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Improved estimation methods for VaR, Expected Shortfall and the risk contributions with high precisions

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Abstract. The (marginal) risk contribution is very useful for analysing the concentration risk in a portfolio. However, it is difficult to estimate the risk contributions for VaR and ES precisely, especially by the Monte Carlo simulation. Against this difficulty, we applied the saddlepoint approximation to estimate the distribution function, so that the difficulty of estimating the risk contributions for VaR was dissolved. In this article we propose new estimation methods for ES and the risk contributions for ES based on the conditional independence and the saddlepoint approximation. Numerical studies confirm that the new methods are much better than the existing ones.

Keywords: Value at Risk, Expected Shortfall, (marginal) risk contribution, additivity of the risk, subfiltration approach, conditional independence, saddle-point approximation.

1 Introduction

The risk management of a portfolio consisting of many financial instruments is very important for the financial institutions. Its importance grows more and more in these days, especially after the worldwide financial crisis. Since the risk management and the capital allocation of the portfolio are based on the amounts of risks, or potential loss, estimated quantitatively by some models, many studies have been done about the risk evaluation methods and models.

One of the most important roles of the risk evaluation models is estimating the potential loss of a portfolio adequately, and the estimated values are monitored and reported to the investors periodically. Additionally, it is desirable that the models can catch the signs of the disastrous crises in the early stage. And, another important role is the concentration risk analysis of the portfolio, that is, measuring which part contributes much to the total risk in the portfolio. Such an analysis will give us important informations about how to deal with the risks and which measures to take first prior to others. The first step of the concentration risk analysis is estimating the volumes of the risk of individual assets (or subportfolios), of course, with taking the diversification effect into consideration. Some risk measures have been proposed for the concentration risk analysis, however, it is very difficult to estimate such risk measures precisely and robustly, especially by the Monte Carlo simulation, which is often used in the practical risk evaluation models.

The (marginal) risk contribution is one of the risk measures useful for the concentration risk analysis. It has the meaning that the sensitivity of the total risk of a portfolio to an infinitesimal change in asset allocation. Litterman (1997) proposed the concept of this sensitivity, pointing out its desirable property that the sum of the sensitivities of all assets is equal to the overall risk of the portfolio if the sensitivities are well-defined. This property of the risk contribution is called as the additivity in this article. Since Litterman (1997), various research papers have addressed about the risk contributions, for example, Tasche (1999), Hallerbach (2002) and Kalkbrener (2005). Now, the concept of the risk contribution advances to a concept for the risk allocation, and it is called as the Euler allocation according to Tasche (2007), in which many findings and practical usages are summarized briefly. However, the robust and precise estimation for the risk contributions is difficult technically except for the risk contributions for the standard deviation.

By the way, a saddlepoint approximation is well-known as an evaluation method of the contour integral in the complex domain, and is often used in engineering. In financial engineering, Arvanitis and Gregory (1998) and Arvanitis et al.(1999) used the saddlepoint approximation in order to build an analytical credit risk evaluation model. Their ideas were extended by Martin et al.(2001a) to estimate the distribution function of the portfolio's potential loss under more realistic settings, and soon later, Martin et al. (2001b) derived an approximated formula for the risk contribution of each asset to the portfolio's VaR (Value at Risk). Summing up and clearing up the above ideas theoretically, Muromachi (2004) proposed a new framework of a risk evaluation model by assuming the conditional independence and using the saddlepoint method, and called it as a hybrid method. He showed that much more reliable estimates of the risk contributions for VaR and ES (Expected Shortfall) could be obtained by the hybrid method than those obtained by the ordinary Monte Carlo simulation.

However, according to our numerical results shown in this article, the estimates of ES and risk contributions for ES are not so reliable than those of VaR and risk contributions for VaR in the hybrid method. Since the ES is one of the coherent risk measures (see Artzner et al. (1999), Acerbi and Tasche (2002), and so on), the precise and robust estimation of ES and the risk contributions for ES is important for the theoretically desirable risk management, therefore, we have been seeking more reliable estimation methods.

In this article, we propose new estimation methods for ES and the risk contributions for ES. The proposed calculation methods are based on a universal mathematical relation between VaR and ES, and uses a saddlepoint approximation in order to calculate them easily and quickly.

This article is organized as follows. In Section 2 we show the generalized hybrid method and some useful expressions; the approximated formulas for the conditional distribution and the risk contributions for VaR and ES. And, in order to estimate ES and risk contributions for ES, two methods are introduced; "old methods" proposed by Muromachi (2004) and "new methods" proposed in this article. In Section 3, we explain how to apply the hybrid method to the risk evaluation model by using an example, and present some numerical results and compare the estimates by the old and new methods, and Section 4 concludes this article.

2 Hybrid method: setting and estimation methods

This section describes the framework of what we call as the hybrid method in more general setting than that described in Muromachi (2004). There exist two approaches for evaluating the potential loss of a portfolio: the simulation approach, and the analytical approach. The hybrid method uses both to calculate the potential loss of a portfolio. It is an extended method of the conditionally independent default model proposed by Martin et al. (2001a).

2.1 The setting

We consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where P is the physical probability measure. Suppose that there are n assets, and that the price of the j–th asset per share or face value at time t is denoted by $X_j(t)$. Here, $t = 0$ means the present, and the risk horizon is T, $T > 0$. Consider a portfolio π, and a_j denotes the holding amount of the j–th asset in the portfolio. Then the time t price of the portfolio π is $X(t) = \sum_{j=1}^{n} a_j X_j(t)$.

We use a "sub-filtration approach" called in credit risk modeling. Let $w_j(t)$, $j = 1, \ldots, m$, denote the basic factors and let

$$\mathbf{W}(t) \;\;=\;\; (w_1(t), \ldots, w_m(t)), \quad t \geq 0.$$

A filtration $\mathcal{G}_t$ is defined as a filtration generated by the process $\mathbf{W}(t)$, that is, $\mathcal{G}_t = \sigma(\mathbf{W}(s), 0 \leq s \leq t)$ [1]. Here, the basic factors are defined as stochastic variables such that $X_j(t|\mathcal{G}_T)$, $0 \leq t \leq T$, $j = 1, \cdots, n$ become conditionally independent when $\mathcal{G}_T$ are given. Here, the conditional independence with respect to $\mathcal{G}_T$ means that $X_j(t)$, $0 \leq t \leq T$, $j = 1, \cdots, n$ are independent given $\mathcal{G}_T$. On the other hand, the filtration generated by all the processes except for $\mathbf{W}(t)$ is denoted by $\mathcal{H}_t$, and the filtration $\mathcal{F}$ is defined as the minimum filtration including $\mathcal{G} \cup \mathcal{H}$, i.e., $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$ for any $t \in \mathcal{R}^+$.

Next, we define some statistics explicitly. Suppose that X is a stochastic variable which shows the future price of a portfolio, and $F_X(x)$ denotes its distribution function, and $f_X(x)$ is its density function if it exists. In this article, the 100α percentile of X, denoted by $Q_X(\alpha)$, $0 < \alpha < 1$, is defined by [2]

$$Q_X(\alpha) \;\;\equiv\;\; \inf\{x | F_X(x) \geq \alpha\} \tag{1}$$

[1] We assume that all the filtrations satisfy the usual conditions in this article.

[2] Definition (1) is a general expression for the inverse function of the distribution function, which can be denoted by $F_X^{-1}(\alpha)$.

and the VaR (Value at Risk) with the confidence level 100α % is defined by

$$\mathrm{VaR}_X(\alpha) \;\equiv\; c - Q_X(1-\alpha) \tag{2}$$

where c is the reference value, for example, today's price of the portfolio. Moreover, the ES (Expected Shortfall) with the confidence level 100α % is defined by

$$\mathrm{ES}_X(\alpha) \;\equiv\; \frac{1}{1-\alpha}\int_{\alpha}^{1} \mathrm{VaR}_X(p)\mathrm{d}p. \tag{3}$$

When $F_X(x)$ is continuous and strictly increasing in x, these can be expressed more easily. That is, we obtain

$$Q_X(\alpha) \;=\; \{x|F_X(x)=\alpha\} \tag{4}$$

$$\mathrm{ES}_X(\alpha) \;=\; c - \mathrm{TCE}_X(Q_X(1-\alpha)) \tag{5}$$

where TCE (Tail Conditional Expectation) is defined by

$$\mathrm{TCE}_X(x) \;\equiv\; E\left[X|X\le x\right]. \tag{6}$$

Rigidly speaking, these definitions are restricted in this article, and other definitions and terms might be used in other articles [3]. Hereafter, for simplicity, we assume that $F_X(x)$ is continuous and strictly increasing in x.

2.2 Estimation of the density and the distribution functions

By assumption, since the conditional prices $X_j(T|\mathcal{G}_T)$, $j=1,\cdots,n$ is conditionally independent given $\mathcal{G}_T$, the conditional moment generating function $M_X(s|\mathcal{G}_T)$ of $X(T|\mathcal{G}_T)=\sum_{j=1}^{n}a_jX_j(T|\mathcal{G}_T)$ is given by

$$M_X(s|\mathcal{G}_T) \;\equiv\; E\left[e^{sX(T|\mathcal{G}_T)}\right]=E\left[e^{s\sum_{j=1}^{n}a_jX_j(T|\mathcal{G}_T)}\right]=\prod_{j=1}^{n}E\left[e^{sa_jX_j(T|\mathcal{G}_T)}\right]. \tag{7}$$

Here, we implicitly assume that the conditional moment generating function $M_X(s|\mathcal{G}_T)$ exists. Since $M_X(s|\mathcal{G}_T)$ is the Laplace transformation [4] of the conditional density function $f_X(u|\mathcal{G}_T)$ of $X(T|\mathcal{G}_T)$, we obtain by the inverse Laplace transformation,

$$f_X(x|\mathcal{G}_T) \;=\; \frac{1}{2\pi i}\int_{\sigma-i_\infty}^{\sigma+i_\infty} M_X(s|\mathcal{G}_T)e^{-xs}\mathrm{d}s=\frac{1}{2\pi i}\int_{\sigma-i_\infty}^{\sigma+i_\infty} e^{K_X(s|\mathcal{G}_T)-xs}\mathrm{d}s \tag{8}$$

[3]For example, some people call TCE defined by (6) as CVaR (Conditional Value at Risk), ES (Expected Shortfall) or TVaR (Tail Value at Risk).

[4]The direction of the contour integral is inverse to the standard definition of the Laplace transformation.

where $K_X(s|\mathcal{G}_T) \equiv \log M_X(s|\mathcal{G}_T)$ is the conditional cumulant generating function of $X(T|\mathcal{G}_T)$, σ is the real value such that the integral in (8) exists, and the contour in (8) is parallel to the imaginary axis in the complex domain.

In order to evaluate (8), we use the saddlepoint approximation method, which is very famous in engineering science. In the saddlepoint method, the integral in the complex domain is approximated by the contribution from the curvilinear integral near the saddlepoint. A detailed description is available in texts on engineering science or statistical science such as Jensen (1995). In this article, we adopt the saddlepoint approximation with an asymptotic expansion derived in Daniels (1987). Using the approximation [5], we obtain from (8),

$$f_X(x|\mathcal{G}_T) \simeq \frac{e^{K_X(\bar{s}|\mathcal{G}_T)-\bar{s}x}}{\sqrt{2\pi K_X^{(2)}(\bar{s}|\mathcal{G}_T)}}\left[1+\frac{1}{8}\lambda_{(4)}(\bar{s}|\mathcal{G}_T)-\frac{5}{24}\lambda_{(3)}^2(\bar{s}|\mathcal{G}_T)\right] \tag{9}$$

where $\bar{s}$ is the saddlepoint of $J_X(s|\mathcal{G}_T) \equiv K_X(s|\mathcal{G}_T) - xs$, which means $\mathrm{d}J_X(\bar{s}|\mathcal{G}_T)/\mathrm{d}s = 0$, $K_X^{(n)}(s|\mathcal{G}_T)$ is the n–th order derivative of $K_X(s|\mathcal{G}_T)$, and $\lambda_{(r)}(s|\mathcal{G}_T) \equiv K_X^{(r)}(s|\mathcal{G}_T)/(K_X^{(2)}(s|\mathcal{G}_T))^{r/2}$. Since $K_X(s|\mathcal{G}_T)$ is a convex function, the saddlepoint $\bar{s}$ is unique and can be searched numerically with ease. We refer to the approximated formula (9) as the first–order approximation, and the formula obtained by ignoring the terms of $\lambda_{(3)}(\bar{s}|\mathcal{G}_T)$ and $\lambda_{(4)}(\bar{s}|\mathcal{G}_T)$ in (9) as the zeroth–order approximation.

Integrating the density function $f_X(v|\mathcal{G}_T)$ from $v = -\infty$ to $v = x$, and using the saddlepoint approximation method with an asymptotic expansion, we obtain the approximated formula for the conditional distribution function as

$$\begin{aligned}
F_X(x|\mathcal{G}_T) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{M_X(s|\mathcal{G}_T)e^{-xs}}{s}\mathrm{d}s \\
&\simeq e^{K_X(\bar{s}|\mathcal{G}_T)-\bar{s}x+\frac{1}{2}\hat{z}^2}\left[\{1-\Phi(\hat{z})\}\left\{1+\frac{1}{6}\lambda_{(3)}(\bar{s}|\mathcal{G}_T)\hat{z}^3\right.\right. \\
&\qquad\left. +\left(\frac{1}{24}\lambda_{(4)}(\bar{s}|\mathcal{G}_T)\hat{z}^4+\frac{1}{72}\lambda_{(3)}^2(\bar{s}|\mathcal{G}_T)\hat{z}^6\right)\right\} \\
&\quad +\phi(\hat{z})\left\{-\frac{1}{6}\lambda_{(3)}(\bar{s}|\mathcal{G}_T)(\hat{z}^2-1)\right. \\
&\qquad\left.\left. -\left(\frac{1}{24}\lambda_{(4)}(\bar{s}|\mathcal{G}_T)(\hat{z}^3-\hat{z})+\frac{1}{72}\lambda_{(3)}^2(\bar{s}|\mathcal{G}_T)(\hat{z}^5-\hat{z}^3+3\hat{z})\right)\right\}\right]
\end{aligned} \tag{10}$$

for $x \leq E[X(T|\mathcal{G}_T)]$, where $\hat{z} \equiv \sqrt{\bar{s}^2 K_X^{(2)}(\bar{s}|\mathcal{G}_T)}$, and $\Phi(\cdot)$ and $\phi(\cdot)$ are the distribution and density functions of the standard normal distribution, respectively [6]. Corresponding to the

[5] According to Jensen (1995), there exist two approximation formulas: for the continuous variable and for the discrete one. In this article we use the former because its form is much simpler.

[6] The approximated formula for $x > E[X(T|\mathcal{G}_T)]$ is obtained by replacing $\lambda_{(3)}(\bar{s}|\mathcal{G}_T)$ with $-\lambda_{(3)}(\bar{s}|\mathcal{G}_T)$.

Edgeworth expansion which is used to derive (9) and (10), the approximated formula (10) is called as the second–order approximation, while the formula obtained by ignoring the terms of $\lambda_{(4)}(\bar{s}|\mathcal{G}_T)$ and $\lambda^2_{(3)}(\bar{s}|\mathcal{G}_T)$ is called as the first–order approximation, and the formula obtained by ignoring all $\lambda_{(r)}(\bar{s}|\mathcal{G}_T)$ terms is the zeroth–order approximation.

Since the approximated values for the conditional distribution, $f_X(x|\mathcal{G}_T)$ and $F_X(x|\mathcal{G}_T)$, are obtained above, the approximated values for the unconditional distribution can be obtained by using the chain rule of the expectation. That is, the unconditional density and distribution functions of $X(T)$ are given by

$$f_X(x) \quad = \quad E\left[f_X(x|\mathcal{G}_T)\right] \tag{11}$$

and

$$F_X(x) \quad = \quad E\left[F_X(x|\mathcal{G}_T)\right]. \tag{12}$$

Here, we can replace $f_X(x|\mathcal{G}_T)$ in (11) and $F_X(x|\mathcal{G}_T)$ in (12) with the approximations (9) and (10), respectively.

In summary, we obtain the approximated values of the unconditional distribution, $f_X(x)$ and $F_X(x)$, by the following procedures:

(1) Generate many sample paths of the basic factors from $t = 0$ to $t = T$, $\{\mathbf{W}(s),\ 0 \leq s \leq T\}$. The Monte Carlo simulation is a powerful candidate to generate sample paths, however, other methods can be used.

(2) Evaluate $f_X(x|\mathbf{W}(s),\ 0 \leq s \leq T)$ and $F_X(x|\mathbf{W}(s),\ 0 \leq s \leq T)$ given $\mathbf{W}(s)$, $0 \leq s \leq T$ approximately by using (9) and (10).

(3) Calculate the right hand sides of (11) and (12) by using all sample paths to obtain the approximated values of $f_X(x)$ and $F_X(x)$.

When the density and the distribution functions are obtained, the VaRs with arbitrary confidence levels can be calculated easily.

Whatever the joint distribution of the basic factors is, the Monte Carlo simulation can be used in order to generate sample paths in Procedure (1), and then the calculation in Procedure (3) can be done numerically. Thus, we refer to our method as hybrid, which means the hybrid of the Monte Carlo simulation in Procedure (1) and the analytical approximation formulas in Procedure (2).

On the other hand, when the joint distribution is described as a combination of some distributions such as the normal distributions and the χ^2 distributions, we can apply the numerical quadratures to take the expectation in Procedure (3). For example, when the basic factors are normally distributed, the Gaussian quadratures can be used. In such cases, the best sample paths generated in Procedure (1) are already known in order to make the quadratures work most efficiently.

2.3 Estimation of the expected shortfall: old method

The method described in this subsection is derived from Studer (2001). Due to the assumption that the distribution function of X is continuous and strictly increasing, we have

$$\mathrm{ES}_X(\alpha) \quad = \quad c - \mathrm{TCE}_X(Q_X(1-\alpha)).$$

Therefore, we focus on estimating $\mathrm{TCE}_{X(T)}(x)$, which is written as follows:

$$\mathrm{TCE}_{X(T)}(x) \quad \equiv \quad E[X(T)|X(T) \leq x] = \frac{1}{P\{X(T) \leq x\}} \int_{-\infty}^{x} v f_{X(T)}(v) dv. \tag{13}$$

Additionally assuming that $X(T) \geq 0$, *a.s.* and $E[X(T)] > 0$ [7], from (13), we have

$$\mathrm{TCE}_{X(T)}(x) \quad = \quad \frac{E[X(T)]}{P\{X(T) \leq x\}} F_h(x) \tag{14}$$

where $F_h(x)$ is the distribution function of $X(T)$ under the equivalent probability measure P_h, under which the density function of $X(T)$ is given by

$$h_{X(T)}(x) \quad \equiv \quad \frac{x f_{X(T)}(x)}{E[X(T)]}.$$

Using the approximated formula (10) for $F_h(x)$ in (14), we can obtain the approximated formula for $\mathrm{TCE}_{X(T)}(x)$. Since the moment generating function of $X(T)$ under the probability measure P_h is given by $M_h(x) = M'_{X(T)}(x)/M'_{X(T)}(0)$, the approximated formula for $F_h(x)$ can be calculated in the same way as (10) easily.

[7] These assumptions can be applied if $X(T)$ is finitely bounded below. When there exists a constant d such that $X(T) \geq d$, *a.s.*, we define $Y(T) \equiv X(T) - d$ and then apply the following discussion to $Y(T)$ instead of $X(T)$.

2.4 Risk contributions

In this subsection, we describe the risk contribution briefly. VaR and ES are used to measure the overall risk of a portfolio in financial institutions in these days, but in order to understand the risk profile of the portfolio in detail, it is necessary to know how much each asset (or subportfolio) in the portfolio contributes to the total risk of the portfolio. One of the common measures of this contribution is the sensitivity of the total risk of the portfolio to an infinitesimal change in asset allocation. From this point of view, various researchers studied about the properties of the sensitivity, for example, Litterman (1997), Tasche (1999) and Hallerbach (2002), Kalkbrener (2005), and so on.

Let R_p be a risk measure of a portfolio, for example, the standard deviation, VaRs and ESs. The risk contribution of the j-th asset, hereafter denoted by RC_j, is defined as the sensitivity to an infinitesimal change of the holding amount of the j–th asset in the portfolio $(\partial R_p/\partial a_j)$, multiplied by the holding amount (a_j), that is,

$$RC_j \equiv a_j \frac{\partial R_p}{\partial a_j}.$$

Notice that $\partial R_p/\partial a_j$ does not always exist. However, if $\partial R_p/\partial a_j$ exists, and additionally if the risk measure R_p satisfies the first–order positive homogeneity [8], then the following equation

$$\sum_{j=1}^{n} RC_j = R_p \tag{15}$$

is always satisfied. Equation (15) is directly derived by the Euler's theorem for the homogeneous function, therefore, its equality is guaranteed mathematically.

Previous studies show clear expressions of risk contributions for some risk measures such as the standard deviation, VaRs and ESs. Some of them are shown below. See Tasche (1999) in detail for the derivation and the necessary conditions. Soppose $X(T) = \sum_{j=1}^{n} a_j X_j(T)$. For example, the risk contribution of the j-th asset for the standard deviation is given by

$$RC_j^{SD} \equiv a_j \frac{\partial \mathrm{SD}_{X(T)}}{\partial a_j} = a_j \frac{\mathrm{Cov}(X(T), X_j(T))}{\mathrm{SD}_{X(T)}}$$

[8]A function f is called as n-th order positive homogeneous when

$$f(\lambda a_1, \cdots, \lambda a_m) = \lambda^n f(a_1, \cdots, a_m), \qquad \lambda > 0$$

is satisfied.

if $\mathrm{SD}_{X(T)} > 0$, where $\mathrm{SD}_{X(T)}$ is the standard deviation of $X(T)$ and $\mathrm{Cov}(X(T), X_j(T))$ is the covariance between $X(T)$ and $X_j(T)$. Notice that RC_j^{SD} is given as a simple form. On the other hand, the risk contribution of the j-th asset for the VaR with confidence level α is given by

$$
\begin{aligned}
RC_j^{VaR}(\alpha) &\equiv a_j \frac{\partial \mathrm{VaR}_{X(T)}(\alpha)}{\partial a_j} = a_j \left[\frac{\partial c}{\partial a_j} - \frac{\partial Q_{X(T)}(1-\alpha)}{\partial a_j} \right] \\
&= a_j \frac{\partial c}{\partial a_j} - a_j E\left[X_j(T) | X(T) = Q_{X(T)}(1-\alpha) \right] \qquad (16)
\end{aligned}
$$

if it exists. Similarly, the risk contribution of the j-th asset for the ES with confidence level α is given by

$$
RC_j^{ES}(\alpha) \equiv a_j \frac{\partial \mathrm{ES}_{X(T)}(\alpha)}{\partial a_j} = a_j \frac{\partial c}{\partial a_j} - a_j E\left[X_j(T) | X(T) \leq Q_{X(T)}(1-\alpha) \right] \qquad (17)
$$

if it exists. For example, if the joint density function of $X_j(T)$, $j = 1, \cdots, n$ is continuous and positive almost everywhere, and there exist arbitrary ordered moments for $X_j(T)$, $j = 1, \cdots, n$, then, (16) and (17) are satisfied for all $\alpha \in (0, 1)$.

Consider the risk contribution for the VaR. The second term in (16) is difficult to calculate because this conditional expectation must be taken on condition that the total future price of the portfolio $X(T)$ is constant; in general, if all the future prices $X_j(T)$, $j = 1, \cdots, n$ are not redundant, the conditional expectation is calculated on a $(n-1)$-dimensional hyperplane in the n-dimensional space. As you see easily, it is very difficult to calculate the expectation, especially by the Monte Carlo simulation. About the estimation by the Monte Carlo simulation, see Glasserman (2005) [9]. So, we use another method to estimate the risk contribution.

2.5 Estimation of the risk contributions for VaR

Consider a VaR with the confidence level α, $0 < \alpha < 1$, in our conditionally independent setting. Hereafter, for simplicity, we drop the argument (T), which means the future time T, from many stochastic variables. For example, $\mathrm{VaR}_X(\alpha)$ means $\mathrm{VaR}_{X(T)}(\alpha)$. Given $\mathcal{G}_T$, the conditional distribution function is given by

$$
F_X(x|\mathcal{G}_T) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{K_X(s|\mathcal{G}_T) - sx}}{s} \mathrm{d}s
$$

[9] According to Glasserman (2005), the Monte Carlo simulation with importance sampling techniques is a hopeful method.

where the contour of the integral is parallel to the imaginary axis and runs to the right of the origin if the pole should be avoided. Then, the unconditional distribution function is given by

$$F_X(x) \;=\; E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{K_X(s|\mathcal{G}_T)-sx}}{s}\mathrm{d}s\right]. \tag{18}$$

Differentiating (18) with respect to a_j, $j=1,\ldots,n$ and assuming the exchangeability between the differentiation and the integration, we have

$$\frac{\partial}{\partial a_j}F_X(x) \;=\; E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{\partial K_X(s|\mathcal{G}_T)}{\partial a_j}-s\frac{\partial x}{\partial a_j}\right\}\frac{e^{K_X(s|\mathcal{G}_T)-sx}}{s}\mathrm{d}s\right]. \tag{19}$$

Keeping $F_X(x)$ constant in (19), we obtain the contribution [10] of the j-th asset to α–percentile $Q_X(\alpha)$, denoted by $RC_j^{Q_X}(\alpha)$, as

$$\begin{aligned} RC_j^{Q_X}(\alpha) \;&\equiv\; a_j\frac{\partial Q_X(\alpha)}{\partial a_j}=a_j\frac{E\left[\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\partial K_X(s|\mathcal{G}_T)}{\partial a_j}\frac{e^{K_X(s|\mathcal{G}_T)-sQ_X(\alpha)}}{s}\mathrm{d}s\right]}{E\left[\int_{\sigma-i\infty}^{\sigma+i\infty}e^{K_X(s|\mathcal{G}_T)-sQ_X(\alpha)}\mathrm{d}s\right]} \\ &=\; a_j\frac{E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\partial K_X(s|\mathcal{G}_T)}{\partial a_j}\frac{e^{K_X(s|\mathcal{G}_T)-sQ_X(\alpha)}}{s}\mathrm{d}s\right]}{f_X(Q_X(\alpha))}. \end{aligned} \tag{20}$$

(20) is a slight modification of the expression derived by Martin et al. (2001b). Formally, the integral in (20) can be obtained if $K_X(s)$ is replaced by

$$K_M(s|\mathcal{G}_T)\equiv K_X(s|\mathcal{G}_T)+\log(\partial K_X(s|\mathcal{G}_T)/\partial a_j)-\log s$$

in (8). Notice that $K_M(s|\mathcal{G}_T)$ is not a cumulant generating function, however, the saddlepoint approximation can be used to give an approximate in (20). Assuming that a reference value c in (16) is differentiable with respect to a_j, we obtain

$$RC_j^{VaR}(\alpha) \;=\; \frac{\partial c}{\partial a_j}-a_j\frac{E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\partial K_X(s|\mathcal{G}_T)}{\partial a_j}\frac{e^{K_X(s|\mathcal{G}_T)-sQ_X(\alpha)}}{s}ds\right]}{f_X(Q_X(\alpha))}. \tag{21}$$

2.6 Estimation of the risk contributions for ES: old method

Next, consider a ES with the confidence level α, $0<\alpha<1$. Here we assume that the differentiation with respect to a_j and the integration are exchangeable.

[10] Although $Q_X(\alpha)$ does not mean a risk directly, we call $RC_j^{Q_X}(\alpha)$ a contribution to $Q_X(\alpha)$ simply.

Given $\mathcal{G}_T$, since the tail conditional expectation $\mathrm{TCE}_X(x|\mathcal{G}_T)$ is given by (14), we have

$$\begin{aligned}
\frac{\partial}{\partial a_j}\mathrm{TCE}_X(x) &= \frac{\partial}{\partial a_j}\left(\frac{E[X(T)]}{F_X(x)}F_h(x)\right) \\
&= \frac{1}{F_X(x)}\left[F_h(x)E[X_j(T)] + E[X(T)]\frac{\partial F_h(x)}{\partial a_j}\right] - \frac{E[X(T)]F_h(x)}{(F_X(x))^2}\frac{\partial F_X(x)}{\partial a_j}.
\end{aligned}$$

Using the following relation

$$\frac{\partial}{\partial a_j}F_h(x) = \frac{\partial}{\partial a_j}E[F_h(x|\mathcal{G}_T)] = E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{\partial K_h(s|\mathcal{G}_T)}{\partial a_j} - s\frac{\partial x}{\partial a_j}\right\}\frac{e^{K_h(s|\mathcal{G}_T)-sx}}{s}\mathrm{d}s\right],$$

where $K_h(s|\mathcal{G}_T)$ is the cumulant generationg function of $X(T)$ under P_h, we have

$$\begin{aligned}
\frac{\partial}{\partial a_j}\mathrm{TCE}_X(x) &= \frac{F_h(x)E[X_j(T)]}{F_X(x)} + \frac{E[X(T)]}{F_X(x)}E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\partial K_h(s|\mathcal{G}_T)}{\partial a_j}\frac{e^{K_h(s|\mathcal{G}_T)-sx}}{s}\mathrm{d}s\right] \\
&\quad -\frac{E[X(T)]}{F_X(x)}\frac{\partial x}{\partial a_j}E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{K_h(s|\mathcal{G}_T)-sx}\mathrm{d}s\right] - \frac{E[X(T)]F_h(x)}{(F_X(x))^2}\frac{\partial F_X(x)}{\partial a_j} \\
&= \frac{F_h(x)E[X_j(T)]}{F_X(x)} + \frac{E[X(T)]}{F_X(x)}E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\partial K_h(s|\mathcal{G}_T)}{\partial a_j}\frac{e^{K_h(s|\mathcal{G}_T)-sx}}{s}\mathrm{d}s\right] \\
&\quad -\frac{xf_X(x)}{F_X(x)}\frac{\partial x}{\partial a_j} - \frac{E[X(T)]F_h(x)}{(F_X(x))^2}\frac{\partial F_X(x)}{\partial a_j}.
\end{aligned}$$

Therefore, considering that $F_X(Q_X(1-\alpha)) = 1-\alpha$ is constant, we obtain

$$\begin{aligned}
\frac{\partial}{\partial a_j}\mathrm{ES}_X(\alpha) &\equiv \left.\frac{\partial}{\partial a_j}\mathrm{TCE}_X(Q_X(1-\alpha))\right|_{\alpha=\mathrm{constant}} \\
&= \frac{F_h(Q_X(1-\alpha))E[X_j(T)]}{\alpha} \\
&\quad +\frac{E[X(T)]}{\alpha}E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\partial K_h(s|\mathcal{G}_T)}{\partial a_j}\frac{e^{K_h(s|\mathcal{G}_T)-sx}}{s}\mathrm{d}s\right] \\
&\quad -\frac{Q_X(1-\alpha)f_X(Q_X(1-\alpha))}{\alpha}\frac{\partial Q_X(1-\alpha)}{\partial a_j}.
\end{aligned} \tag{22}$$

The first and third terms in (22) are obtained before, and the second term is calculated by the same method as that used in (20).

2.7 New estimation methods for ES and risk contribution for ES

As described in above, the explicit expression (22) of the risk contribution for ES is obtained. However, due to our numerical results shown later, (22) does not give us so reliable estimates. Moreover, in fact, (14) does not give us so reliable estimates, either. Therefore, we propose other estimation methods of ES and the risk contribution for ES.

The concept of the proposed methods is very simple. As described in (3), the following relation

$$\mathrm{ES}_X(\alpha) \equiv \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_X(p)\mathrm{d}p \tag{23}$$

exists. If we assume the exchangeability between the derivative with respect to a_j and the integral with respect to p, we obtain the following relation:

$$\frac{\partial}{\partial a_j}\mathrm{ES}_X(\alpha) = \frac{1}{1-\alpha}\int_\alpha^1 \frac{\partial \mathrm{VaR}_X(p)}{\partial a_j}\mathrm{d}p = \frac{1}{(1-\alpha)a_j}\int_\alpha^1 RC_j^{VaR}(p)\mathrm{d}p. \tag{24}$$

Since the numerical results shown below imply that our method gives good estimates for $\mathrm{VaR}_X(p)$ and $RC_j^{VaR}(\alpha)$, it is expected that the combination of (24) and (21) would give us good estimates for $RC_j^{ES}(\alpha)$. Similarly, the combination of (23) and the estimation method of the distribution function $F_X(x)$ described in section 2.2 would give good estimates for $\mathrm{ES}_X(\alpha)$.

3 Numerical examples

In this section we provide numerical examples for estimating VaR, ES, and their risk contributions based on the distribution of the future value of a portfolio. Here, we always use the second-order approximations in (9) and (10) when we use the hybrid method.

3.1 Applying the hybrid methods to KM model

In this subsection, we describe how to apply the hybrid method described in Section 2 to the risk evaluation model. As an example of a risk evaluation model, we use Kijima and Muromachi model (hereafter, abbreviated by KM Model). KM model is a synthetic risk evaluation model of a portfolio, which means that the market and the credit risk can be evaluated simultaneously and synthetically. We describe the model briefly in Appendix A. All the notations here are the same in Appendix A.

Consider the situation where $\mathcal{G}_T$ is given. Then, the sample paths of the hazard rates $h_j(s)$, $0 \le s \le T$, $j = 1, \cdots, n$ are determined uniquely, so that the conditional survival probabilities up to time t, $0 \le t \le T$ under the physical probability measure P are given by

$$S_j(t|\mathcal{G}_T) = \exp\{-H_j(0,t)\}$$

where the cumulative hazard rate $H_j(t,T)$ is defined by

$$H_j(t,T) \equiv \int_t^T h_j(s)\mathrm{d}s.$$

Setting $v_j(T,T_j^M)$ in (A.10) as $X_j(T|\mathcal{G}_T)$ in (7), we have

$$E\left[e^{sa_jX_j(T|\mathcal{G}_T)}\right] = S_j(t|\mathcal{G}_T)e^{sa_jv_j^S(T,T_j^M)} + (1-S_j(t|\mathcal{G}_T))e^{sa_jv_j^D(T,T_j^M)}$$

where $v_j^S(T,T_j^M)$ is the time T price of the j-th defaultable discount bond when the bond survives at T, and $v_j^D(T,T_j^M)$ is the time T price when the bond defaults before T, respectively, and from (A.10), they are given by

$$v_j^S(T,T_j^M) = v_0(T,T_j^M)\left[\delta_j + (1-\delta_j)L_j(T,T_j^M)P_T\left\{\tau_j > T_j^M\right\}\right]$$

and

$$v_j^D(T,T_j^M) = \delta_j v_0(T,T_j^M).$$

Given $\mathcal{G}_T$, the vector $(r(T),h_1(T),\cdots,h_n(T))$ is uniquely determined, and from (A.10) $(v_j^S(T,T_j^M),\ v_j^D(T,T_j^M))$, $j=1,\cdots,n$ are all determined. Therefore, we can calculate the conditional moment generating function $M_X(s|\mathcal{G}_T)$ easily, and we can apply the hybrid method described in Section 2 to KM model.

Notice that given $\mathcal{G}_T$, we cannot determine $v_j(T,T_j^M)$ because of the lack of the information about $1_{\{\tau_j>T\}}$, however, we can calculate $M_X(s|\mathcal{G}_T)$ because we already obtain $S_j(t|\mathcal{G}_T)$, $v_j^S(T,T_j^M)$ and $v_j^D(T,T_j^M)$. In this model, giving $\mathcal{G}_T$ corresponds to generating a random vector $(r(T),h_1(T),\cdots,h_n(T),H_1(0,T),\cdots,H_n(0,T))$ according to the stochastic model described in Appendix A as a scenario, and taking expectation with respect to $\mathcal{G}_T$, for example, in (11) and (12), corresponds to taking expectation with respect to all the generated scenarios $(r(T),h_1(T),\cdots,h_n(T),H_1(0,T),\cdots,H_n(0,T))$.

3.2 The setting for calculation

We calculate the risk of a bond portfolio at the risk horizon $T=1$ year. The bond portfolio consists of 100 corporate discount bonds ($n=100$) with maturity 5 years and zero recovery rates. Each bond is issued from different firms, and these bonds have various credit ratings; Aaa-rated 10 bonds, Aa-rated 10 bonds, A-rated 10 bonds, Baa-rated 10 bonds, Ba-rated

30 bonds, and B-rated 30 bonds. The face values of bonds in Aaa, Aa, A, and Baa are 3, 6, 9, $\cdots$, 30, and those in Ba and B are 1, 2, 3, $\cdots$, 30.

The parameters used here are almost the same as those in Kijima and Muromachi (2000). For example, parameters of the initial forward rate curves for credit ratings (including the default-free bonds) and the stochastic differential equations describing the future changes of the instantaneous default-free spot rate and hazard rates of individual bonds, the recovery rates, and additionally, the initial term structures of the hazard rates under the physical probability measure P. If you want to know the parameterization in this model and their values used in the calculation, see Kijima and Muromachi (2000). Among the parameters, we set the volatility of the default-free spot rate process $\sigma_0 = 0.01$ (1%) [11], and the volatilities of the hazard rate processes $\sigma_j = 0$, $j = 1, \cdots, n$ for this article.

We do a Monte Carlo simulation with 500,000 scenarios and calculate VaRs, ESs and their risk contributions with many confidence levels. In detail, we create ten sets of 50,000 simulation runs, and estimate VaRs, ESs and their risk contributions with many confidence levels in each set, and calculate the average and the standard deviations from the ten estimates. And we set the reference value for calculating VaRs and ESs, $c = 1095.959$, the average value of the portfolio at the risk horizon T.

Hereafter, we use the estimates by the Monte Carlo simulation as reference values, and compare the estimates by the hybrid method with the reference values. Especially, the estimates of ESs and the risk contributions for ESs are calculated by two methods: the old methods described before Section 2.6 and the new methods proposed in Section 2.7.

3.3 VaR and ES

First, we compare the VaRs of the future portfolio value $X(T)$. Figure 1 shows the curve of VaR with many confidence levels estimated by the hybrid method ("VaR (IS-H)") [12] and the range within the standard deviation from the average calculated from ten sets of Monte Carlo simulations each with 50,000 samples (left thin curve "VaR-SD" and right thin curve "VaR+SD"). This figure shows that the estimated VaRs are within the standard deviation range in all confidence levels, and in more detail, they are very close to the average of the

[11] Due to this setting, the joint density function of all bond prices at T becomes continuous and non-negative theoretically.

[12] The notation "(IS-H)" means the hybrid method with importance sampling techniques. This is not a simple hybrid method, however, more explanation is omitted in this article. See Muromachi (2004) in detail.

Monte Carlo's estimates. These results imply that the hybrid method gives us good estimates for VaRs.

Figure 2 shows the curve of ES with many confidence levels estimated by the hybrid method ("ES (IS-H)") and the range within the standard deviation from the average calculated from ten sets of Monte Carlo simulations each with 50,000 samples (left thin curve "ES-SD" and right thin curve "ES+SD"). In this figure, the estimated ESs are out of the standard deviation range under 99% confidence level, and in the higher confidence level the estimated ESs are within the range. We think that this is partly because the accuracy of the saddlepoint approximation increases with the confidence level, but partly because the standard deviation becomes larger. Therefore, these results imply that the old hybrid method described in Section 2.3 does not give us so good estimates for ESs.

On the other hand, it seems that the new hybrid method described in Section 2.7 gives us good estimates for ESs. Figure 3 shows the curve of ES with many confidence levels estimated by the new hybrid method ("ES (from VaR)") and the same range in Figure 2. This figure shows that the estimated ESs are within the standard deviation range in all confidence levels, and in more detail, they are close to, but a slightly larger than the average of the Monte Carlo's estimates. We think that this slight overestimation derives from the concavity of the VaR curve in the high confidence level.

3.4 Additivity of risk contributions

Next, we check whether the estimated risk contributions satisfy the additivity precisely or not. Here, the additivity means (15), that is, the sum of the risk contributions of all assets is equal to the total risk of the portfolio.

Our numerical example shows that the differences between the VaRs and the sums of the risk contributions of all assets are less than 0.0031 in all confidence levels, and the differences become smaller with the increase of the confidence levels, for example, about 0.001 with 99% confidence level [13]. Therefore, we conclude that the estimated risk contributions for VaR by

[13]Notice that in order to calculate the differences mentioned above, we do not use the estimates by the Monte Carlo simulation as the VaR estimates. This is because in the hybid method, we give a level (x, $x \sim 1 \times 10^3$), and estimate the distribution function at the level ($F_X(x)$) by the hybrid method. Therefore, the difference is calculated between the level and the sum of the contributions RC_j^{Qx} in (20), that is, $\sum_{j=1}^{n} RC_j^{Qx}(F_X(x)) - x$. The numerical result show that the relative absolute value of the difference to x is less than 3×10^{-6}.

the hybrid method satisfy the additivity with a significant precision for the practical use. Based on the conclusion, in Figure 4, we compare (1) the difference between the estimated VaR and ES, $\mathrm{ES}_X(\alpha) - \mathrm{VaR}_X(\alpha)$, and (2) the sum of the differences between the estimated risk contributions for VaRs and ESs, $\sum_{j=1}^{n}(RC_j^{ES}(\alpha) - RC_j^{VaR}(\alpha))$, for the old and new methods, respectively. The figure shows that the risk contributions for ES estimated by the old method (the sum is "total RC (IS-H)") is consistent with the ESs estimated by the old method denoted by "IS-H", and that the risk contributions for ES estimated by the new method (the sum is "total RC (from VaR)") is consistent with the ESs estimated by the new method denoted by "from VaR", and that there exist significant differences between the estimates by the two methods, "IS-H" and "from VaR".

Moreover, we check the total sum of the risk conrtibutions for ES estimated by the new method. Figure 5 shows the differences defined by "total RC(from VaR)" minus "from VaR" in Figure 4. From this figure, we conclude that the risk contributions for ES estimated by the new method satisfy the additivity with high precisions enough to use in practice.

In Figure 5, notice that the diffecences decrease down (the absolute values of the differences grow up) sharply in the region where the confidence level is slightly less than 100%. This is because the estimated values of ES at the highest confidence level is not good. In calculating the right hand side of (24), we use the simple trapezoidal method. Then, at the highest confidence level, we use the following approximation:

$$\int_{\alpha_H}^{1} RC_j^{VaR}(p)\mathrm{d}p \ \simeq \ (1-\alpha_H) \times RC_j^{ES,old}(\alpha_H) \tag{25}$$

where α_H is the "highest" confidence level in the calculation for (24) and $RC_j^{ES,old}(\alpha_H)$ is the risk contribution for ES at $100\alpha_H\%$ estimated by the old method [14]. Therefore, as the confidence level approaches to 100%, the precision of the additivity becomes worse sharply. In order to improve the precision, the VaRs and the risk contributions for VaR should be calculated in higher confidence levels and used for estimating the ESs and the risk contributions for ES.

[14]We cannot use the estimates by the new method to calculate the left hand side of (25) because no VaR with the confidence level higher than α_H is estimated in this calculation.

4 Concluding remarks

In this article, we summarize the hybrid method for estimating VaR, ES and their risk contributions generally by assuming the conditional independence and using the saddlepoint approximation method. According to our numerical studies, the estimates of VaRs and the risk contributions for VaRs by the hybrid method are much more reliable than the estimates by the ordinary Monte Carlo simulations. However, we also show that the estimates of ESs in Section 2.3 are not so much better that those by the ordinary Monte Carlo simulations, and that the estimates of the risk contributions for ES in Section 2.6 are not so good and does not satisfy the additivity so precisely. Therefore, we propose new methods for estimating ESs and the risk contributions for ESs in Section 2.7, and confirm much improvements of the new methods from the old methods numerically.

Since the concentration risk analysis becomes more important than before in financial institutions, more precise and robust estimation methods for the risk contributions are needed in practice, and it is necessary to do much more researches about this topic. We think that the combination of the saddlepoint approximation method and the conditional independent risk evaluation model is one of the hopeful technical tools for the concentration risk analysis.

Appendix

A Kijima and Muromachi model

In this section we briefly describe the Kijima and Muromachi model (KM model), which is a synthetic risk evaluation model for the market and credit risk of a portfolio. In this model, the processes of the default-free interest rate and hazard rates for default are expressed as stochastic differential equations (hereafter, abbreviated by SDE) with correlations. Based on these basic equations, many sample paths are generated up to the risk horizon T by the Monte Carlo simulation, and the future price of each asset at time T is evaluated by the arbitrage-free pricing method on each sample path. From the future prices of all assets on all sample paths, the distribution of the future value of the portfolio is constructed numerically.

A.1 Basic equations

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ where P is the physical probability measure, $\mathcal{F}_t$ is a filtration up to time t generated by all the stochastic processes described below, and $\mathcal{F} = \mathcal{F}_\infty$. We assume that there exists a unique risk-neutral probability measure $\widetilde{P}$. Here, we show a simple model described as an example in Kijima and Muromachi (2000).

Suppose that there are n different firms. The default-free instantaneous spot rate at time t is $r(t)$, and the hazard rate for default of the j–th firm at time t is $h_j(t)$, $j = 1, \cdots, n$. Under the physical probability measure P, these stochastic variables follow the SDEs

$$dr(t) = (b_0(t) - a_0 r(t))dt + \sigma_0 dz_0(t) \tag{A.1}$$

$$dh_j(t) = (b_j(t) - a_j h_j(t))dt + \sigma_j dz_j(t), \qquad j = 1, \cdots, n \tag{A.2}$$

where a_j, σ_j, $j = 0, \cdots, n$ are positive constants, $b_j(t)$, $j = 0, \cdots, n$ are deterministic functions of time t, and $z_j(t)$, $j = 0, \cdots, n$ are $(n+1)$–dimensional standard Brownian motions under P. (A.1) and (A.2) are the extended Vasicek model proposed by Hull and White (1990).

Under the risk-neutral probability measure $\widetilde{P}$, the default-free instantaneous spot rate at time t, $r(t)$, follows the SDE

$$dr(t) = (\phi(t) - a_0 r(t))dt + \sigma_0 d\widetilde{z}_0(t) \tag{A.3}$$

where $\phi(t)$ is also a deterministic function of time t and $\widetilde{z}_0(t)$ is a standard Brownian motion under $\widetilde{P}$. (A.1) and (A.3) imply that the market price of risk for $z_0(t)$ is also a deterministic function of time, given by $\lambda(t) = (b_0(t) - \phi(t))/\sigma_0$.

Let P^τ be the forward-neutral probability measure such that the forward price of a risky asset at time t, denoted by $S^\tau(t) \equiv S(t)/v_0(t, \tau)$, becomes a martingale under P^τ where $S(t)$ is the time t price of the risky asset and $v_0(t, T)$, $t \leq T$ is the time t price of a default-free discount bond with maturity T. Under P^τ, the hazard rate at time t , denoted by $h_j^\tau(t)$, is assumed to be

$$h_j^\tau(t) = h_j(t) + \ell_j(t), \quad j = 1, \cdots, n \tag{A.4}$$

where $\ell_j(t)$ is a deterministic function of time t and independent of τ.

In this setting, the filtration generated by $(n+1)$-dimensional standard Brownian motions $\mathbf{z}(t) = (z_0(t), \cdots, z_n(t))$ is denoted by $\mathcal{G}_t$, that is, $\mathcal{G}_t = \sigma(\mathbf{z}(s), 0 \leq s \leq t)$, and $\mathcal{G} = \mathcal{G}_\infty$.

Let τ_j, $j = 1, \cdots, n$ be the default time of the j-th asset, and the filtration generated by the default process $H_j(t) = 1_{\{\tau_j \le t\}}$, $j = 1, \cdots, n$ is denoted by $\mathcal{H}^j_t = \sigma(H_j(s), 0 \le s \le t)$, where 1_A designates the indication function, meaing that $1_A = 1$ if the event A is true and $1_A = 0$ otherwise. And, define $\mathcal{H} = \mathcal{H}^1 \vee \cdots \vee \mathcal{H}^n$ and $\mathcal{F} = \mathcal{G} \vee \mathcal{H}$.

Additionally, we assume that the default times τ_j, $j = 1, \cdots, n$ are conditionally independent with respect to the filtration $\mathcal{G}$ under P, that is, given $\mathcal{G}_T$, $T \ge \max_j t_j$, the following equation

$$P\{\tau_1 > t_1, \cdots, \tau_n > t_n | \mathcal{G}_T\} = \prod_{j=1}^{n} P\{\tau_j > t_j | \mathcal{G}_T\} \tag{A.5}$$

is satisfied. This assumption means that the hazard rates of different firms may correlate with each other through the information included in $\mathcal{G}$ under P, however, the default event of each firm occurs independently. For further details about the conditional independence, see Bielecki and Rutkowski (2002).

A.2 Valuation of each asset at present and at future

For simplicity, consider a portfolio with discount bonds only.

According to the risk–neutral valuation method, the time t price of the default-free discount bond with maturity τ is given by

$$v_0(t, \tau) = \widetilde{E}_t \left[\exp\left\{ -\int_t^\tau r(s) ds \right\} \right] = A_0(t, \tau) e^{-B(a_0, t, \tau) r(t)} \tag{A.6}$$

where $\widetilde{E}_t$ is the conditional expectation operator under the risk-neutral probability measure $\widetilde{P}$, and

$$\begin{aligned} A_0(t, \tau) &= \exp\left\{ \frac{\sigma_0^2}{2a_0^2} (\tau - t - 2B(a_0, t, \tau) + B(2a_0, t, \tau)) - \int_t^\tau \phi(u) B(a_0, u, \tau) du \right\} \\ B(a, t, \tau) &= \frac{1 - e^{-a(\tau - t)}}{a}. \end{aligned}$$

Consider a defaultable discount bond with maturity τ issued by firm j. Let δ_j be the recovery rate of this bond and δ_j be constant. We assume that the holder of this bond receives at τ either δ if it defaults before τ or 1 if it does not [15]. Then, according to the forward-neutral valuation method, the time t price of this defaultable discount bond, $v_j(t, \tau)$,

[15] This assumption is consistent with the setting in Jarrow and Turnbull (1995).

is given by

$$\begin{aligned} v_j(t,\tau) &= v_0(t,\tau)E_t^\tau\left[1_{\{\tau_j>\tau\}}+\delta_j 1_{\{\tau_j\le\tau\}}\right] = v_0(t,\tau)\left[\delta_j+(1-\delta_j)P_t^\tau\{\tau_j>\tau\}\right] \\ &= v_0(t,\tau)\left[\delta_j+(1-\delta_j)L_j(t,\tau)P_t\{\tau_j>\tau\}\right] \qquad \text{(A.7)} \end{aligned}$$

where $P_t\{\tau_j>\tau\}$ and $P_t^\tau\{\tau_j>\tau\}$ are conditional survival probabilities of firm j under P and P^τ given the information up to time t, respectively, and

$$\begin{aligned} P_t^\tau\{\tau_j>\tau\} &= E_t^\tau\left[\exp\left\{-\int_t^\tau h_j^\tau(s)ds\right\}\right] = L_j(t,\tau)P_t\{\tau_j>\tau\} \qquad \text{(A.8)} \\ P_t\{\tau_j>\tau\} &= E_t\left[\exp\left\{-\int_t^\tau h_j(s)ds\right\}\right] \qquad \text{(A.9)} \\ L_j(t,T) &= \exp\left\{-\int_t^T \ell_j(s)ds\right\}. \end{aligned}$$

(A.8) is derived from the assumption (A.4).

From (A.6), $v_0(t,\tau)$ depends on $r(t)$, and (A.9) implies that $P_t\{\tau_j>\tau\}$ depends on $h_j(t)$. Thus, $v_j(t,\tau)$ depends on $r(t)$ and $h_j(t)$ from (A.7).

A.3 Distribution of portfolio's value at risk horizon

Since (A.6) and (A.7) can be used not only for the present ($t=0$) but for the future ($t>0$), the time T (risk horizon) price of the defaultable discount bond issued by firm j with maturity T_j^M, $T_j^M \ge T$, is given by

$$v_j(T,T_j^M) = v_0(T,T_j^M)\left[\delta_j+(1-\delta_j)L_j(T,T_j^M)P_T\left\{\tau_j>T_j^M\right\}1_{\{\tau_j>T\}}\right]. \qquad \text{(A.10)}$$

Since (A.10) shows that $v_j(T,T_j^M)$ is a function of the three stochastic variables $r(T)$, $h_j(T)$ and $1_{\{\tau_j>T\}}$, the total value of the portfolio at time T, denoted by $\pi(T)$, is a function of stochastic variables $(r(T),h_1(T),\cdots,h_n(T),1_{\{\tau_1>T\}},\cdots,1_{\{\tau_n>T\}})$. Therefore, in order to obtain the distribution of the future value of the portfolio at the risk horizon T, we have the following simulation algorithm:

(1) Generate a scenario of $(r(T),h_1(T),\cdots,h_n(T),1_{\{\tau_1>T\}},\cdots,1_{\{\tau_n>T\}})$ under P. [16]

[16] In this model, it is not necessary to generate the detailed paths of the hazard rates to obtain the default probabilities. Since $(r(T),h_1(T),\cdots,h_n(T),H_1(0,T),\cdots,H_n(0,T))$ are subject to the $(2n+1)$-variate normal distribution in this model, these scenarios can be easily generated. In each scenario, the conditional default probability of the j-th asset up to time T is given by $1-e^{-H_j(0,T)}$. Since the conditional independence is assumed, we can generate the realizations of $1_{\{\tau_j>T\}}$, $j=1,\cdots,n$ by using independent random variables U_j, $j=1,\cdots,n$ subject to the uniform distribution on $[0,1]$. If $U_j<1-e^{-H_j(0,T)}$, then $1_{\{\tau_j>T\}}=0$, otherwise $1_{\{\tau_j>T\}}=1$.

(2) Calculate $v_j(T, T_j^M)$ using (A.10) for each discount bond, and then calculate the value of the portfolio, $\pi(T)$.

(3) If enough numbers of scenarios are obtained, go to (4). Otherwise, go to (1).

(4) Analyse the obtained scenarios statistically.

Since the joint distribution function of the future prices of all assets are obtained numerically, the distribution function of the portfolio's price is estimated, so that any risk measures can be calculated. For example, VaR and ES with the confidence level $100\alpha\%$ are calculated by using (1), (2), (5) and (6). If you would like to know more details about this model, see Kijima and Muromachi (2000).

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Figure 1. Estimated VaR curve of the future value of the portfolio by hybrid method (“VaR(IS-H)”, thick broken line). The thin lines (left “VaR-SD” and right “VaR+SD”) show the range within the standard deviation from the average calculated from ten sets of 50,000 simulation runs.

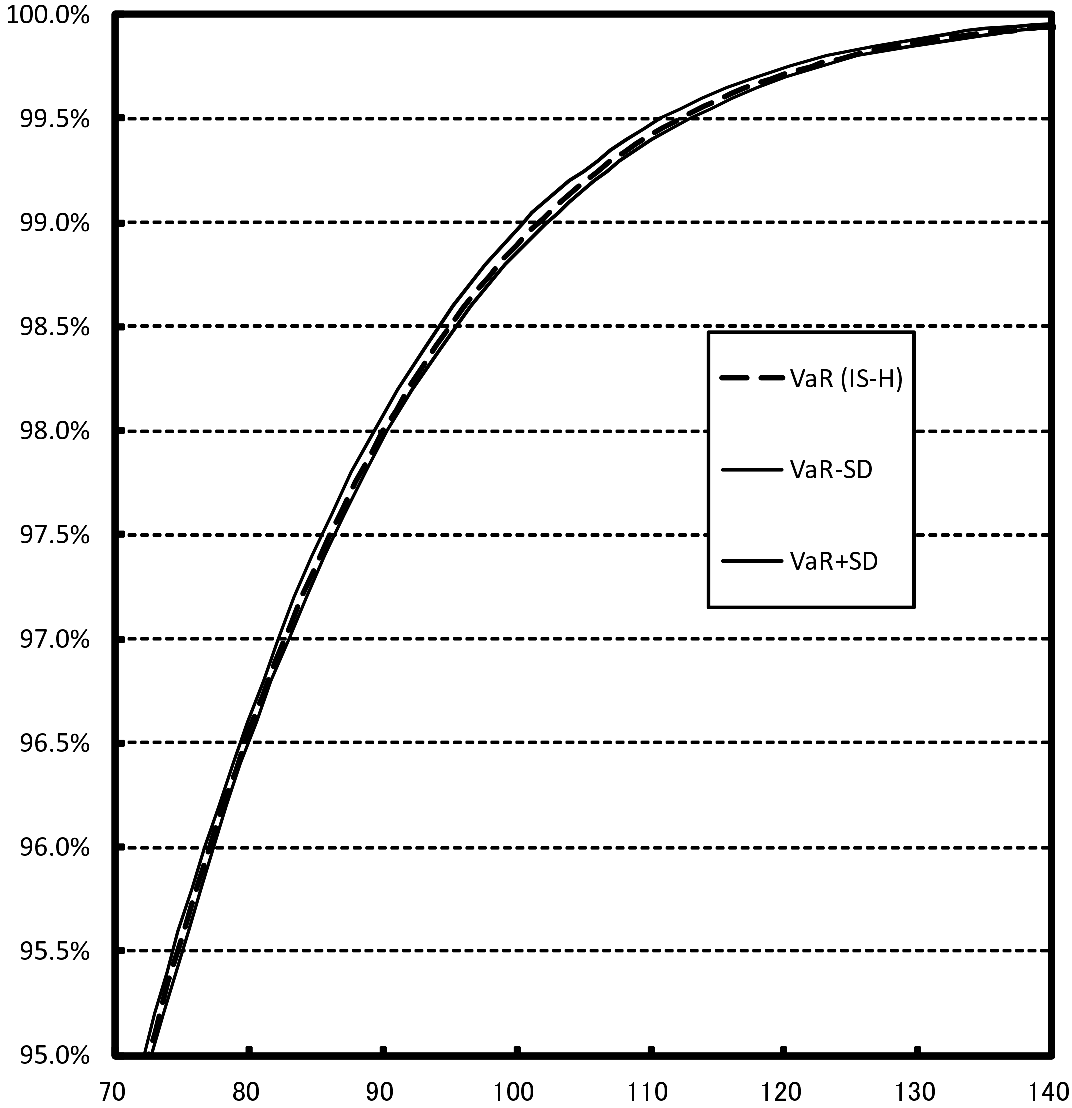

Figure 1: Estimated VaR curve of the future value of the portfolio by hybrid method.

Figure 2. Estimated ES curve of the future value of the portfolio by the old method ("ES(IS-H)", thick broken line). The thin lines (left "ES-SD" and right "ES+SD") show the range within the standard deviation from the average calculated from ten sets of 50,000 simulation runs.

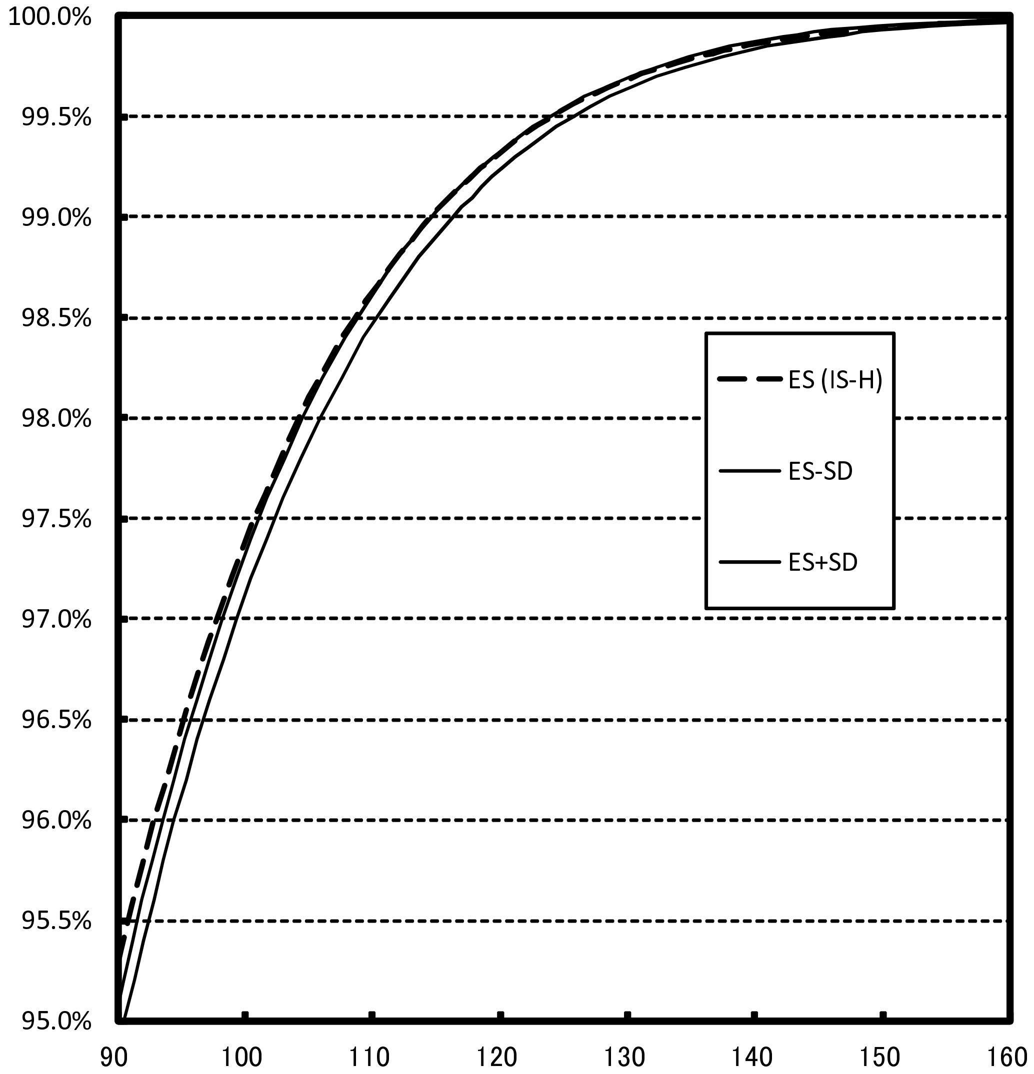

Figure 2: Estimated ES curve of the future value of the portfolio by old method.

Figure 3. Estimated ES curve of the future value of the portfolio by the new method ("ES(from VaR)", thick broken line). The thin lines (left "ES-SD" and right "ES+SD") show the range within the standard deviation from the average calculated from ten sets of 50,000 simulation runs.

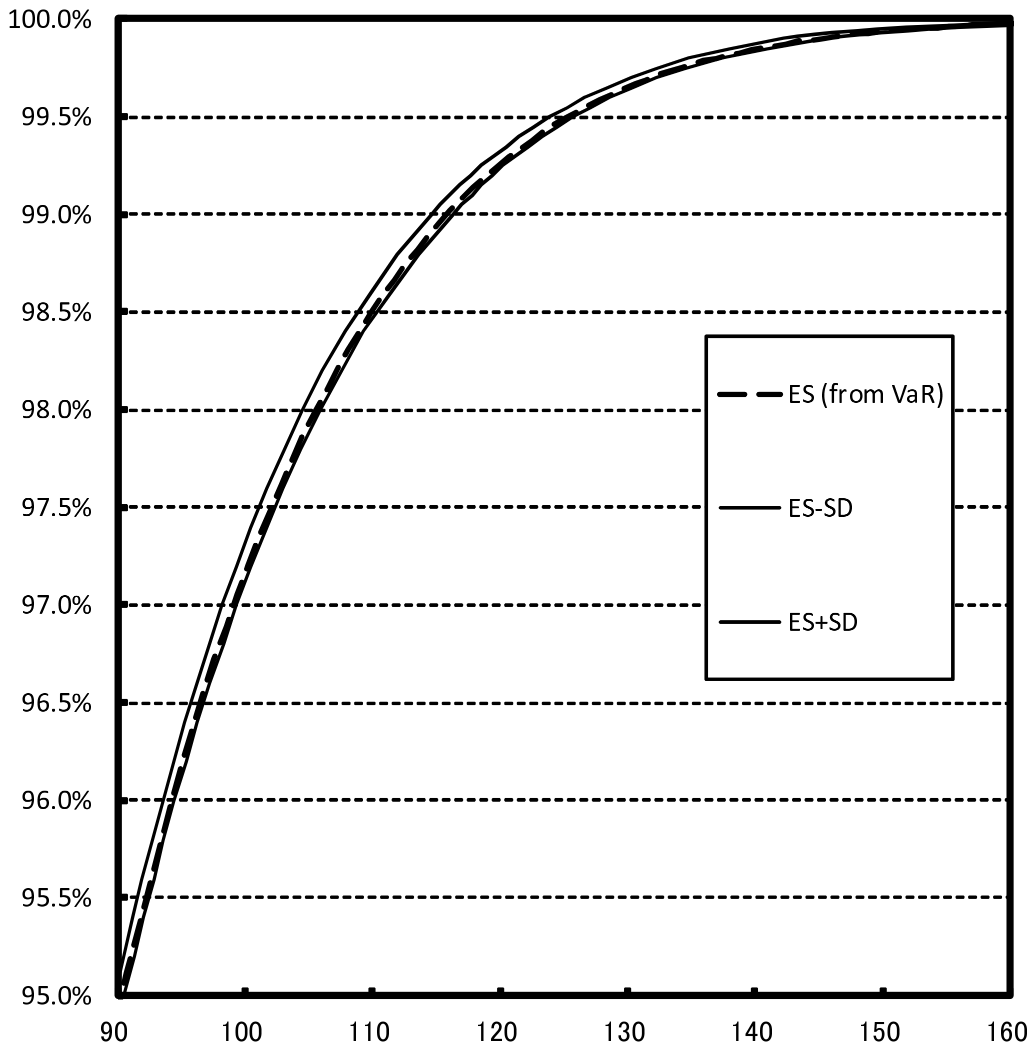

Figure 3: Estimated ES curve of the future value of the portfolio by new method.

Figure 4. Differences between ES and VaR, and the sums of the differences between the risk contributions for ES and VaR by the old and new methods. "IS-H" and "from VaR" denote the differences between ES and VaR by the old and new methods, respectively. "total RC(IS-H)" and "total RC(from VaR)" denotes the sums of the differences between the risk contributions for ES and VaR by the old and new methods, respectively.

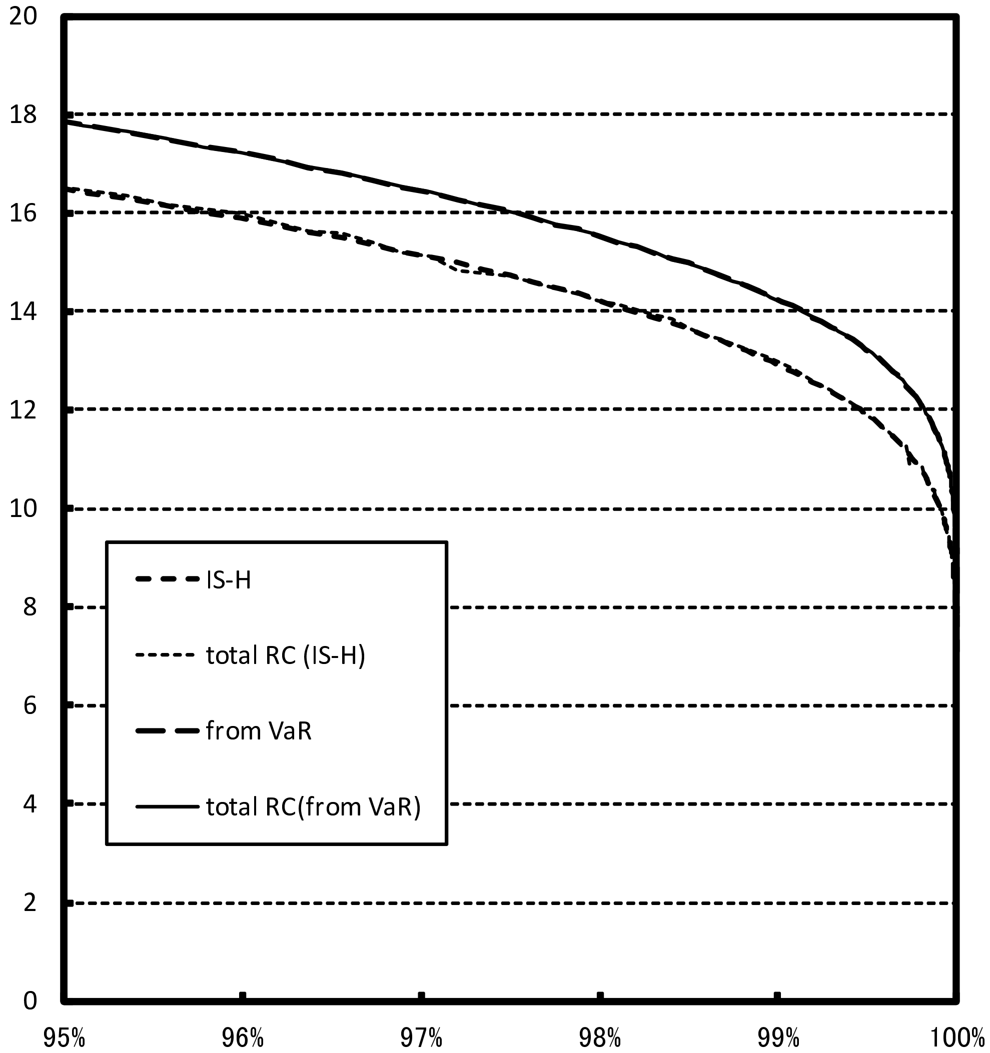

Figure 4: Differences between ES and VaR, and sums of the differences between the risk contributions for ES and VaR.

Figure 5. Check of the sum of the risk contributions for ES. The thick curve shows the differences between “total RC(from VaR)” and “from VaR” estimated by new methods.

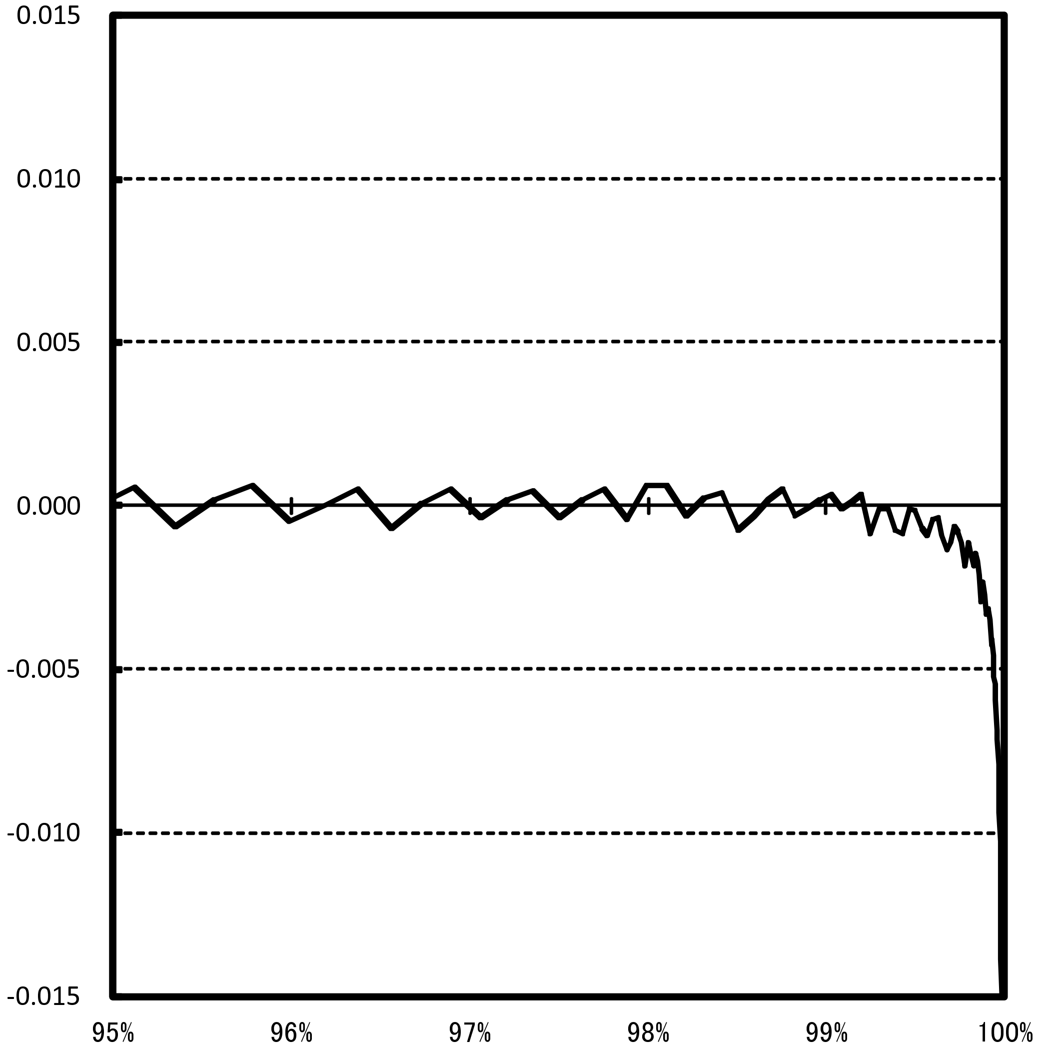

Figure 5: Check of the sum of the risk contributions for ES.