ge 0\}$ is a compound Poisson process with constant arrival rate λ_j^X and random jump sizes Y_j^X with distribution $\nu_j^X(\mathrm{d}y)$, where

$$\nu_j^X(\mathrm{d}y) = \lambda_j^X(p_j^X\eta_{j1}^X\mathrm{e}^{-\eta_{j1}^X y}1_{\{y\ge 0\}} + (1-p_j^X)\eta_{j2}^X\mathrm{e}^{\eta_{j2}^X y}1_{\{y<0\}})\mathrm{d}y \tag{3.2}$$

with $\eta_{j1}^X > 1$, $\eta_{j2}^X > 0$, and $0 \le p_j^X \le 1$.

The Laplace exponent (2.4) now takes the form

$$\mathbf{K}^X[u] \equiv \{\kappa_j^X(u)\}_{\text{diag}} + \mathbf{Q}, \tag{3.3}$$

where

$$\kappa_j^X(u) = b_j^X u + \frac{(\sigma_j^X u)^2}{2} + \lambda_j^X\left(\frac{p_j^X\eta_{j1}^X}{\eta_{j1}^X - u} + \frac{(1-p_j^X)\eta_{j2}^X}{\eta_{j2}^X + u} - 1\right). \tag{3.4}$$

3.1. **First-passage-time distribution for the case of two regimes.** To study the first-passage-time distribution under the regime-switching jump-diffusion model, we recall some results from Kijima and Siu (2011).

Define the first passage time τ by (2.6), and assume $X_0 > L$ and $J_0 = i$, $i = 1,2$. We want to calculate

$$\mathbb{E}_i[\mathrm{e}^{-a\tau+bX_\tau}; J_\tau | J_0 = i]$$

for $a > 0$ and $b \in \mathbb{R}\backslash\{\eta_{i1}^X, -\eta_{i2}^X,\ i = 1,2\}$.

To this end, we shall introduce few notations. Define

$$\pi_{(i,j)}^{(-,L)}[a] = \mathbb{E}_i\left[\mathrm{e}^{-a\tau}1_{\{J_\tau=j\, X_\tau<L\}}\right], \qquad \pi_{(i,j)}^{(0,L)}[a] = \mathbb{E}_i\left[\mathrm{e}^{-a\tau}1_{\{J_\tau=j\, X_\tau=L\}}\right].$$

For each l, $l = 1, ..., 4$, let $\varrho_{l,a}$ be the solutions[4] of the equation

$$(\kappa_1^X(u) - a - q_1)(\kappa_2^X(u) - a - q_2) = q_1 q_2,$$

where $\kappa_j^X(u)$ is defined in (3.4), such that

$$-\infty < \varrho_{1,a} < \varrho_{2,a} < \varrho_{3,a} < \varrho_{4,a} < 0.$$

Let

$$\gamma_l = \frac{\kappa_2(\varrho_{l,a}) - a - q_2}{q_2},$$

and define

$$h_{(1,0)}^l = \gamma_l, \quad h_{(1,-)}^l = \frac{\gamma_l\eta_{12}^X}{\eta_{12}^X + \varrho_{l,a}}; \qquad h_{(2,0)}^l = -1, \quad h_{(2,-)}^l = \frac{-\eta_{22}^X}{\eta_{22}^X + \varrho_{l,a}}. \tag{3.5}$$

Also, define

$$\mathbf{x} = \begin{pmatrix} \mathrm{e}^{-\varrho_{1,a}L}h_{(i,0)}^1 \\ \vdots \\ \mathrm{e}^{-\varrho_{4,a}L}h_{(i,0)}^4 \end{pmatrix}, \quad \boldsymbol{\pi} = \begin{pmatrix} \pi_{(i,1)}^{(-,L)} \\ \vdots \\ \pi_{(i,2)}^{(0,L)} \end{pmatrix}, \quad H = \begin{pmatrix} h_{(1,-)}^1 & \cdots & h_{(2,0)}^1 \\ \vdots & \ddots & \vdots \\ h_{(1,-)}^4 & \cdots & h_{(2,0)}^4 \end{pmatrix}. \tag{3.6}$$

According to Theorem 1 of Kijima and Siu (2011), we have

$$H\boldsymbol{\pi} = \mathbf{x}. \tag{3.7}$$

[4]See Kijima and Siu (2010) for the existence of these solutions.

Parallel to the case of regime-switching Brownian motion developed in Guo (2001), the matrix H takes the form of a special Alternant matrix, called the (squared) Vandermonde matrix, which is invertible as the roots in H are distinct. In order to obtain $\mathbb{E}_i[\mathrm{e}^{-a\tau+bX_\tau};\, J_\tau]$, define

$$\widehat{f}_{(j,-)}[b] = \frac{\eta^X_{j2}}{\eta^X_{j2}+b}\mathrm{e}^{bL}, \qquad \widehat{f}_{(j,0)}[b] = \mathrm{e}^{bL}$$

for $j = 1, 2$. Thanks to the conditional memoryless and independence properties, we have

$$\mathbb{E}_i[\mathrm{e}^{-a\tau+bX_\tau}1_{\{J_\tau=j,\, X_\tau<L\}}] = \pi^{(-,L)}_{(i,j)}\widehat{f}_{(j,-)}[b]$$

and

$$\mathbb{E}_i[\mathrm{e}^{-a\tau+bX_\tau}1_{\{J_\tau=j,\, X_\tau=L\}}] = \pi^{(0,L)}_{(i,j)}\widehat{f}_{(j,0)}[b].$$

Therefore, by invoking Corollary 2 of Kijima and Siu (2011), we obtain

$$\mathbb{E}_i[\mathrm{e}^{-a\tau+bX_\tau};\; J_\tau] = \sum_j\left(\pi^{(-,L)}_{(i,j)}\widehat{f}_{(j,-)}[b] + \pi^{(0,L)}_{(i,j)}\widehat{f}_{(j,0)}[b]\right). \tag{3.8}$$

In what follows, we shall see that the tractable first-passage-time distribution for the regime-switching double-exponential jump-diffusion greatly enhances the analysis of the credit and equity products within our framework.

4. Credit Default Swap

In this section, we consider the credit default swap (CDS for short) of a corporate firm. CDS is a bilateral contract between two parties, protection buyer and protection seller. In actual credit markets, liquidity of CDS's of corporate firms is significantly higher than that of the corporate bonds issued by firms.

As CDS has a nature of a swap, there are payment exchanges between two parties during the life of the contract. Upon entering a CDS contract with recovery rate R, notional amount N, and maturity T, the protection buyer will make payment c_T, known as the CDS premium, to the protection seller on the pre-specified payment dates, provided that the reference entity has not defaulted by the time of payment. If the reference entity defaults, the protection seller will pay the protection buyer the amount $(1-R)N$ and the contract terminates.

Assuming that there are no counterparty risks, the above description can be compactly summarized by the following two equations:

$$\begin{aligned}\text{Protection Buyer's Leg} &= \mathbb{E}_i\left[\int_0^T c^{(i)}_T N\mathrm{e}^{-rt}1_{\{\tau>t\}}\mathrm{d}t\right]\\ &= \mathbb{E}_i\left[\int_0^{T\wedge\tau}\mathrm{e}^{-rt}c^{(i)}_T N\mathrm{d}t\right]\\ &= c^{(i)}_T N\frac{\mathbb{E}_i[1-\mathrm{e}^{-r(\tau\wedge T)}]}{r}\\ &= c^{(i)}_T N\frac{1-\mathbb{E}_i[\mathrm{e}^{-r\tau}1_{\{\tau<T\}}]-\mathrm{e}^{-rT}\mathbb{P}_i(\tau>T)}{r}\end{aligned}$$

and

$$\begin{aligned}\text{Protection Payer's Leg} &= (1-R)N\int_0^T\mathrm{e}^{-rt}\mathrm{d}\mathbb{P}_i(\tau\le t)\\ &= (1-R)N\mathbb{E}_i\left[\mathrm{e}^{-r\tau}1_{\{\tau<T\}}\right],\end{aligned}$$

where $c^{(i)}_T$ denote the CDS premium under regime i and τ is defined as (2.6). To exclude any arbitrage opportunity, the CDS premium $c^{(i)}_T$ must be calculated by equating the protection buyer's leg and the protection seller's leg, so that

$$\begin{aligned} c_T^{(i)} &= (1-R)\frac{\int_0^T \mathrm{e}^{-rt}\mathrm{d}\mathbb{P}_i(\tau \le t)}{\int_0^T \mathrm{e}^{-rt}\mathbb{P}_i(\tau > t)\mathrm{d}t} \\ &= (1-R)r\frac{\mathbb{E}_i\left[\mathrm{e}^{-r\tau}1_{\{\tau<T\}}\right]}{1-\mathbb{E}_i[\mathrm{e}^{-r\tau}1_{\{\tau<T\}}] - \mathrm{e}^{-rT}\mathbb{P}_i(\tau > T)}. \end{aligned} \tag{4.1}$$

Observe that, by the definition of default time τ, the above formulation of the CDS only takes the actual firm value process into an account. This is the same as traditional structural models.

What makes the latent firm model different from the traditional models is that securities issued by the firm are embedded with credit quality of the firm. Although the actual firm value is unobservable, we can use the actively traded CDS premiums to extract credit quality of the firm.[5] This is in contrast with the intent of the original latent structural model in Kijima et al. (2009). In the original latent structural model, credit quality from the corporate bonds is extracted from the tangible asset correlated to the actual firm value. Tangible assets are the asset shown on the accounting book of the firm and thus can only be revealed during periodic corporate announcement. Hence, tangible assets on the accounting book are seen as the lagging factor on the credit quality of the firm. On the other hand, CDS represents not only the credit quality of the firm but also the market perception of the credit quality of the firm for some time in the *future*. Hence, CDS provides us the forward-looking indicator of the credit quality of the firm.

We shall now move to provide an explicit calculation of the CDS premium when X_t is the candidate process as in (3.1). To compute the CDS premium $c_T^{(i)}$ from (4.1), it is clear that $c_T^{(i)}$ can be computed once we compute the first-passage-time probability $\mathbb{P}_i(\tau > T)$ for each T. For the case of Brownian motion, this first-passage-time probability is well known and has a closed-form solution. Unfortunately, the closed-form solution is absent when one makes a departure from the Brownian motion framework. Nevertheless, by taking the Laplace transform of $\mathbb{P}_i(\tau > t)$ with respect to t yields the following result.

Before proceeding to compute CDS at any time t, let us first investigate the asymptotic behaviors of the CDS premium with respect to time. In particular, we shall now investigate the effect on the CDS premium as time approaches to 0 and ∞ under the general jump-diffusion processes with regime-switch. This analysis collectively can be seen as an extension of the results of Ruf and Scherer (2011) to include the regime-switching case.

Lemma 4.1. Denote $x = -\log(\frac{L}{A_0})$ and $J_0 = i$. Then, we have

$$\lim_{T\downarrow 0} c_T^{(i)} = (1-R)r\nu_i^X((-\infty, x]), \tag{4.2}$$

where $\nu^{(i)}$ denotes the Lévy measure under regime i.

Proof. See Appendix A.3.

Lemma 4.1 conveys a very important message. The non-zero credit spread is induced *only* by the Lévy measure under the initial regime J_0 and the diffusion component plays no role. Additionally, regime-switching intensity does *not* enter into the picture at $t = 0$. This is in line with the concept of holding time of a Markov chain presented in (2.1). Hence, contrary to the previous studies, CDS premium computed under the regime-switching Brownian motion alone cannot produce non-zero credit spread! This further justifies the role of Lévy measure in explaining the *short-term* behavior of the CDS curve.

[5]The default event of the CDS is essentially the default event of the underlying firm. High liquidity of the CDS market may provide us an updated inference of the actual firm value under our framework.

By the simple limiting argument, the almost surely finiteness of the default time τ, and the stationary distribution of the Markov chain J_t, we also have the following result. The proof is omitted.

Lemma 4.2. Assume $\mathbb{P}_i[\tau < \infty] = 1$ and $J_0 = i$. Then,

$$\lim_{T\to\infty} c_T^{(i)} = (1-R)r\frac{\mathbb{E}_\Pi[e^{-r\tau}]}{1-\mathbb{E}_\Pi[e^{-r\tau}]}, \tag{4.3}$$

where Π denotes the stationary distribution of the Markov chain J_t.

Contrary to the behavior of the CDS premium at $T \to 0$, the effect of the regime-switching enters into the picture in Lemma 4.2. As we shall soon see in the numerical examples, the presence of the regime-switching factor pulls the value of the CDS premium computed under the initial high and low regimes together and Lemma 4.2 provides a limiting value of which the prices of CDS under different regimes converge at the far end of the time horizon.

For the value of CDS at any time $t \in (0,\infty)$, we shall proceed to compute it via Laplace transform. For the rest of the paper, we shall denote by $\mathcal{L}(\cdot)$ and $\mathcal{L}^{-1}(\cdot)$ the Laplace transform and inverse Laplace transform operators, respectively. That is, for univariate functions $f(t)$ and $\hat{f}(\alpha)$, we denote

$$\mathcal{L}_\alpha(f(t)) = \int_{-\infty}^{\infty} \mathrm{e}^{-at} f(t)\mathrm{d}t, \qquad \mathcal{L}_t^{-1}(\hat{f}(\alpha)) = \lim_{R\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{-R}^{R}\mathrm{e}^{\alpha t}\hat{f}(\alpha)\mathrm{d}\alpha,$$

if the limit exists.

Similarly, for bivariate functions $g(x,y)$ and $\hat{g}(\alpha,\beta)$, we denote

$$\mathcal{L}_{\alpha,\beta}(g(x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\alpha x-\beta y}g(x,y)\mathrm{d}x\mathrm{d}y$$

and

$$\mathcal{L}_{x,y}^{-1}(\hat{g}(\alpha,\beta)) = \lim_{R'\to\infty}\lim_{R\to\infty}\frac{-1}{4\pi^2}\int_{-R'}^{R'}\int_{-R}^{R}\mathrm{e}^{\alpha x+\beta y}\hat{g}(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta,$$

if the limit exists.

Lemma 4.3. The Laplace transform of $\mathbb{P}_i(\tau > T)$ with respect to T is given by

$$\mathcal{L}_a[\mathbb{P}_i(\tau > T)] = \frac{1}{a} - \frac{\mathbb{E}_i\left[\mathrm{e}^{-a\tau}\right]}{a}, \tag{4.4}$$

whereas the Laplace transform of $\mathbb{E}_i[\mathrm{e}^{-r\tau}1_{\{\tau<T\}}]$ with respect to T is

$$\mathcal{L}_a[\mathbb{E}_i[\mathrm{e}^{-r\tau}1_{\{\tau<T\}}]] = \frac{\mathbb{E}_i[\mathrm{e}^{-(r+a)\tau}]}{a}. \tag{4.5}$$

Proof. Direct calculation shows that

$$\begin{aligned}
\mathcal{L}_a[\mathbb{P}(\tau > T)] &= \mathcal{L}_a\left[\mathbb{E}_i\left[1_{\{\tau>T\}}\right]\right] \\
&= \mathbb{E}_i\left[\int_0^\tau \mathrm{e}^{-aT}\mathrm{d}T\right] \\
&= \frac{1}{a} - \frac{\mathbb{E}_i\left[\mathrm{e}^{-a\tau}\right]}{a}
\end{aligned}$$

and

$$\mathcal{L}_a[\mathbb{E}_i[\mathrm{e}^{-r\tau}1_{\{\tau<T\}}]] = \mathbb{E}_i\left[\int_\tau^\infty \mathrm{e}^{-at-r\tau}\mathrm{d}T\right] = \frac{\mathbb{E}_i[\mathrm{e}^{-(r+a)\tau}]}{a}.$$

□

Hence, we can recover the values of $\mathbb{P}_i(\tau > T)$ and $\mathbb{E}_i[\mathrm{e}^{-r\tau}1_{\{\tau<T\}}]$ by performing the numerical Laplace inversion. In this paper, we adopt the Abate-Whitt (1992) algorithm to perform the numerical inversion.

4.1. **Regime-switching jump-diffusion model.** For the case of regime-switching jump-diffusion model with two regimes, by inserting equation (3.7) to Lemma 4.3, we have the following expression for the CDS premium, after inverting the Laplace transform.

Corollary 4.1. Assume that the actual firm value process follows the latent model described in (3.1) with $J_0 = i$. Then, the CDS premium is given by

$$c_T^{(i)} = (1-R)r\frac{P_2^{RS,i}}{1-P_2^{RS,i}-\mathrm{e}^{-rT}P_1^{RS,i}}, \quad i \in E, \tag{4.6}$$

where we define

$$P_1^{RS,i} = \mathcal{L}_T^{-1}\left(\frac{1}{a} - \frac{1}{a}\sum_{k=1}^{4}\mathrm{e}^{-\varrho_{k,a}L}h_{(i,0)}^{k}\left(\sum_{m=1}^{4}b_{mk}\right)\right)$$

with $B \equiv (b_{ij}) = H^{-1}$ and H being the form of (3.6), and where

$$P_2^{RS,i} = \mathcal{L}_T^{-1}\left(\frac{1}{a}\sum_{k=1}^{4}\mathrm{e}^{-\varrho_{k,a+r}L}h_{(i,0)}^{k}\left(\sum_{m=1}^{4}\tilde{b}_{mk}\right)\right)$$

with $\tilde{B} = (\tilde{b}_{ij}) = \tilde{H}^{-1}$ and $\tilde{H}$ being the form of (3.6) with r being replaced by $r+a$.

Proof. Since $B = (b_{ij}) = H^{-1}$, we have from (3.7) that $\boldsymbol{\pi} = B\mathbf{x}$. After some simple algebraic manipulation, we have

$$\mathbb{E}_i[\mathrm{e}^{-a\tau}; J_\tau] = \sum_{k=1}^{4}\mathrm{e}^{-\varrho_{k,a}L}h_{(i,0)}^{k}\left(\sum_{m=1}^{4}b_{mk}\right).$$

The result follows by noting from (4.4) that

$$\begin{aligned}\mathbb{P}_i(\tau > t) &= \mathcal{L}_T^{-1}\left(\frac{1}{a} - \frac{1}{a}\mathbb{E}_i[\mathrm{e}^{-a\tau;J_\tau}]\right)\\ &= \mathcal{L}_T^{-1}\left(\left(\frac{1}{a} - \frac{1}{a}\sum_{k=1}^{4}\mathrm{e}^{-\varrho_{k,a}L}h_{(i,0)}^{k}\left(\sum_{m=1}^{4}b_{mk}\right)\right)\right).\end{aligned}$$

The computation of $P_2^{RS,i}$ is similar by replacing r with $r+a$. □

4.2. **Randomized default barrier.** Up to now, we have assumed that the default barrier remains constant over time. In reality, the default barrier may vary as the creditworthiness of firm changes over time. In fact, the original CreditGrades model proposes the concept of randomized default barrier as means to cure the close-to-zero credit-spread when evaluating the CDS under a modified Black-Cox model, where the Brownian motion is the only random source. See the original CreditGrades model for the detailed coverage of the randomized default barrier under the modified Black-Cox model.[6]

For the case of regime-switching jump-diffusion model, the inclusion of randomized default barrier is straightforward. To see this, we revisit the computations of $\mathbb{P}_i(\tau < t)$ under the case of randomized barrier L.

[6]Sepps (2006) also considers the randomized default barrier as an extension of the extended CreditGrades model with jumps or uncorrelated stochastic volatility.

Corollary 4.2. Let L be the random default barrier with Laplace exponent $\Psi_L(u)$, $u \in \mathbb{R}$. If $J_0 = i$, then we have

$$\mathbb{P}_i(\tau < t) = \mathcal{L}_T^{-1}\left(\sum_{k=1}^{4} \mathrm{e}^{\Psi_L(-\varrho_{k,a})} h_{(i,0)}^k \left(\sum_{m=1}^{4} b_{mk}\right)\right).$$

Proof. Observe that

$$\begin{aligned}
\mathbb{E}_i[\mathrm{e}^{-a\tau}; J_\tau] &= \mathbb{E}_i[\mathbb{E}_i[\mathrm{e}^{-a\tau}; J_\tau | L]] \\
&= \sum_{k=1}^{4} \mathbb{E}_i[\mathrm{e}^{-\varrho_{k,a} L}] h_{(i,0)}^k \left(\sum_{m=1}^{4} b_{mk}\right) \\
&= \sum_{k=1}^{4} \mathrm{e}^{\Psi_L(-\varrho_{k,a})} h_{(i,0)}^k \left(\sum_{m=1}^{4} b_{mk}\right).
\end{aligned}$$

The result follows by noting that

$$\begin{aligned}
\mathbb{P}_i(\tau < t) &= \mathcal{L}_T^{-1}\left(\frac{1}{a}\mathbb{E}_i[\mathrm{e}^{-a\tau}; J_\tau]\right) \\
&= \mathcal{L}_T^{-1}\left(\frac{1}{a}\mathbb{E}_i[\mathbb{E}_i[\mathrm{e}^{-a\tau}; J_\tau | L]]\right) \\
&= \mathcal{L}_T^{-1}\left(\frac{1}{a}\sum_{k=1}^{4} \mathrm{e}^{\Psi_L(-\varrho_{k,a})} h_{(i,0)}^k \left(\sum_{m=1}^{4} b_{mk}\right)\right).
\end{aligned}$$

□

5. Equity Options

Since the celebrated Black-Scholes (1973) formula emerged, many security models have been introduced to alleviate the deficiency of the Brownian motion assumption and to capture more realistic market phenomena. Despite the plethora of equity pricing models available, most of them have no connection to the credit quality of the firm.[7] Because the underlying equity process is tied closely with the credit quality of the firm, as observed in the actual market, the option price itself should reflect the connection to the actual firm value.

In the case of structural models, on the other hand, many of them can be used for the pricing of equity options with default features; however, they have serious drawbacks. The firm by itself is not a tradable asset and the parameters of typical structural models, such as the mean return and volatility, are not directly observable. The original latent structural model in Kijima et al. (2009) remedies this defect by introducing the observable tangible asset of the firm as means to extract information on the actual firm value. In their framework, equity is then expressed as a residual amount after payment has been made to the debt holders in case of default or at maturity, whichever comes first. As mentioned in the introduction, this setting makes an implicit assumption of equity with a maturity. Moreover, the resulting formulation of the equity process takes rather complicated form that is difficult to price equity options even under the standard Brownian motion.

As introduced in Section 2, we embrace the idea in Kijima et al. (2009) by making use of the correlated marker process to induce credit information of a firm. More specifically, we shall use equity process as the correlated marker process of the actual firm value in the form of (2.7). As we shall soon see, this formulation provides us a great deal of convenience in pricing

[7] An exception is the jump-to-default model proposed by Carr et al. (2006) and later extended by Mendoza-Arriaga et al. (2010).

equity options with default feature. In practice, parameters of the equity process are imputed by calibrating the option price computed in this section against the option data in the market.

Consider a defaultable call option with strike price K and maturity T.[8] The call option price written on S is given by

$$\begin{aligned} C_i(S,K,T) &= \mathbb{E}_i[\mathrm{e}^{-rT}(S_T-K)^+ 1_{\{\tau>T\}}] \\ &= \mathbb{E}_i[\mathrm{e}^{-rT}(S_T-K)^+] - \mathbb{E}_i[\mathrm{e}^{-rT}(S_T-K)^+ 1_{\{\tau\le T\}}], \quad i\in E, \end{aligned} \tag{5.1}$$

or, equivalently,

$$\text{Defaultable call} = \text{Non-defaultable call} - \text{Down-and-in call},$$

so that the defautable call option can be replicated by holding one unit of a non-defaultable call option and selling one unit of a down-and-in call simultaneously. The major difference from the non-defaultable case is that the down-and-in feature is triggered by the latent firm value, *not* by the equity value itself.

Parallel to the methodology used in the pricing of CDS, we shall price the defaultable call option (5.1) via the Laplace transform. More precisely, let $k=-\log K$, and define $\mathbb{I}$ to be an identity matrix. The non-defaultable call option is priced via the Laplace transform with respect to k, whereas the down-and-in call option is given in terms of the *double* Laplace transform with respect to k and T.

Theorem 5.1. Let $\xi,\beta\in\mathbb{R}$ satisfy

$$0<\xi<\min_{j\in E}\left\{\frac{\eta^X_{j2}}{\rho}-1\right\},\ \ \min_{j\in E}\{\kappa^X_j(\rho(\xi+1))+\kappa^Z_j(\xi+1)\}>0,$$

$$\beta>\max_{j\in E}\{\max\{\kappa^Z_j(\xi+1)-r,0\}\}.$$

Then, the Laplace transform of $\mathbb{E}_i[\mathrm{e}^{-rT}(S_T-K)^+]$ with respect to k is given by

$$\begin{aligned} &\mathcal{L}_\xi(\mathbb{E}_i[\mathrm{e}^{-rT}(S_T-K)^+]) \\ &= \frac{\mathrm{e}^{-rT}S_0^{\xi+1}}{\xi(\xi+1)}\sum_j\left\{\exp\left(\left(\{\kappa^Z_j(\xi+1)+\kappa^X_j(\rho(\xi+1))\}_{\text{diag}}+\mathbf{Q}\right)T\right)\right\}_{ij} \end{aligned} \tag{5.2}$$

and the double Laplace transform of $\mathbb{E}_i[\mathrm{e}^{-rT}(S_T-K)^+1_{\{\tau\le T\}}]$ with respect to k and T is obtained as

$$\begin{aligned} &\mathcal{L}_{\xi,\beta}(\mathbb{E}_i[\mathrm{e}^{-rT}(S_T-K)^+1_{\{\tau\le T\}}]) \\ &= \frac{S_0^{\xi+1}}{\xi(\xi+1)}\sum_j\tilde{\mathbb{E}}_i\left[\mathrm{e}^{-((\beta+r)-\kappa^Z_j(\xi+1))\tau+(\xi+1)\rho X_\tau}1_{\{J_\tau=j\}}\right] \\ &\quad\times\sum_n\left((r+\beta)\mathbb{I}-\left(\left\{\kappa^Z_j(\xi+1)+\kappa^X_j(\rho(\xi+1))\right\}_{\text{diag}}+\mathbf{Q}\right)\right)^{-1}_{jn},\ \ i\in E, \end{aligned} \tag{5.3}$$

where $\tilde{\mathbb{E}}_i$ is the expectation under $\tilde{\mathbb{P}}_i$ defined by (2.11).

Proof. For the first part, we take the Laplace transform with respect to k. For $\xi>0$, in order to solve for

$$\mathbb{E}_i\left[\mathrm{e}^{-rT}S_0(\exp(\rho X_T+Z_T)-K)^+\right],$$

[8] Johnson and Stulz (1987) consider a *vulnerable* option whose payoff is given by $\min\{\hat{V}_T,(S_T-K)^+\}$, where $\hat{V}$ is the value of the total assets of issuer. In our model, default is triggered by the actual firm value and no counterparty risk of the issuer is considered.

define $k = -\ln(K)$ and take the single Laplace transform with respect to k. Then we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{e}^{-\xi k} \sum_j \mathbb{E}_i[\mathrm{e}^{-rT}(S_0 \mathrm{e}^{\rho X_T + Z_T} - \mathrm{e}^{-k})^+; J_T = j]\mathrm{d}k \\
= \;& \mathrm{e}^{-rT} \sum_j \mathbb{E}_i \left[\int_{-\ln S_0 - \rho X_T - Z_T}^{\infty} \mathrm{e}^{-\xi k}(S_0 \mathrm{e}^{\rho X_T + Z_T} - \mathrm{e}^{-k})\mathrm{d}k; J_T = j \right] \\
= \;& \mathrm{e}^{-rT} \sum_j \mathbb{E}_i \left[\int_{-\ln S_0 - \rho X_T - Z_T}^{\infty} S_0 \mathrm{e}^{\rho X_T + Z_T} e^{-\xi k} - \mathrm{e}^{-(\xi+1)k}\mathrm{d}k; J_T = j \right] \\
= \;& \mathrm{e}^{-rT} \sum_j \mathbb{E}_i \left[\frac{S_0^{\xi+1} \mathrm{e}^{(\xi+1)(\rho X_T + Z_T)}}{\xi} - \frac{S(0)^{\xi+1} \mathrm{e}^{(\xi+1)(\rho X_T + Z_T)}}{\xi+1}; J_T = j \right] \\
= \;& \frac{\mathrm{e}^{-rT} S_0^{\xi+1}}{\xi(\xi+1)} \sum_j \mathbb{E}_i \left[\mathrm{e}^{(\xi+1)\rho X_T + (\xi+1) Z_T}; J_T = j \right] \\
= \;& \frac{\mathrm{e}^{-rT} S_0^{\xi+1}}{\xi(\xi+1)} \sum_j \tilde{\mathbb{E}}_i \left[\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \mathrm{e}^{(\xi+1)\rho X_T + (\xi+1) Z_T}; J_T = j \right] \\
= \;& \frac{\mathrm{e}^{-rT} S_0^{\xi+1}}{\xi(\xi+1)} \sum_j \tilde{\mathbb{E}}_i \left[\mathrm{e}^{\kappa_j^Z(\xi+1)T + \rho(\xi+1)X_T}; J_T = j \right] \\
= \;& \frac{\mathrm{e}^{-rT} S_0^{\xi+1}}{\xi(\xi+1)} \sum_j \left\{ \exp\left(\left(\{\kappa_j^Z(\xi+1) + \kappa_j^X(\rho(\xi+1))\}_{\text{diag}} + \mathbf{Q} \right) T \right) \right\}_{ij},
\end{aligned}
$$

where the sixth equality follows by invoking the Randon-Nikodym derivative of the form of (2.11).

Next, the second part (5.3) is proved by taking the double Laplace transform with respect to k and T. To solve for $\mathbb{E}_i \left[\mathrm{e}^{-rT} S_0(\exp(\rho X_T + Z_T) - K)^+ 1_{\{\tau < T\}} \right]$, define $k = -\ln(K)$ and take the double Laplace transform with respect to k and T. Then, we have

$$
\begin{aligned}
&\int_0^{\infty} \mathrm{e}^{-\beta T} \int_{-\infty}^{\infty} \mathrm{e}^{-\xi k} \mathbb{E}_i[\mathrm{e}^{-rT}(S_0 \mathrm{e}^{\rho X_T + Z_T} - \mathrm{e}^{-k}) 1_{\{\tau \le T,\, -\ln S_0 - \rho X_T - Z_T < k\}}]\mathrm{d}k\mathrm{d}T \\
= \;& \mathbb{E}_i \left[\int_{\tau}^{\infty} \mathrm{e}^{-(\beta+r)T} \int_{-\ln S_0 - \rho X_T - Z_T}^{\infty} \mathrm{e}^{-\xi k}(S_0 \mathrm{e}^{\rho X_T - Z_T} - \mathrm{e}^{-k})\mathrm{d}k\mathrm{d}T \right] \\
= \;& \tilde{\mathbb{E}}_i \left[\int_{\tau}^{\infty} \mathrm{e}^{-(\beta+r)T} \frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \frac{S_0^{\xi+1} \mathrm{e}^{(\xi+1)Z_T + \rho(\xi+1)X_T}}{\xi(\xi+1)} \mathrm{d}T \right] \\
= \;& \tilde{\mathbb{E}}_i \left[\int_{\tau}^{\infty} \mathrm{e}^{-(\beta+r)T} \frac{S_0^{\xi+1} \mathrm{e}^{\kappa_{(J_T)}^Z(\xi+1)T + \rho(\xi+1)X_T}}{\xi(\xi+1)} \mathrm{d}T \right] \\
= \;& \frac{S_0^{\xi+1}}{\xi(\xi+1)} \tilde{\mathbb{E}}_i \left[\int_{\tau}^{\infty} \mathrm{e}^{-(\beta+r-\kappa_{(J_T)}^Z(\xi+1))T + \rho(\xi+1)X_T} \mathrm{d}T \right] \\
= \;& \frac{S_0^{\xi+1}}{\xi(\xi+1)} \tilde{\mathbb{E}}_i \Big[\mathrm{e}^{-(\beta+r-\kappa_{(J_\tau)}^Z(\xi+1))\tau + \rho(\xi+1)X_\tau} \\
& \times \int_0^{\infty} \mathrm{e}^{-(\beta+r)s} \tilde{\mathbb{E}} \left[\exp\left(\kappa_{(J_{\tau+s})}^Z(\xi+1)(\tau+s) - \kappa_{(J_\tau)}^Z(\xi+1)\tau + \rho(\xi+1)(X_{\tau+s} - X_\tau) \right) |\mathcal{F}_\tau \right] \mathrm{d}s \Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{S_0^{\xi+1}}{\xi(\xi+1)}\tilde{\mathbb{E}}_i\left[\mathrm{e}^{-((\beta+r)-\kappa^Z_{(J_\tau)}(\xi+1)\tau+(\xi+1)X_\tau}\right.\\
&\quad \left.\times \int_0^\infty \mathrm{e}^{-(\beta+r)s}\tilde{\mathbb{E}}_{J_\tau}\left[\exp\left(\kappa^Z_{(J_s)}(\xi+1)s+\rho(\xi+1)X_s\right)|\mathcal{F}_\tau\right]\mathrm{d}s\right]\\
&= \frac{S_0^{\xi+1}}{\xi(\xi+1)}\tilde{\mathbb{E}}_i\left[\mathrm{e}^{-((\beta+r)-\kappa^Z_{(J_\tau)}(\xi+1))\tau+\rho(\xi+1)X_\tau}\right.\\
&\quad \left.\times \int_0^\infty \mathrm{e}^{-(\beta+r)s\mathbb{I}}\exp\left(\left(\left\{\kappa^Z_k(\xi+1)+\kappa^X_k(\rho(\xi+1))\right\}_{\text{diag}}+\mathbf{Q}\right)s\right)\mathrm{d}s\right]\\
&= \frac{S_0^{\xi+1}}{\xi(\xi+1)}\sum_j \tilde{\mathbb{E}}_i\left[\mathrm{e}^{-((\beta+r)-\kappa^Z_j(\xi+1))\tau+\rho(\xi+1)X_\tau}1_{\{J_\tau=j\}}\right]\\
&\quad \times \sum_n\left((r+\beta)\mathbb{I}-\left(\left\{\kappa^Z_k(\xi+1)+\kappa^X_k(\rho(\xi+1))\right\}_{\text{diag}}+\mathbf{Q}\right)\right)^{-1}_{jn},
\end{aligned}
$$

completing the proof. □

5.1. **Randomized default barrier.** Parallel to the pricing of CDS, the concept of randomized default barrier can also be applied to the case of equity options. Assume that X^J_t is the regime-switching jump-diffusion model (3.1) with two regimes. Then, we have the following.

Corollary 5.1. Let L be the random default barrier with Laplace exponent $\Psi_L(u)$, $u\in\mathbb{R}$. Then, the price of defaultable call option is given by

$$
\begin{aligned}
& C(S,K,T)\\
=\ & \mathcal{L}^{-1}_k\left(\frac{\mathrm{e}^{-rT}S_0^{\xi+1}}{\xi(\xi+1)}\sum_j\left\{\exp\left(\left(\{\kappa^Z_j(\xi+1)+\kappa^X_j(\rho(\xi+1))\}_{\text{diag}}+\mathbf{Q}\right)T\right)\right\}_{ij}\right)\\
-\ & \mathcal{L}^{-1}_{k,T}\left(\frac{S_0^{\xi+1}}{\xi(\xi+1)}\left[\left(\sum_{k=1}^4 b_{1k}\mathrm{e}^{\Psi_L(b-\varrho_{k,a})}h^k_{(i,0)}\right)\frac{\eta_{12}}{\eta_{12}+b}+\left(\sum_{k=1}^4 b_{2k}\mathrm{e}^{\Psi_L(b-\varrho_{k,a})}h^k_{(i,0)}\right)\right]\sum_{n=1}^2\mathbf{A}_{1n}\right.\\
& \left.+\frac{S_0^{\xi+1}}{\xi(\xi+1)}\left[\left(\sum_{k=1}^4 b_{3k}\mathrm{e}^{\Psi_L(b-\varrho_{k,a})}h^k_{(i,0)}\right)\frac{\eta_{22}}{\eta_{22}+b}+\left(\sum_{k=1}^4 b_{4k}\mathrm{e}^{\Psi_L(b-\varrho_{k,a})}h^k_{(i,0)}\right)\right]\sum_{n=1}^2\mathbf{A}_{2n}\right),
\end{aligned}
$$

where

$$
a=(\beta+r)-\kappa^Z_j(\xi+1),\quad \mathbf{A}=\left((r+\beta)\mathbb{I}-\left(\left\{\kappa^Z_k(\xi+1)+\kappa^X_k(\rho(\xi+1))\right\}_{\text{diag}}+\mathbf{Q}\right)\right)^{-1}.
$$

Proof. Note that the random term L only appears in

$$
\mathbb{E}_i\left[\mathrm{e}^{(\kappa^Z_j(\xi+1)-(\beta+r))\tau+\rho(\xi+1)X^J_\tau}1_{\{J_\tau=j\}}\right].
$$

By combining the results of (3.8) and (5.1) and defining

$$
a=-\kappa^Z_j(\xi+1)+(\beta+r),\qquad b=\rho(\xi+1),
$$

TABLE 1. Base case parameters for the firm-value process X_t: Kou (2002) Model with two regimes

V_0	T	r	b_1^X	σ_1^X	η_{11}^X	η_{12}^X	p_1^X	λ_1^X	L	R	b_2^X	σ_2^X	η_{21}^X	η_{22}^X	λ_2^X	p_2^X	q_1	q_2
100	1	0.05	0.05	0.4	3	2	0.5	0.5	30	0.5	0.05	0.2	8	6	1	0.6	0.5	0.5

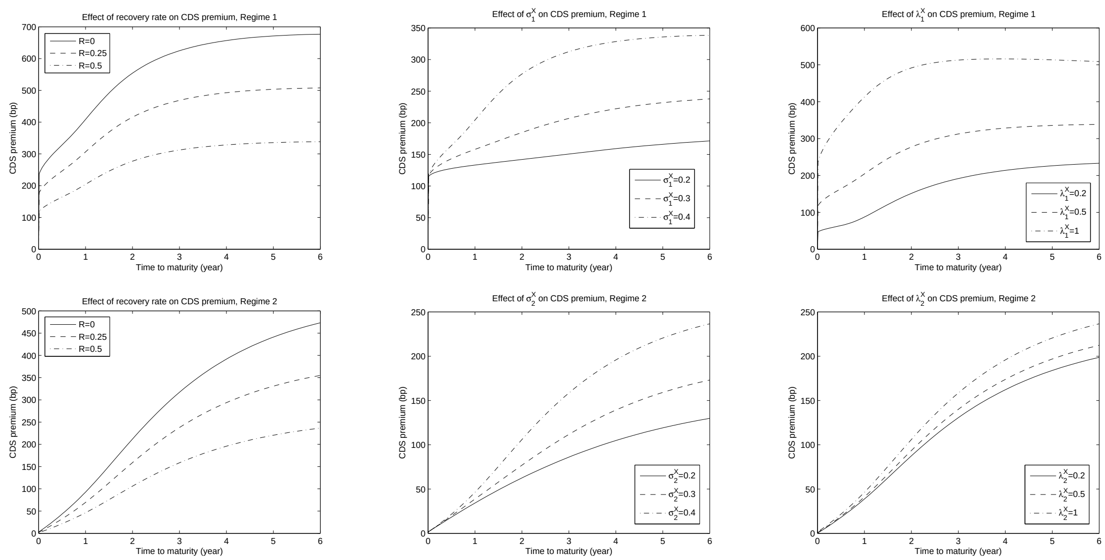

FIGURE 1. Impact of various parameters on the CDS premium under Kou process (2002) with two regimes: Regime 1 (First Row) and Regime 2 (Second Row)

we have

$$\begin{aligned}
\mathbb{E}_i\left[\mathrm{e}^{-a\tau+bX_\tau^J}; J_\tau\right] &= \mathbb{E}_i\left[\mathbb{E}_i\left[\mathrm{e}^{-a\tau+bX_\tau^J}; J_\tau | L\right]\right] \\
&= \left(\sum_{k=1}^{4} b_{1k}\mathrm{e}^{\Psi_L(b-\varrho_{k,a})}h_{(i,0)}^k\right)\frac{\eta_{12}}{\eta_{12}+b} \\
&\quad + \left(\sum_{k=1}^{4} b_{2k}\mathrm{e}^{\Psi_L(b-\varrho_{k,a})}h_{(i,0)}^k\right) + \left(\sum_{k=1}^{4} b_{3k}\mathrm{e}^{\Psi_L(b-\varrho_{k,a})}h_{(i,0)}^k\right)\frac{\eta_{22}}{\eta_{22}+b} \\
&\quad + \left(\sum_{k=1}^{4} b_{4k}\mathrm{e}^{\Psi_L(b-\varrho_{k,a})}h_{(i,0)}^k\right).
\end{aligned}$$

The result follows by plugging the above equation into (5.1). □

6. NUMERICAL EXAMPLES

In this section, we assume that the model parameters are chosen a priori with the intention to illustrate the parameter effect on the prices of CDS and defaultable equity options.

6.1. **CDS.** In this subsection, we shall study the impact of the regime-switching and jump-diffusion components of X_t to the CDS premium. The subsequent numerical studies are performed according to the base case parameters in Table 1. From Table 1, it is clear that Regime 1 indicates the regime under which X_t has both high volatility and infrequent jumps with big jump sizes, whereas in Regime 2, X_t has small volatility and relatively frequent jumps with moderate jump sizes.

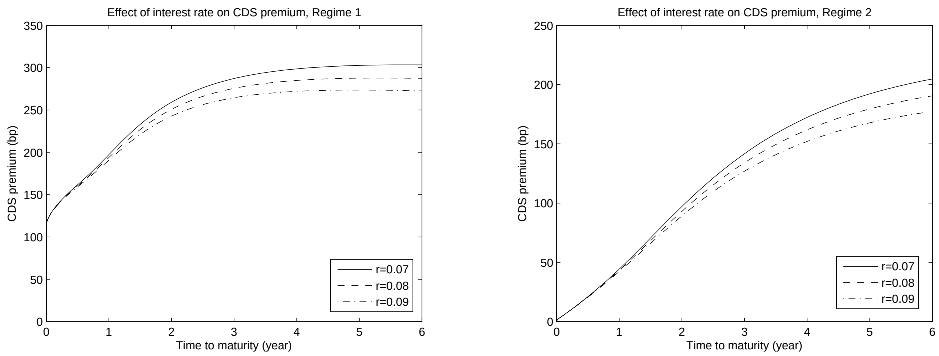

FIGURE 2. Impact of interest rate on the CDS premium under Kou process (2002) with two regimes: Regime 1 (Left) and Regime 2 (Right)

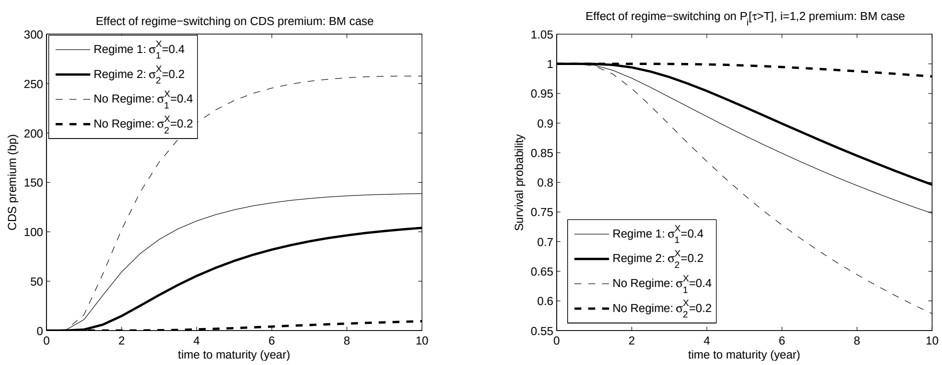

FIGURE 3. Impact of regime-switching on the CDS premium under Brownian motion with two regimes: CDS premium (Left) and Survival Probability (Right)

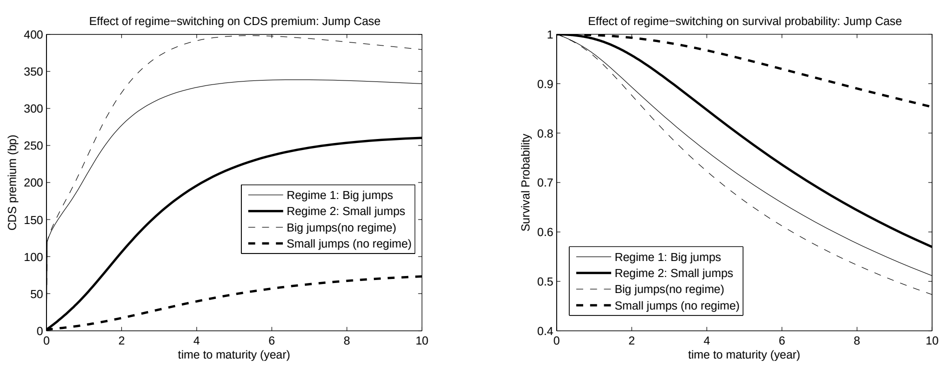

FIGURE 4. Impact of regime-switching to the CDS premium under Kou (2002) process with two regimes $(\sigma_1^X = \sigma_2^X = 0.2)$: CDS premium (Left) and Survival Probability (Right)

Figure 1 summarizes the impact of recovery rate, diffusion volatility, and the jump frequency of the firm value process X_t to the CDS premium for the cases that the process X_t starts from high and low regimes. The general patterns of CDS curves are in a close agreement with those found in Chen and Kou (2009). Decreasing recovery rate R has an upward-level effect on

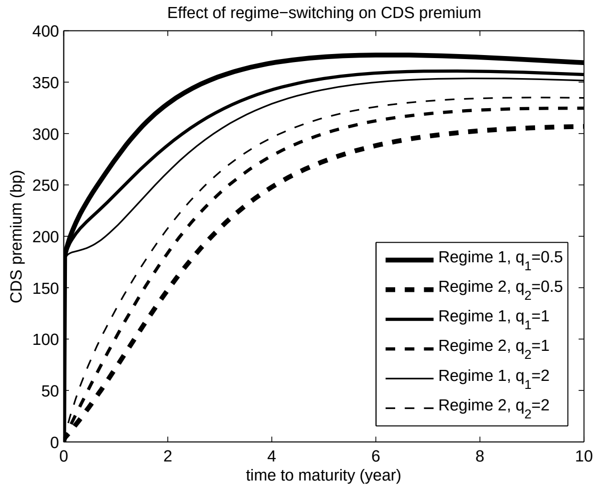

FIGURE 5. Effects of switching intensity to the CDS premium under Kou process (2002) with two regimes: The CDS curve denoted with regime i indicates that the initial regime of the firm is i.

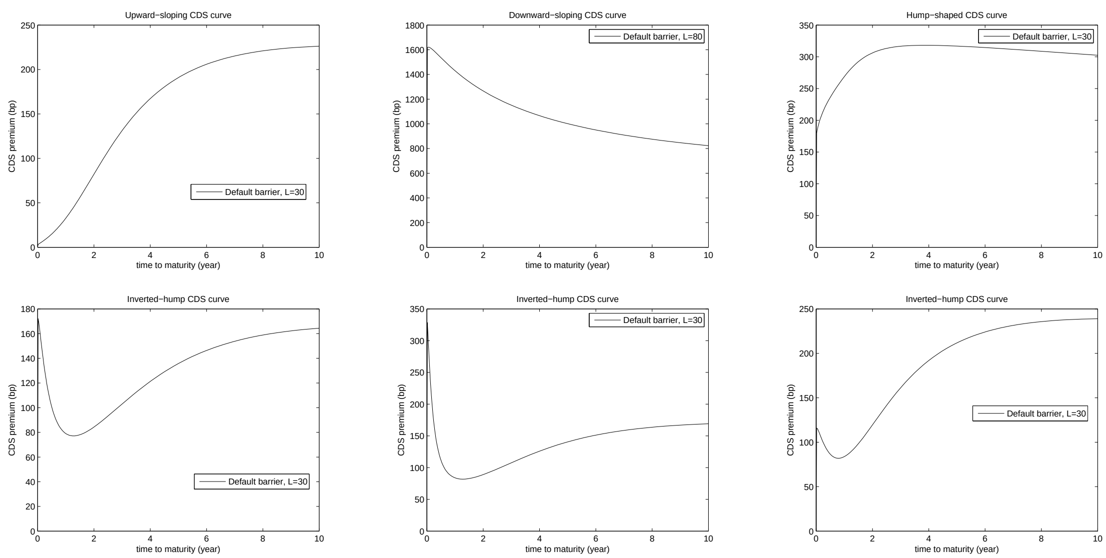

FIGURE 6. Various shapes of the CDS curve under Kou process (2002) with two regimes

the CDS curve, since the amount issuer is required to pay upon default is greater; the CDS premium must then be higher to preclude any arbitrage opportunity. Diffusion volatility indicates the volatile behavior of the firm and thus high σ_i^X, $i = 1, 2$, would imply higher CDS premium. Similar effects are observed in the case of higher jump frequencies.

Another important factor that impacts the CDS premium is the level of risk-free interest rate. The interest rate effect on the CDS premium for Regimes 1 and 2 is displayed in Figure 2. Higher interest rate leads to lower CDS premium. This is consistent with the empirical studies mentioned in Chen and Kou (2009).

To study the impact of the regime-switching factor to the CDS premium, we turn to Figures 3 and 4, where the underlying process in Figure 3 is the Brownian motion alone and the process in Figure 4 is the Kou process (2002). The general pattern of regime-switching on the left-hand sides of Figures 3 and 4 is that the introduction of regime-switching pulls the CDS premium in the Regimes 1 and 2 closer together, with the difference between them diminishing as the maturity of CDS lengthens. In fact, their values converge to the common value predicted in Lemma 4.2. The reason behind the convergence of CDS premiums under Regimes 1 and 2 can be explained by the survival probability curves on the right-hand sides of Figures 3 and 4. The effect of regime-switching elevates the survival probability under the Regime 1 while it decreases the survival probability under the Regime 2. Higher survival probability under Regime 1 results in lowering its CDS premium, and the opposite impact occurs for the case of Regime 2.

Figures 3 and 4 also indicate that the regime-switching factor has more influence on the medium maturities than the short maturities. In fact, Figure 3 indicates that the regime-switching Brownian motion alone cannot produce the non-zero short-term credit spreads, as predicted in Lemma 4.1. All CDS curves are emanated from the origin when the firm value is modeled by the regime-switching Brownian motion alone. On the other hand, Figure 4 studies the case of regime-switching factor with jumps under each regime, while leaving the diffusion volatility to be constant across both regimes. Figure 4 has an interesting observation that the CDS premium at the near-zero maturity region coincides with the jump-diffusion process without any regime-switching effect, other things equal. In summary, Figures 3 and 4 serve as the visual justification of Lemma 4.1.

Figure 5 studies the effect of switching intensity on the CDS premium. Figure 5 shows that increasing the regime-switching intensity would result in the CDS curves under initial Regimes 1 and 2 converging to a common value. The common value is precisely computed in Lemma 4.2. Moreover, observe that when the regime-switching intensities are fixed, the CDS curves under initial Regimes 1 and 2 also converge to a common value, albeit at a lower speed for the case of small switching intensities. We shall return to this general pattern later when we study the impact of the regime-switching intensities to the implied volatilities of defaultable European options.

He et al. (2000) provide an in-depth empirical investigation on the general shapes of the credit spread curves. By grouping the corporate bonds with various maturities, they classify various shapes that the credit spread curve can take under different credit-ratings. Hence, it is of great interest to investigate if the model proposed in this paper can generate the credit spread curves taking similar shapes as those in He et al. (2000). Figure 6 provides various shapes of the CDS curves. On the first row of Figure 6, we recover the shapes of CDS curves similar to those found in Chen and Kou (2009). More specifically, the upward-sloped CDS curve corresponds to the CDS with a high credit-rating. Downward-sloped CDS portrays the CDS with relatively low credit-rating and hump-shaped CDS captures the CDS with the credit-rating that lies in between.

In addition to the hump-shaped CDS curves, the study of He et al. (2000) indicates that other shapes of the CDS curves can also be found, albeit less frequently observed than the hump-shaped curves. In particular, from the study of He et al. (2000), it appears that credit spread curve with medium credit ratings can also have the inverted-hump shapes. Second row of Figure 6 displays the CDS curve with an inverted-hump shapes. These inverted-hump shapes are generated from the asymmetric regime-switching intensities. More specifically, by increasing the switching intensity of Regime 1, q_1, the firm value process leaves Regime 1 more frequently than it does in Regime 2. From the parameters in Table 1, Regime 2 refers to the state of which firm has a relatively lower total volatility. With $q_1 > q_2$, the firm spends more time on average in a lower volatility environment than it does in a higher volatility region. This results in a lower CDS term structure, even when the firm was initially at the high volatility state (Regime 1) at the inception of the CDS contract. Interestingly, the depth of the inverted-hump is determined only by the regime-switching intensity of the Markov chain J_t. Higher the regime-switching intensity of Regime 1, q_1, deeper the inverted-hump shape would result. This translates into the fact that the persistence of the firm in a particular regime is influential to the medium part of the

TABLE 2. Base case parameters: Kou (2002) Model

S_0	K	V_0	T	r	b^X	σ_X	η_1^X	η_2^X	p^X	λ^X	σ_Z	η_1^Z	η_2^Z	p^Z	λ^Z	L
100	90	100	1	0.05	0.05	0.4	10	4	0.4	0.5	0.1	40	40	0.6	3	30

TABLE 3. Defaultable European call prices with varying K and L, respectively (LT=Laplace Transform, MC=Monte Carlo): Kou model (2002)

$(L = 30)$	$\rho = 0.5$		$\rho = 1$	
K	LT	MC	LT	MC
50	52.3496	52.3743	53.2586	53.2084
60	43.0395	43.0833	44.8539	44.8087
70	33.9613	33.9352	37.2213	37.1980
80	25.5212	25.5580	30.4963	30.4254
90	18.1890	18.1917	24.7249	24.7196
100	12.2968	12.2479	19.8787	19.8668
120	4.8799	4.8658	12.6252	12.6182
$(K = 90)$	$\rho = 0.5$		$\rho = 1$	
L	LT	MC	LT	MC
10	18.1894	18.2062	24.7249	24.7147
30	18.1890	18.1229	24.7249	24.6617
50	18.0799	18.0637	24.6979	24.6470
70	15.9291	15.9271	23.2404	23.1921

TABLE 4. Defaultable European call prices with varying ρ: Kou model (2002)

ρ	0	0.1	0.2	0.3	0.4	0.5	0.6
	14.5801	14.8855	15.4048	16.1484	17.0930	18.1890	19.3907

ρ	0.7	0.8	0.9	1
	20.6646	21.9881	23.3452	24.7249

CDS term structure. Intuitively, in addition to the present credit condition of the firm, investors also take their perception on the average time the firm staying in the present condition when purchasing the CDS.

Similar results are also found in the work of Hatgioannides and Petropoulos (2007).They observe the inverted-hump shaped credit curves in the European corporate bond markets for the corporate bonds with credit ratings between AA- and B. The depth in inverted-hump shapes is more pronounced in BB and B rating bonds. They argue that these complex credit curves cannot be generated by stochastic processes with constant volatility. To capture the stochastic behavior of the credit spread dynamics, Hatgioannides and Petropoulos (2007) model the credit spread dynamics directly by means of a two-factor Longstaff and Schwartz (1992) process. They conclude that the role of the stochastic volatility in a credit spread model results in better fitting to the credit curves observed in the market. In this paper, we provide one explanation behind these complex CDS curves via the regime-switching jump-diffusion model.

6.2. **Equity Options.**

In this subsection, we shall study the effects of parameters of the model without and with regime switching to defaultable equity options through numerical examples. We investigate the equity options under the case when both X_t and Z_t take the form of the double-exponential jump-diffusion processes. Observe that the choice of Z_t here is not a strict necessity. In fact,

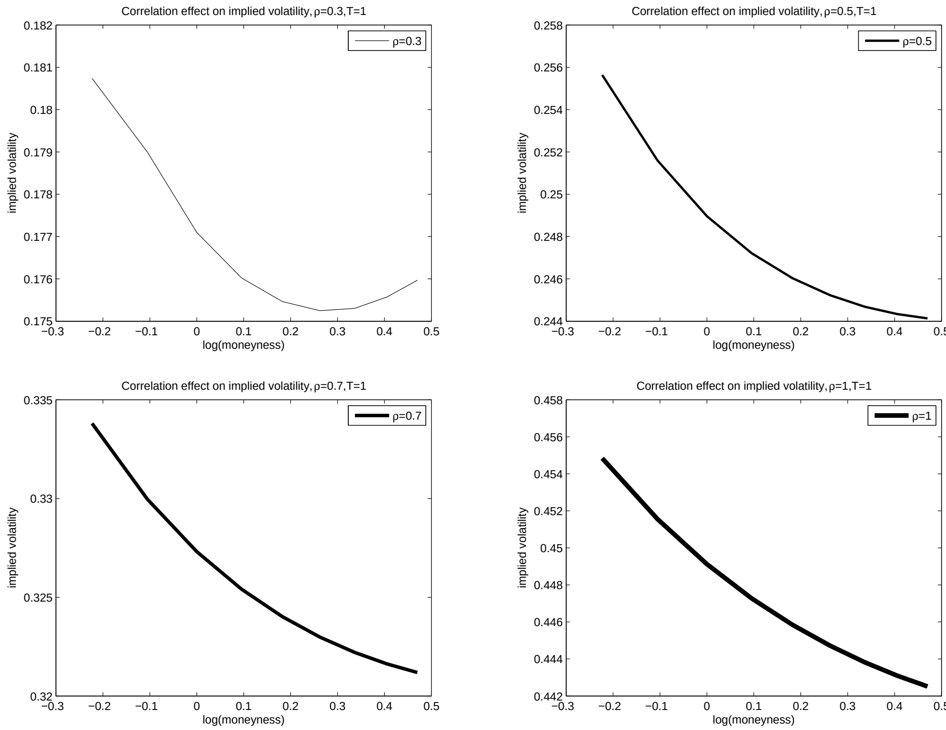

FIGURE 7. Effects of correlation ρ to the implied volatility of a defaultable European call: Kou model (2002)

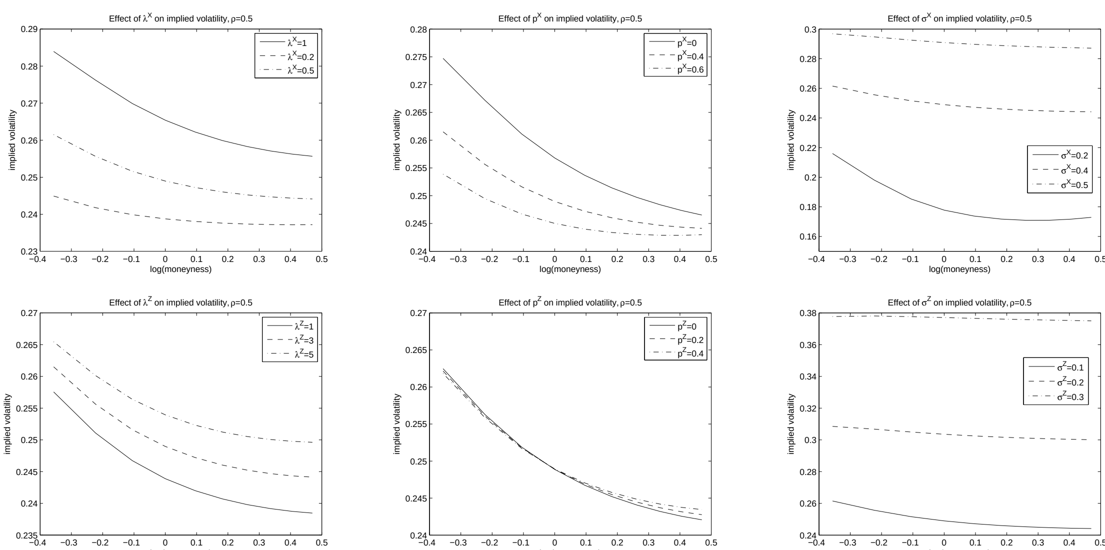

FIGURE 8. Effects of various parameters to the implied volatility of a defaultable European call ($\rho = 0.5$): Kou model (2002)

Z_t can be any Lévy process and we just choose Z_t to take the form of Kou process (2002) for simplicity.

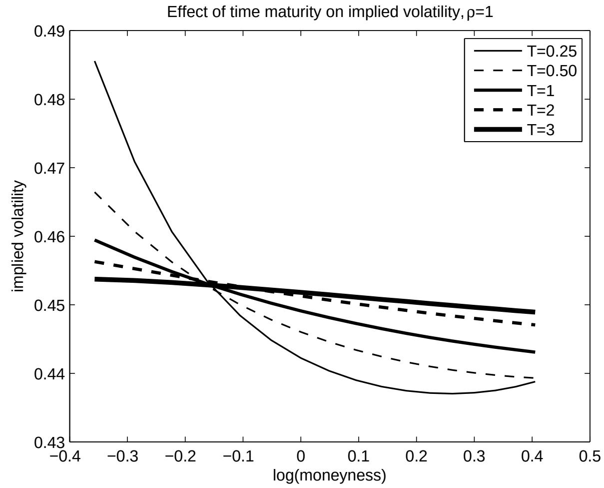

FIGURE 9. Effects of time-to-maturity to the implied volatility of a defaultable European call: Kou model (2002)

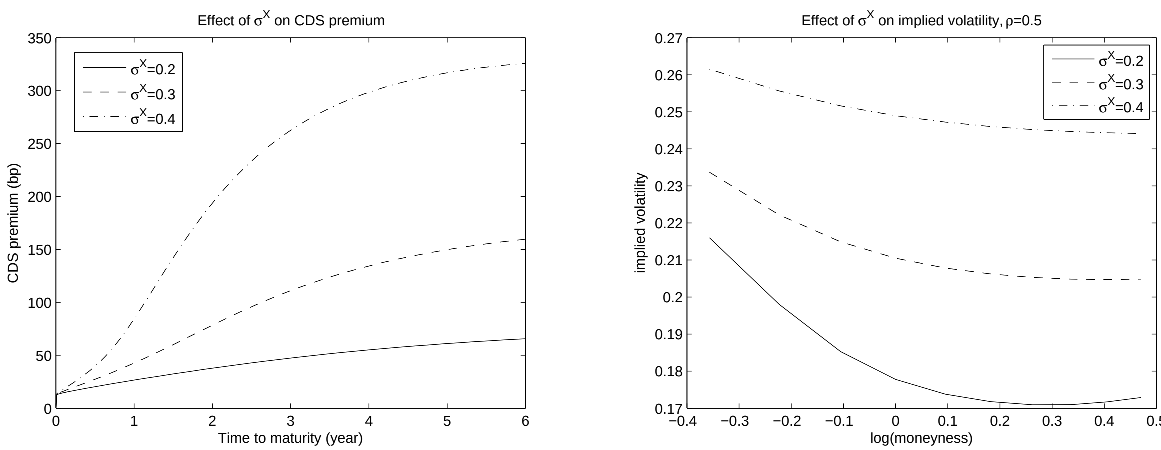

FIGURE 10. Effect of σ_X to CDS premium and the defaultable European call: CDS (left) and defaultable European call (right) in Kou model (2002)

6.2.1. *Without Regime switching.* Unless otherwise stated, the numerical examples in this subsection are performed in accordance to the parameters given in Table 2.[9] In Table 2, the parameters of X_t indicate its dominant role in the equity process. This is logical as the firm value should have a critical influence on its equity value. The jump sizes induced by X_t are bigger than those induced by Z_t. This can be reasoned as jumps from X_t represent corporate restructuring or change of its credit rating, whereas jumps from Z_t denote the sudden change of the investor's preference to the equity due to some non-firm specific factors.

Table 3 provides the prices of the defaultable European call with various values of strike price K and default barrier L. The computations are done by both the numerical Laplace inversion of the result in Theorem 5.1 and by the Monte Carlo simulations.[10] The closeness in values

[9]Observe that the drift term b^Z is left unspecified. It is uniquely determined by (2.15).

[10]The Monte Carlo simulation in this paper are done by taking $50,000$ sample paths, simulating on the jump times. Simulating on jump times instead of regular time steps makes use of the concept of Brownian bridge and the first-passage-time density function of the Brownian motion. See Chapter 5 of Cont and Tankov (2004) for details of this approach.

TABLE 5. Base case parameters: Kou (2002) Model with two regimes

S_0	K	V_0	T	r	b_1^X	σ_1^X	η_{11}^X	η_{12}^X	p_1^X	λ_1^X	σ_1^Z	η_{11}^Z	η_{12}^Z	p_1^Z	λ_1^Z	L
100	90	100	1	0.05	0.05	0.4	10	4	0.4	0.5	0.1	40	40	0.6	3	30

b_2^X	σ_2^X	η_{21}^X	η_{22}^X	p_2^X	λ_2^X	σ_2^Z	η_{21}^Z	η_{22}^Z	p_2^Z	λ_2^Z	q_1	q_2
0.05	0.1	20	10	0.4	1	0.1	60	60	0.4	4	0.5	0.5

by these two methods delineates the strength of the Laplace transform technique as the values can be computed within few seconds.

Table 4 captures the effect of ρ to the values of call options. It shows that increasing ρ has a positive effect on the value of the call option. More specifically, higher ρ implies the greater impact of the firm's default on its equity value. Equivalently, we can find the positive effect of ρ to option prices from Figure 7. Implied volatilities in Figure 7 increase in level as the correlation factor ρ increases. In addition, greater ρ reduces the curvature of implied volatility while it increases the negative skewness. This can be explained by the fact that ρ provides a direct linkage between the firm's credit condition and its equity value. Since higher ρ increases the impact of the default nature of the firm on its equity, higher ρ induces investors' demand to cover their short positions by means of longing the in-the-money calls. Consequently, the price of the in-the-money call (equivalently, out-of-the-money put) is higher than that of the out-of-the-money call (in-the-money put). This price differential between in-the-money and out-of-the-money options intensifies as the dependence parameter ρ increases.

Figure 8 displays the effects of various parameters of X_t and Z_t to the implied volatilities. Increase in jump intensities in both X_t and Z_t elevates the levels of the implied volatility. The effect is particularly dominant for the case of X_t, as increase in λ^X implies higher frequency of jumps, which in turn implies higher probability of default. Since p^X denotes the probability of an upward jump, diminishing value of p^X implies higher chance of X_t jumping downward, thereby increasing the probability of hitting the default barrier L. The resulting effect elevates the implied volatilities, with greater gap at in-the-money region. Interestingly enough, the effect of p^Z has a rather sluggish effect on the implied volatility. This is due to the fact that jump sizes from Z_t are small relative to that from X_t and that jumps in Z_t contribute nothing to the probability of default in X_t, due to independent assumption on X_t and Z_t. Finally, the impacts of σ^X and σ^Z on the implied volatility are greater than that of their jump components. This can be explained as follows. The total variance of the jump-diffusion process is equal to the sum of the variance of its volatility and jump components. With higher values of σ^X or σ^Z contributes more to the total variance of the process than their jump counterparts. As the role of jump component diminishes, the high volatility process becomes closer to the case of geometric Brownian motion. Therefore, the implied volatility curves flatten as σ^X and σ^Z increase.

Figure 9 shows the effect of time-to-maturity on the implied volatilities. Implied volatility curve flattens as the maturity of the option lengthens. This is in agreement to the actual implied volatility curve observed in the market (see, e.g., Cont and Tankov (2004)). However, as discussed in Cont and Tankov (2004), the phenomenon of implied volatility curve flattening is particularly pronounced for most of the Lévy processes encountered in the finance literature. We shall return to this issue later when we study the effect of maturity on the regime-switching jump-diffusion processes.

Figure 10 depicts the relationship between the CDS spread and the implied volatility of the equity options. Increase in σ^X results in both increase in CDS premium and implied volatility of its equity options. This is consistent with the reality that distressed firm usually has both high CDS premium and high implied volatility on its equity option as investors become exceedingly concerned with high possibility of default in the near future.

TABLE 6. Defaultable European call prices with varying K and L: Kou model with two regimes. (LT=Laplace Transform, MC=Monte Carlo)

$(L = 30,\ \rho = 0.5)$	Regime 1		Regime 2	
K	LT	MC	LT	MC
50	52.4063	52.4602	52.3704	52.3616
60	43.0150	43.0710	42.9596	42.9490
70	33.8436	33.8720	33.4950	33.4470
80	25.2278	25.2891	24.2984	24.2596
90	17.6605	17.6096	15.7826	15.7468
100	11.5841	11.5500	8.8754	8.9143
120	4.2256	4.2862	1.8643	1.8652
$(K = 90,\ \rho = 0.5)$	Regime 1		Regime 2	
L	LT	MC	LT	MC
10	17.6608	17.7218	15.7827	15.8179
30	17.6605	17.6669	15.7826	15.7797
50	17.5804	17.5100	15.7698	15.7346
70	15.8148	15.8886	15.2496	15.2491

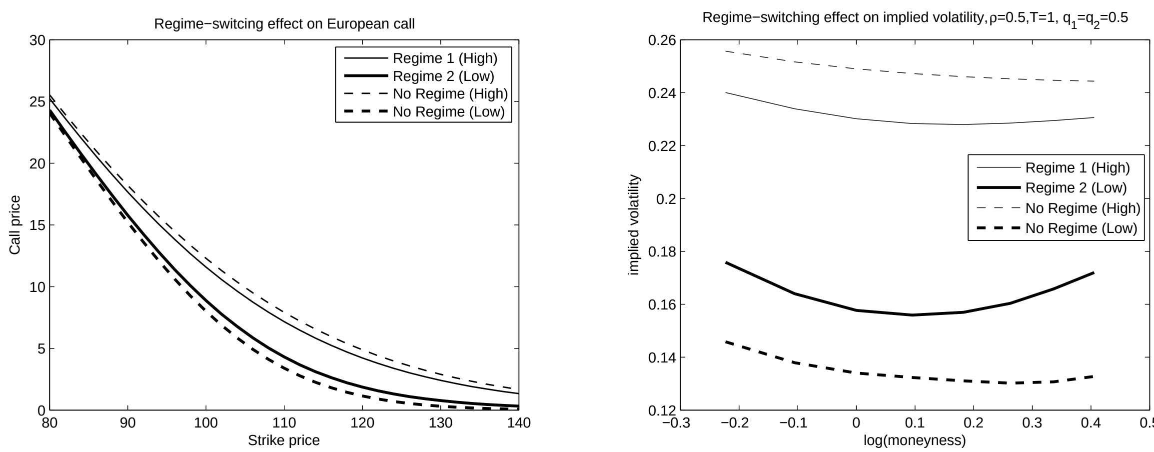

FIGURE 11. Regime-switching effect on the implied volatility of defaultable European call: Implied volatility (Left) and defaultable European call (Right)

6.2.2. *Regime Switching case.* As illustrated in Konikov and Madan (2002), the independent and stationary increments of Lévy processes make it inflexible in capturing the implied volatility curve across different maturities. In particular, the property of the independent increments of Lévy processes makes the implied volatility be deterministic with respect to time (Cont and Tankov, 2004), a feature that is in strong disagreement with reality. For this reason, we shall now proceed to study the equity options with an inclusion of regime-switching effect. For simplicity, we shall only focus on the case that Z_t is also the Kou process (2002) under each regime. The parameters for X_t and Z_t for the case of two regimes are given in Table 5. In Table 5, it is clear that the values of X_t and Z_t in Regime 1 indicate that Regime 1 has higher total volatility than Regime 2.

Similar to the last subsection, we compute the defaultable options under various values of strike price K and default barrier L by means of the Laplace transform and Monte Carlo simulation for high and low regimes. Observe that we have similar patterns on the equity options with respect to K and L as those shown in Table 3. In particular, higher K increases the chance of the defaultable European call to be exercised out-of-the-money, resulting in its lower

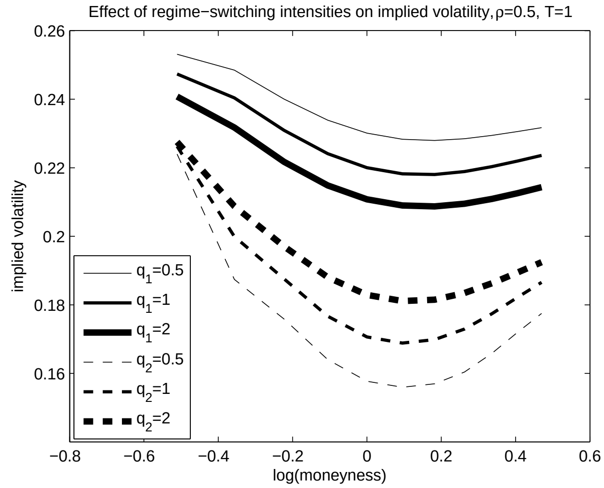

FIGURE 12. Effect on switching intensity for initial Regimes 1 (solid lines) and initial Regime 2 (dotted lines), with same switching intensities, i.e. $q_1 = q_2$.

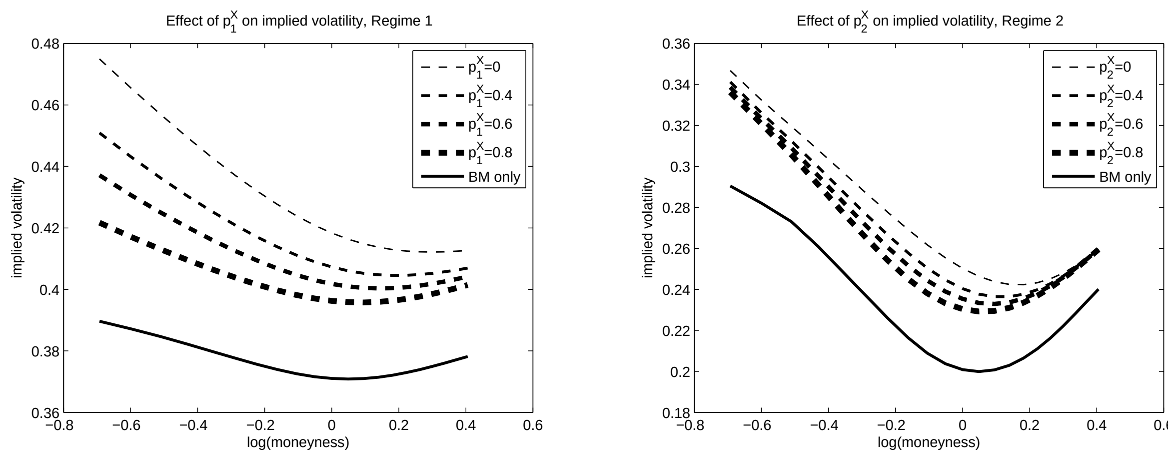

FIGURE 13. Effect of p_i^X, $i = 1, 2$, on the skewness of implied volatility ($\rho = 1$): Initial Regime 1 (Left) and Initial Regime 2 (Right)

current value. As the defaultable European call can be seen as the down-and-out call, increasing the level of L would certainly decrease the value of the options.

Another interesting aspect of regime-switching from the view of Table 3 is the difference between the prices under Regimes 1 and 2 with respect to the option's moneyness. When the option is deep in-the-money, the prices under Regimes 1 and 2 differ in less than 1 decimal point. As option moves from deep in-the-money to deep out-of-the-money, the difference between high and low regimes gains its momentum gradually. The visual effect is displayed on the left-hand side of Figure 11.

To study the regime-switching effect to the implied volatility, we now turn our focus to the right-hand side of Figure 11. The presence of Markov chain J_t increases the curvature of the implied volatility curve, comparing to the case with no regime-switching. This phenomenon is parallel to the case of the continuous-time stochastic volatility models. The presence of the stochastic volatility factor introduces the persistence in total volatility of the process. This persistence in turn slows down the decay in the curvature of the implied volatility curve.

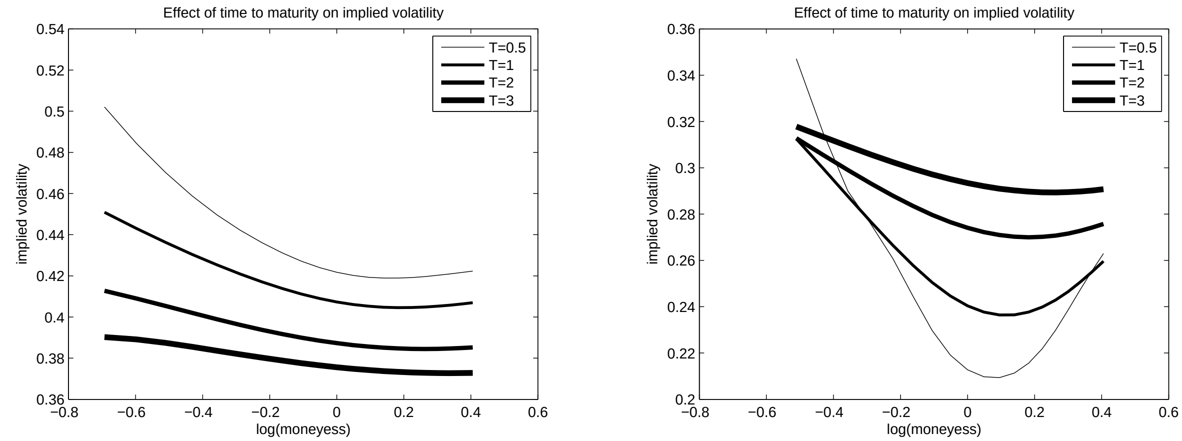

FIGURE 14. Effect of time-to-maturity on implied volatility ($\rho = 1$): Initial Regime 1 (Left) and Initial Regime 2 (Right)

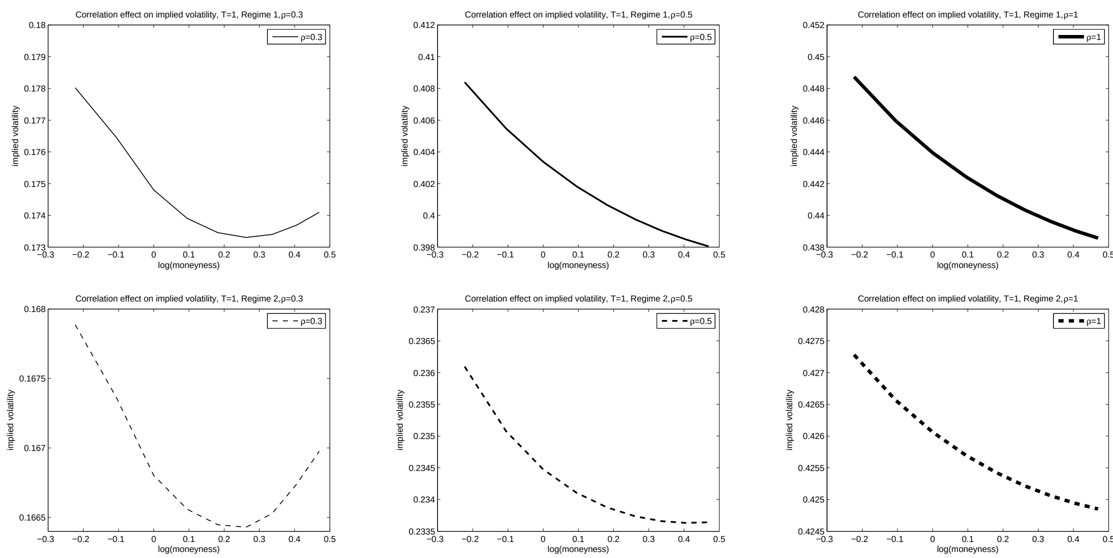

FIGURE 15. Effects of correlation ρ on the implied volatility of defaultable European call in Kou model (2002): Initial Regime 1 (First row) and Initial Regime 2 (Second row)

Figure 12 captures the role of regime-switching intensity to the implied volatility curve. In Figure 12, increasing the regime-switching intensities brings the implied volatilities of the Regimes 1 and 2 together. More specifically, the solid lines represent the implied volatilities under initial Regime 1, whereas the dash lines represent the implied volatilities under initial Regime 2, with different regime-switching intensities. The introduction of the regime-switching factor results in lowering the implied volatility of Regime 1 while elevating the implied volatility of Regime 2. As switching intensity dictates the average time of the process to stay in one regime, speeding up the switching intensity on both regimes results in Regimes 1 and 2 converge to a common regime. Consequently, the difference in the option values under Regimes 1 and 2 reduces. When the difference in prices of high and low regime diminishes, so do their corresponding implied volatilities.

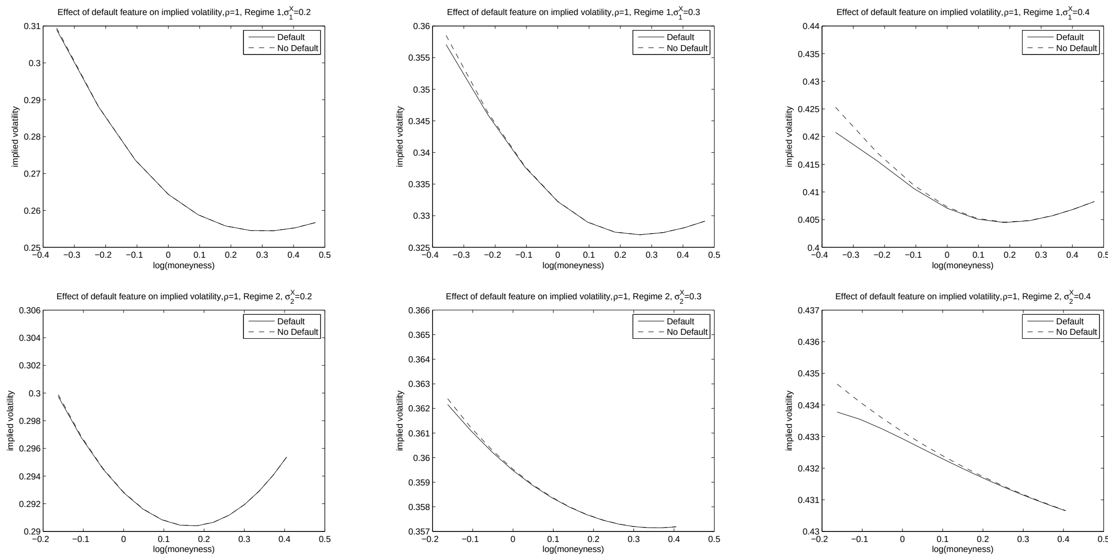

FIGURE 16. Effects of default feature on the implied volatility of defaultable European call ($L = 50$): Initial Regime 1 (First row) and initial Regime 2 (Second row)

Parallel to our study on the CDS curve, we would like to investigate the combined effects of the jump-diffusion and regime-switching factors on the equity options. From Bollen (1998), we know that option pricing under the regime-switching Brownian motion can also generate volatility smile. This certainly begs a question on the impacts of regime-switching and jump-diffusion components to the shape of the implied volatility curve. Figure 13 studies the shapes of implied volatility curve under the case of regime-switching Brownian motion with and without the jump components. In both Regimes 1 and 2, it appears that the regime-switching Brownian motion produces symmetric smiles, whereas the inclusion of jump components introduces the negative skewness in volatility smile. In fact, the negative skewness is more pronounced as the probability of upper jumps of X_t (i.e. p_i^X, $i = 1, 2$) decreases. From Cont and Tankov (2004), we know that the negative skewness of implied volatility can be captured by the distribution of downward jumps of the underlying process. The creation of the symmetric volatility smile implies that regime-switching Brownian motion can produce fat tail distributions, i.e. excessive kurtosis. However, the symmetry in volatility smile reflects that regime-switching Brownian motion cannot create skewed distributions. Since negative skewness is a common feature found in the equity markets, it appears that regime-switching Brownian motion alone is not flexible enough to introduce skewness to the implied volatility.

Time effect on the shape of implied volatility is one important feature to investigate when pricing options under any stochastic process. In the previous subsection, we see that implied volatility flattens as the maturity of the option increases. Figure 14 studies the effect of the option's maturity on the implied volatility on Regimes 1 and 2. While we see that the volatility curve flattens as the maturity elongates, it appears that the speed of flattening seems to be slower. At $T = 3$, smile effect remains highly visible in comparison to the one at $T = 0.25$. In addition, the implied volatility curves with initial Regimes 1 and 2 behave differently with increasing maturities. In Regime 1, the implied volatility curve moves downward as it flattens. In Regime 2, the implied volatility curve elevates as its curvature decreases. This effect is strikingly similar to the case when we study the effect of the implied volatility curve with respect to changing regime-switching intensities. This is not a mere coincidence but is a consequence of long-term behavior of an ergodic Markov chain. As discussed in Fouque et al. (2000), the long-term behavior of the ergodic Markov chain is governed completely by its invariant distribution, which can be expressed as the products of the switching intensities q_i and the time T. This indicates that increasing the regime-switching intensities or speeding up the time has the same impact as the invariant distribution of the underlying Markov chain. Therefore, it is no surprise that the implied volatilities of Regimes 1 and 2 converge to a common implied volatility curve as T lengthens, as captured in Figure 14.

Figure 15 studies the correlation effect of X_t on the equity options. Similar to the case of Kou process (2002) in the previous subsection, higher ρ has an upward shift to the implied volatility across Regimes 1 and 2. Increase in ρ augments the negative skewness of the implied volatility curve across Regimes 1 and 2. As explained in the case of no regime-switch, the negative skewness reflects the credit nature of the equity. The dependence parameter ρ controls the degree of the firm's credit exposure to equity. Since the implied volatility of the put option with the same strike and maturity coincides that of the call option, we can also interpret the increase in negative skewness through the eyes of the equity holders. To hedge against the possibility of the firm's default before the option's maturity, equity holders can purchase the deep out-of-the-money put to lock in their loss in the case of adverse situations. This increases the demand of the out-of-the-money puts than that of the in-the-money puts, thereby augmenting negative skewness of the implied volatility curve.

Figure 16 compares the difference of implied volatility curves between the defaultable and non-defaultable options. In Figure 16, the implied volatility curves indicated by "No Default" mean that they are generated without taking the assumption of default feature into account. For the low volatility case, the implied volatility curves of "Default" and "No Default" are almost indistinguishable. In the higher volatility case, the difference between implied volatility curves of "Default" and "No Default" becomes greater in in-the-money region and the two curves remain indistinguishable in at-the-money and out-of-the-money regions. This is in the line with the implied volatility curves of vanilla and down-and-out calls observed in the actual market. Lower volatility implies that the firm is not in a volatile state and the probability of default is slim. In the high volatility case, higher probability of default results in greater difference between the non-defaultable and defaultable options, resulting in difference in the implied volatility when the options are currently in the money. When the option is currently deep out-of-the-money, the chance of exercising the option is slim, regardless of the default feature, and hence the prices of non-defaultable and defaultable optons have similar values, resulting in overlapping implied volatility curves in that region.

7. Conclusion

With an increasing evidence of the linkage between equity and credit aspects of a corporate firm, we propose an extended version of the latent firm value model first proposed by Kijima et al. (2009) so as to include jumps and regime-switching dynamics. As with the original latent firm value model, the extended latent model assumes that the equity and actual firm value processes are correlated. Numerical examples on the CDS premiums confirm our intuition that jumps and regime-switching dynamics can generate more realistic credit spread, especially near the short and median parts of the term structure.

Following the work of Kijima et al. (2009), we assume that the actual firm value is not directly observable to the investors. Different from the framework of Kijima et al. (2009), equity process comes as an input to our framework instead of an output. We use the firm's equity as means to extract information on the actual firm value. The high liquidity of equity market provides great convenience in analyzing the effect of the creditworthiness of the firm on its securities. Equity serves as a correlated marker process of the actual firm value in our framework. The correlation captures the impact of the credit quality of the actual firm value on its equity. Moreover, the definition of the equity as the sum of the actual firm value and another independent stochastic process allows us to include the firm-independent factor under the joint framework. Once the latent information of the firm is extracted from the equity market data, we illustrate the use of our equity-credit model via pricing the firm's credit default swap (CDS) and equity option. The richness of the framework does not reduce the model's tractability as we can price the defaultable European option via the Laplace transform efficiently.

On a technical level, this paper provides an extensive investigation of the dominant effects of the jump-diffusion and the regime-switching factors. More specifically, we prove analytically that regime-switching Brownian motion alone does not create the non-zero credit spread as the maturity of the CDS approaches to 0. Lévy jump factor plays a dominant role of the CDS spread near zero maturity. On the other hand, in the longer time horizon, we see that the CDS spreads of the high and low regimes come closer together. These observations confirm our intuition that Lévy jumps explain the short term behavior whereas regime-switching factor captures the long-run effect of CDS dynamics. Numerical studies also demonstrate the presence of the regime-switching results in greater flexibility of generating different credit spread curves, in addition to those generated by jump-diffusion alone.

Regime-switching factor also plays a significant role in the equity option with default feature. In particular, the implied volatility curve against strike price of the high regime decreases while the implied volatility curve under the low regime increases as the switching-intensity or the maturity of the option lengthens. The symmetric smile generated by the regime-switching Brownian motion tells us that regime-switching Brownian motion is not flexible enough in capturing the skewness of the implied volatility curve, especially those under short maturities. Cross comparison of the CDS curve against implied volatility provides a strong linkage of high volatility of the firm value that creates an upward momentum in both the CDS and the implied volatility curve. Finally, the versatility of the Laplace transform becomes particularly apparent when pricing under the case of randomized default barrier, since it requires no extra effort than the case of constant default barrier.

For the illustration purpose, this paper only considers the pricing of CDS and defaultable equity option under the joint framework. The pricing of other securities issued by a firm using our model deserves further investigation. For example, the pricing of equity default swap, which has both the equity and credit components of a firm, is the subject of great interest. This issue will be addressed in the future work.

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A. PROOFS

A.1. **Proof of Lemma 2.1.** Let $\tilde{Z}^{(j)}$ be a Lévy process with the triplet $(g(t,j)+\zeta b_j^Z, \zeta\sigma_j^Z, \zeta\nu_j^Z)$ and Lévy exponent $\overline{\kappa_j}^Z(z)$, where

$$\overline{\kappa_j}^Z(z) = zg(t,j) + \kappa_j^Z(\zeta z).$$

Define the process $\tilde{Z}_t$ by $\mathrm{d}\tilde{Z}_t = \mathrm{d}\tilde{Z}_t^{(J_t)}$. Then, we see that

$$\begin{aligned}\mathbb{E}_i[\exp(\tilde{Z}_t)|\mathcal{F}_u] &= \exp(\tilde{Z}_u)\mathbb{E}_i\left[\mathbf{e}_{J_t}^\top \exp\left(\int_u^t \mathrm{d}\tilde{Z}_s\right)\mathbf{1}_d|\mathcal{F}_u\right] \\ &= \exp(\tilde{Z}_u)\mathbf{e}_{J_u}^\top \exp\left(\left(\mathbf{Q}+\{\overline{\kappa_j}^Z(1)\}_{\text{diag}}\right)(t-u)\right)\mathbf{1}_d,\end{aligned}$$

where $\mathbf{1}_d = (1,\ldots,1)^\top \in \mathbb{R}^d$. Hence, $V_t = \exp(\tilde{Z}_t)$ is an $\{\mathcal{F}_t\}$-martingale if and only if it holds that

$$\mathbf{e}_{J_u}^\top \exp\left(\left(\mathbf{Q}+\{\overline{\kappa_j}^Z(1)\}_{\text{diag}}\right)(t-u)\right)\mathbf{1}_d = 1, \quad \forall u \le t.$$

We claim that the above condition is equivalent to $\overline{\kappa_j}^Z(1) = 0$ for all t and j.

Suppose first that, for all j,

$$\overline{\kappa_j}^Z(1) = 0.$$

As $\mathbf{Q}$ has an eigenvalue of zero with the eigenvector $\mathbf{1}_d$, we see $\mathbf{Q}^n\mathbf{1}_d = \mathbf{0}_d$ for $n \ge 1$, whence

$$\exp\left(\left(\mathbf{Q}+\{\overline{\kappa_j}^Z(1)\}_{\text{diag}}\right)(t-u)\right)\mathbf{1}_d = \exp\left(\mathbf{Q}(t-u)\right)\mathbf{1}_d = \mathbf{1}_d.$$

Therefore, the above condition holds.

Conversely, suppose that the above condition holds. Then

$$\begin{aligned}0 &= \lim_{u\to t}\mathbf{e}_{J_u}^\top\frac{1}{t-u}\left[\exp\left(\left(\mathbf{Q}+\{\overline{\kappa_j}^Z(1)\}_{\text{diag}}\right)(t-u)\right)-\mathbf{I}_d\right]\mathbf{1}_d \\ &= \mathbf{e}_{J_t}^\top\left(\mathbf{Q}+\{\overline{\kappa_j}^Z(1)\}_{\text{diag}}\right)\mathbf{1}_d \\ &= \overline{\kappa_{(J_t)}}^Z(1).\end{aligned}$$

Since it must hold for any J_t, we have $\overline{\kappa_j}^Z(1)$ for all j and t, proving the claim.

A.2. **Proof of Corollary 2.1.** We shall only prove (2.12), as (2.13) follows analogously. Define

$$\begin{aligned}a_{ij}(0,t) &= \tilde{\mathbb{E}}\left[\exp\left(\kappa_{(J_t)}^Z(\zeta)t+\gamma X_t\right)1_{\{J_t=j\}} \mid J_0=i\right] \\ &= \mathbb{E}\left[\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\exp\left(\kappa_{(J_t)}^Z(\zeta)t+\gamma X_t\right)1_{\{J_t=j\}} \mid J_0=i\right] \\ &= \mathbb{E}\left[\exp\left(\zeta Z_t+\gamma X_t\right)1_{\{J_t=j\}} \mid J_0=i\right].\end{aligned}$$

Up to $o(h)$ terms, we have

$$
\begin{aligned}
& a_{ij}(0,t+h) \\
= \ & \mathbb{E}\left[\exp\left(\int_0^{t+h}\zeta \mathrm{d}Z_s + \int_0^{t+h}\gamma \mathrm{d}X_s\right); J_t = j, J_{t+h} = j \mid J_0 = i\right] \\
& + \sum_{k\neq j}\mathbb{E}\left[\exp\left(\int_0^{t+h}\zeta \mathrm{d}Z_s + \int_0^{t+h}\gamma \mathrm{d}X_s\right); J_t = k, J_{t+h} = j \mid J_0 = i\right] \\
= \ & \mathbb{E}\left[\exp\left(\int_0^{t}\zeta \mathrm{d}Z_s + \int_0^{t}\gamma \mathrm{d}X_s\right) 1_{\{J_t=j\}}\right. \\
& \left.\times \exp\left(\int_t^{t+h}\zeta \mathrm{d}Z_s^{(j)} + \int_t^{t+h}\gamma \mathrm{d}X_s^{(j)}\right) 1_{\{J_t=j\}}1_{\{J_{t+h}=j\}} \mid J_0 = i\right] \\
& + \sum_{k\neq j}\mathbb{E}\left[\exp\left(\int_0^{t}\zeta \mathrm{d}Z_s + \int_0^{t}\gamma \mathrm{d}X_s\right) 1_{\{J_t=k\}}\right. \\
& \left.\times \exp\left(\int_t^{t+h}\zeta \mathrm{d}Z_s^{(k)} + \int_t^{t+h}\gamma \mathrm{d}X_s^{(k)}\right) 1_{\{J_t=k\}}1_{\{J_{t+h}=j\}} \mid J_0 = i\right] \\
= \ & \mathbb{E}\left[\exp\left(\int_0^{t}\zeta \mathrm{d}Z_s + \int_0^{t}\gamma \mathrm{d}X_s\right) 1_{\{J_t=j\}} \mid J_0 = i\right] \\
& \times\mathbb{E}\left[(1+q_{jj}h)\exp\left(\int_t^{t+h}\zeta \mathrm{d}Z_s^{(j)} + \int_t^{t+h}\gamma \mathrm{d}X_s^{(j)}\right) \mid J_t = j\right] \\
& + \sum_{k\neq j}\mathbb{E}\left[\exp\left(\int_0^{t}\zeta \mathrm{d}Z_s + \int_0^{t}\gamma \mathrm{d}X_s\right) 1_{\{J_t=k\}} \mid J_0 = i\right] \\
& \times\mathbb{E}\left[q_{kj}h\exp\left(\int_t^{t+h}\zeta \mathrm{d}Z_s^{(k)} + \int_t^{t+h}\gamma \mathrm{d}X_s^{(k)}\right) \mid J_t = k\right] \equiv \mathcal{K}
\end{aligned}
$$

Given $J_t = k$, $X_t^{(k)}$ and $Z_t^{(k)}$ are independent and we then have

$$
\begin{aligned}
\mathcal{K} \ = \ & \mathbb{E}\left[\exp\left(\int_0^{t}\zeta \mathrm{d}Z_s + \int_0^{t}\gamma \mathrm{d}X_s\right) 1_{\{J_t=j\}} \mid J_0 = i\right] \\
& \times(1+q_{jj}h)\tilde{\mathbb{E}}\left[\exp\left(\int_t^{t+h}\zeta \mathrm{d}Z_s^{(j)} + \int_t^{t+h}\gamma \mathrm{d}X_s^{(j)}\right) \mid J_t = j\right] \\
& + \sum_{k\neq j}\mathbb{E}\left[\exp\left(\int_0^{t}\zeta \mathrm{d}Z_s + \int_0^{t}\gamma \mathrm{d}X_s\right) 1_{\{J_t=k\}} \mid J_0 = i\right] \\
& \times q_{kj}h\tilde{\mathbb{E}}\left[\exp\left(\int_0^{t}\zeta \mathrm{d}Z_s^{(k)} + \int_0^{t}\gamma \mathrm{d}X_s^{(k)}\right) \mid J_t = k\right] \\
= \ & a_{ij}(0,t)(1+q_{jj}h)\exp\left((\kappa_j^Z(\zeta)+\kappa_j^X(\gamma))h\right) + \sum_{k\neq j} a_{ik}(t,T)q_{kj}h\exp\left((\kappa_k^Z(\zeta)+\kappa_k^X(\gamma))h\right) \\
= \ & a_{ij}(0,t)\exp\left((\kappa_j^Z(\zeta)+\kappa_j^X(\gamma))h\right) + h\sum_{k\in E} a_{ik}(0,t)q_{kj}\exp\left((\kappa_k^Z(\zeta)+\kappa_k^X(\gamma))h\right),
\end{aligned}
$$

where we have made use of the independent increments of each $X^{(j)}$ and $Z^{(j)}$ on the third equality and Laplace exponents for $X^{(j)}$ and $Z^{(j)}$ on the fourth equality. It follows that

$$\begin{aligned}\frac{a_{ij}(0,t+h)-a_{ij}(0,t)}{h} &= \frac{a_{ij}(t,T)}{h}\left(\exp\left((\kappa_j^Z(\zeta)+\kappa_j^X(\gamma))h\right)-1\right)\\ &\quad+\sum_{k\in E} a_{ik}(0,t)q_{kj}\exp\left((\kappa_k^Z(\zeta)+\kappa_k^X(\gamma))h\right),\end{aligned}$$

which implies

$$\frac{\partial}{\partial T}a_{ij}(0,t)=a_{ij}(0,t)(\kappa_j^Z(\zeta)+\kappa_j^X(\gamma))+\sum_{k\in E}a_{ik}(0,t)q_{kj}.$$

With the matrix formulation $\mathbf{F}(0,t)=\{a_{ij}(0,t)\}_{ij}$, and$\mathbf{Q}=\{q_{ij}\}_{ij}$, we then have

$$\frac{\partial}{\partial T}\mathbf{F}(0,t)=\mathbf{F}(0,t)\left(\mathbf{Q}+\{\kappa_j^Z(\zeta)+\kappa_j^X(\gamma)\}_{\text{diag}}\right).$$

Since $F(0,0)=I$, the matrix equation is solved as

$$\mathbf{F}(0,t)=\exp\left(\left(\mathbf{Q}+\{\kappa_j^Z(\zeta)+\kappa_j^X(\gamma)\}_{\text{diag}}\right)t\right).$$

A.3. **Proof of Lemma 4.1.** Since J_t is non-explosive, there exists only finite number of regime switches in any compact interval. Define

$$\chi(h)\equiv\inf\{t\in[0,h],J_t\neq i\},$$

as the first time that the Markov chain J_t makes its first jump. Together with the assumption that the probability of the jump-diffusion and the Markov chain jumping together is zero, we have, up to $o(h)$ term,

$$\begin{aligned}&\frac{1}{h}\mathbb{P}_i[\tau<h]\\ =\;&\frac{(1-q_{ii}h)}{h}\sum_{n=0}^{\infty}\mathbb{P}(N_h^X(i)=n)\mathbb{P}_i\left[\inf_{0\le s<\chi(h)}X_s^{(i)}<-x\right]\\ &+\frac{1}{h}\sum_{j\neq i}q_{ij}h\sum_{n=0}^{\infty}\mathbb{P}(N_h^X(j)=n)\mathbb{P}_j\left[\inf_{\chi(h)\le s<h}X_s^{(j)}<-x\right]+o(h) \qquad \text{(A.1)}\\ =\;&\frac{1}{h}\sum_{n=0}^{\infty}\mathbb{P}(N_h^X(i)=n)\mathbb{P}_i\left[\inf_{0\le s<\chi(h)}X_s^{(i)}<-x|N_h^X(i)=n\right]\\ &-q_{ii}\sum_{n=0}^{\infty}\mathbb{P}(N_h^X(i)=n)\mathbb{P}_i\left[\inf_{0\le s<\chi(h)}X_s^{(i)}<-x|N_h^X(i)=n\right]\\ &+\sum_{j\neq i}q_{ij}\sum_{n=0}^{\infty}\mathbb{P}(N_h^X(j)=n)\mathbb{P}_j\left[\inf_{\chi(h)\le s<h}X_s^{(j)}<-x|N_h^X(j)=n\right]+o(h)\end{aligned}$$

The first part of equation (A.1) represents the default to occur before the first regime-switch while the second component denotes the probability of default to occur after the first regime-switch.

Denote $\vartheta_i(h)$ as the first jump time of Poisson process $N_h^X(i)$. Observe that

$$
\begin{aligned}
& \frac{1}{h}\sum_{n=0}^{\infty}\mathbb{P}(N_h^X(i)=n)\mathbb{P}_i\left[\inf_{0\le s<\chi(h)} X_s^{(i)} < -x | N_h^X(i)=n\right] \\
= \;& \frac{\mathrm{e}^{-\lambda_i^X h}}{h}\mathbb{P}_i\left[\inf_{0\le s<\chi(h)}\left(b_i^X s+\sigma_i^X W_s^X\right) < -x\right] \\
& +\lambda_i^X \mathrm{e}^{-\lambda_i^X h}\mathbb{P}_i\left[\inf_{0\le s<\chi(h)}\left(b_i^X s+\sigma_i^X W_s^X + Y_i^X 1_{\{s\ge\vartheta_i(h)\}}\right) < -x\right] \\
& +\frac{1}{h}\sum_{n=2}^{\infty}\frac{\mathrm{e}^{-\lambda_i^X h}(\lambda_i^X h)^n}{n!}\mathbb{P}_i\left[\inf_{0\le s<\chi(h)}\left(b_i^X s+\sigma_i^X W_s^X + N_s^X(i) Y_i^X\right) < -x | N_h^X(i)=n\right].
\end{aligned}
$$

Following the arguments in Ruf and Scherer (2011), we have

$$
\lim_{h\downarrow 0}\frac{1}{h}\sum_{n=0}^{\infty}\mathbb{P}(N_h^X(i)=n)\mathbb{P}_i\left[\inf_{0\le s<\chi(h)} X_s^{(i)} < -x | N_h^X(i)=n\right] = \nu_i^X((-\infty,x]).
$$

By similar arguments, we also have

$$
\lim_{h\downarrow 0} q_{ii}\sum_{n=0}^{\infty}\mathbb{P}(N_h^X(i)=n)\mathbb{P}_i\left[\inf_{0\le s<\chi(h)} X_s^{(i)} < -x | N_h^X(i)=n\right] = 0
$$

and

$$
\lim_{h\downarrow 0}\sum_{j\ne i} q_{ij}\sum_{n=0}^{\infty}\mathbb{P}(N_h^X(j)=n)\mathbb{P}_j\left[\inf_{\chi(h)\le s<h} X_s^{(j)} < -x | N_h^X(j)=n\right] = 0.
$$

The result follows by recalling the definition of CDS premium and L'Hôpital's rule, i.e.,

$$
\lim_{T\downarrow 0} c_T^{(i)} = \lim_{T\downarrow 0}\frac{\frac{1}{T}(1-R)\int_0^T \mathrm{e}^{-rt}\mathrm{d}\mathbb{P}_i(\tau\le t)}{\frac{1}{T}\int_0^T \mathrm{e}^{-rt}\mathbb{P}_i(\tau>t)\mathrm{d}t} = (1-R)r\nu_i^X((-\infty,x]). \tag{A.2}
$$