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Bühlmann’s economic premium principle in an incomplete market

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BÜHLMANN'S ECONOMIC PREMIUM PRINCIPLE IN AN INCOMPLETE MARKET

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ABSTRACT. This paper examines the Bühlmann's equilibrium pricing model (1980) in an incomplete market setting and derives the (multivariate) Esscher transform within the framework under some assumptions. The result reveals that the Esscher transform is an appropriate probability transform for the pricing of insurance risks whose market is presumably incomplete.

Keywords: Equilibrium pricing, Equilibrium allocation, Incomplete market, Esscher transform, State price density

1. INTRODUCTION

In the finance literature, the theory of asset pricing has been developed; the theory is well-developed for the so-called *complete* market while there are still many blanks for *incomplete* markets. The insurance market is presumably incomplete; new attempts are necessary for the development of economically sound pricing methods.

In the actuarial literature, on the other hand, there have been developed many probability transforms for the pricing of insurance risks. Such methods include the variance loading, the standard deviation loading, and the exponential principle. Among them, the most popular pricing method for actuaries seems the *Esscher transform* given by

$$\pi(Y) = \frac{\mathbb{E}[Y\mathrm{e}^{-\theta Y}]}{\mathbb{E}[\mathrm{e}^{-\theta Y}]} \tag{1.1}$$

for random variable Y that represents risk, where θ is a positive constant[1] and $\mathbb{E}$ is an expectation operator under the physical probability measure $\mathbb{P}$.

The pricing methods developed in the actuarial literature are often criticized by economic researchers, because they are not based on economic considerations. As pointed out by Bühlmann (1980), the premiums calculated from the actuarial methods depend only on the risk, while in economics premiums are not only depending on the risk but also on market conditions.

Bühlmann (1980) considers a pure risk exchange market where there are N agents. Each agent is characterized by his/her utility function, initial wealth and potential loss, and is willing to buy/sell a risk exchange so as to maximize the expected utility. An equilibrium price for the risk is obtained under the market clearing condition. The Esscher transform (1.1) is derived from the equilibrium price when exponential utilities are assumed. Hence, the Esscher transform is not just an exponential tilting (or exponential change of measure), but has a sound economic interpretation. See also Wang (2002), Kijima (2006) and Kijima and Muromachi (2008) for further discussions on the Esscher transform and their economic interpretations.

Although not mentioned explicitly, the risk exchange market considered in Bühlmann (1980) is *complete*, while actual insurance markets are presumably *incomplete*. Recall that a market is

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[1]This paper treats risk as an asset. A liability with loss variable X can be viewed as a negative asset with gain $Y = -X$. See Wang (2002) for details.

complete if and only if any asset is duplicated by other existing assets in the market (see, e.g., Kijima, 2002). In other words, agents can use any assets in order to maximize their expected utilities. The aim of this paper is to extend the Bühlmann's result (1980) to an incomplete market setting, thereby giving a further economic interpretation to the Esscher transform (1.1) and its variants.

The present paper is organized as follows. In the next section, we review the Bühlmann's equilibrium pricing model (1980) under the complete market setting. Section 3 considers the same problem under an incomplete market setting where the rates of return of all the risks are normally distributed. It is shown that the Bühlmann's economic premium principle is derived when exponential utilities are assumed. Based on the result, the CAPM (capital asset pricing model) is refined and the pricing of derivative securities is considered. Section 4 discusses some related topics. In particular, we consider the problem under which conditions the same equilibrium allocation as Bühlmann (1980) holds. Finally, Section 5 concludes this paper by giving a counterexample that the Bühlmann's economic premium principle does not hold.

Throughout the paper, we assume that agent i possesses initial risk X_i and utility function $u_i(x)$. The risk X_i may be a portfolio of assets traded in a market and/or other types of nontraded assets. As usual, we consider a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that $u_i' > 0$ and $u_i'' < 0$. The set of tradable assets in the market is denoted by $\mathcal{M}$.

2. BÜHLMANN'S EQUILIBRIUM PRICING MODEL

In this section, we review the Bühlmann's equilibrium pricing model (1980) under the *complete* market setting.

Suppose that there are N agents in the market characterized by pairs (X_i, u_i), $i = 1, 2, \ldots, N$. We want to derive an equilibrium price $\pi(Y)$, $Y \in \mathcal{M}$, satisfying

$$Y_i^* = \operatorname*{argmax}_{Y \in \mathcal{M}} \mathbb{E}\left[u_i(X_i + Y - \pi(Y))\right], \qquad i = 1, 2, \ldots, N, \tag{2.1}$$

$$\sum_i Y_i^* = 0. \tag{2.2}$$

The optimal $Y^* = (Y_1^*, \ldots, Y_N^*)$ is called an *equilibrium risk exchange* and $X + Y^*$ an *equilibrium risk allocation*, where $X = (X_1, \ldots, X_N)$. The condition (2.2) is called the *market clearing* condition. Note that this paper does not consider budget constraints and initial wealths for simplicity. Also, the riskfree interest rate is assumed to be zero, unless stated otherwise.

Suppose that the market is arbitrage-free and complete. That is, any asset Y is traded in the market without arbitrage opportunities. Then, there exists a state price density $\eta > 0$ such that $\pi(Y) = \mathbb{E}[\eta Y]$ and $\mathbb{E}[\eta] = 1$. The problem is now to derive the state price density η satisfying

$$Y_i^* = \operatorname*{argmax}_{Y \in \mathcal{M}} \mathbb{E}\left[u_i(X_i + Y - \mathbb{E}[\eta Y])\right], \qquad i = 1, 2, \ldots, N, \tag{2.3}$$

and the market clearing condition (2.2).

The first order condition (FOC for short) of (2.3) with respect to $Y(\omega)$, $\omega \in \Omega$, is given by

$$\begin{aligned} & u_i'(X_i(\omega) + Y_i^*(\omega) - \mathbb{E}[\eta Y_i^*]) \\ & \quad = \eta(\omega)\mathbb{E}[u_i'(X_i + Y_i^* - \mathbb{E}[\eta Y_i^*])] \equiv C_i \eta(\omega). \end{aligned} \tag{2.4}$$

Conversely, this condition implies (2.3). To see this, since u_i is concave, we observe that

$$\begin{aligned} u_i(X_i + Y - \mathbb{E}[\eta Y]) \le{}& u_i(X_i + Y_i^* - \mathbb{E}[\eta Y_i^*]) \\ & + u_i'(X_i + Y_i^* - \mathbb{E}[\eta Y_i^*])(Y - \mathbb{E}[\eta Y] - Y_i^* + \mathbb{E}[\eta Y_i^*]), \end{aligned}$$

which can be rewritten using (2.4) as

$$\begin{aligned} u_i(X_i + Y - \mathbb{E}[\eta Y]) \le{}& u_i(X_i + Y_i^* - \mathbb{E}[\eta Y_i^*]) \\ & + C_i \eta (Y - \mathbb{E}[\eta Y] - Y_i^* + \mathbb{E}[\eta Y_i^*]). \end{aligned}$$

Taking expectation on both sides and utilizing the fact $\mathbb{E}[\eta] = 1$ yield the desired result.

Now, let us denote the inverse function of u_i' by $I_i = (u_i')^{-1}$. Then, from the FOC (2.4), we have

$$X_i + Y_i^* - \mathbb{E}[\eta Y_i^*] = I_i(\eta C_i), \qquad i = 1, 2, \dots, N.$$

Summing over i and utilizing the market clearing condition (2.2), we obtain

$$\sum_i X_i = \sum_i I_i(\eta C_i). \tag{2.5}$$

Define

$$Z \equiv \sum_i X_i, \qquad I(\eta C) \equiv \sum_i I_i(\eta C_i). \tag{2.6}$$

Also, denote the inverse function of $I(x)$ by $u'(x)$.[2] It follows from (2.5) and (2.6) that

$$\eta = \frac{1}{C} u'(Z).$$

Since $\mathbb{E}[\eta] = 1$, we finally obtain the equilibrium price as

$$\pi(Y) = \mathbb{E}[\eta Y] = \frac{\mathbb{E}[Y u'(Z)]}{\mathbb{E}[u'(Z)]}, \qquad Z = \sum_i X_i, \tag{2.7}$$

for any $Y \in \mathcal{M}$. The equilibrium risk allocation is given by

$$X_i + Y_i^* = \pi(Y_i^*) + I_i(\eta C_i), \qquad i = 1, 2, \dots, N. \tag{2.8}$$

Note that the expressions (2.7) and (2.8) are not explicit, because they involve unknown quantities C_i, $i = 1, 2, \dots, N$.

2.1. **Special case: Exponential utility.** When all the agents have exponential utility functions, the above problem can be solved explicitly. Suppose that

$$u_i(x) = -\frac{1}{\lambda_i} \mathrm{e}^{-\lambda_i x}, \quad \lambda_i > 0, \qquad i = 1, 2, \dots, N,$$

and the rest of the assumptions remains the same. Then, since $u_i'(x) = \mathrm{e}^{-\lambda_i x}$, the FOC (2.4) can be written as

$$\mathrm{e}^{-\lambda_i (X_i + Y_i^*)} = \eta \mathbb{E}\left[\mathrm{e}^{-\lambda_i (X_i + Y_i^*)}\right] \equiv \eta \hat{C}_i.$$

It follows that $\eta = \mathrm{e}^{-\lambda_i (X_i + Y_i^*)} / \hat{C}_i$, whence we obtain

$$X_i + Y_i^* = \frac{-1}{\lambda_i} (\log \eta + \log \hat{C}_i), \qquad i = 1, 2, \dots, N. \tag{2.9}$$

Summing over i and utilizing the market clearing condition (2.2), we have

$$Z = -\frac{1}{\lambda} (\log \eta + \log \hat{C}), \tag{2.10}$$

where we put

$$\frac{1}{\lambda} = \sum_i \frac{1}{\lambda_i}, \qquad \log \hat{C} = \sum_i \frac{\lambda}{\lambda_i} \log \hat{C}_i. \tag{2.11}$$

It is readily checked from (2.10) and the fact $\mathbb{E}[\eta] = 1$ that we have $\hat{C} = \mathbb{E}[\mathrm{e}^{-\lambda Z}]$. Therefore, from (2.10), the equilibrium price (2.7) is given by

$$\pi(Y) = \frac{\mathbb{E}[Y \mathrm{e}^{-\lambda Z}]}{\mathbb{E}[\mathrm{e}^{-\lambda Z}]}, \qquad Z = \sum_i X_i, \tag{2.12}$$

[2] The inverse function exists under the condition $u_i'' < 0$ for all i. The function $u'(x)$ can be seen as the marginal utility function of a *representative agent* in the market.

for any $Y \in \mathcal{M}$. The equilibrium pricing formula (2.12) is explicit, because Z and λ are defined only through the given quantities X_i and λ_i, respectively.

The equilibrium risk allocation (2.8) can be also obtained explicitly from (2.9) and (2.10). Namely, we have

$$X_i + Y_i^* = \frac{\lambda}{\lambda_i} Z + \frac{1}{\lambda_i} \log \frac{\hat{C}}{\hat{C}_i}.$$

Taking the pricing functional π on both sides, we get

$$\pi(X_i + Y_i^*) = \frac{\lambda}{\lambda_i} \pi(Z) + \frac{1}{\lambda_i} \log \frac{\hat{C}}{\hat{C}_i},$$

since π is a linear functional. It follows that

$$X_i + Y_i^* - \pi(X_i + Y_i^*) = \frac{\lambda}{\lambda_i}(Z - \pi(Z)),$$

so that the equilibrium risk allocation is given by

$$X_i + Y_i^* = \frac{\lambda}{\lambda_i} Z, \qquad i = 1, 2, \ldots, N. \tag{2.13}$$

Note that the allocation $X_i + Y_i^*$ is proportional to the aggregated risk Z with weight $\lambda/\lambda_i > 0$, where $\sum_i \lambda/\lambda_i = 1$, for the exponential utility case.

Finally, note that, when $Z = Y + \xi$ with Y and ξ being mutually independent, the equilibrium price (2.12) coincides with the Esscher transform (1.1) for risk Y, as claimed by Bühlmann (1980). When $X_i \in \mathcal{M}$, Kijima (2006) called the equilibrium price (2.12) the *multivariate* Esscher transform.

3. An Equilibrium Model for Incomplete Market

In this section, we propose an equilibrium pricing model in an *incomplete* market setting. While, as before, there are N agents in the market $\mathcal{M}$ characterized by pairs (X_i, u_i), $i = 1, 2, \ldots, N$, we assume that only K assets are tradable. The time-1 (future) value of asset j, $j = 1, 2, \ldots, K$, is denoted by S_j and its time-0 (present) value by s_j. In this setting, any traded portfolio for agent i is written as

$$Y_i = \sum_{j=1}^{K} y_j^i S_j, \qquad i = 1, 2, \ldots, N,$$

where the quantity y_j^i represents the number of asset j traded by agent i at time 0. Of course, $y_j^i > 0$ implies that agent i purchases asset j, whereas $y_j^i = 0$ and $y_j^i < 0$ mean no trade and a sell of asset j, respectively.

The initial risks X_i consist of tradable assets and nontradable risks. More specifically, we assume that the initial risk of agent i is given by

$$X_i = \sum_{j=1}^{K} x_j^i S_j + \varepsilon_i, \qquad i = 1, 2, \ldots, N, \tag{3.1}$$

where the quantity x_j^i represents the number of asset j held by agent i at time 0 and ε_i denotes the residual risk. In the following, we denote the total number of asset j issued in the market by

$$A_j \equiv \sum_{i=1}^{N} x_j^i, \qquad j = 1, 2, \ldots, K.$$

We want to derive an equilibrium price $s_j = \pi(S_j)$, $j = 1, 2, \ldots, K$, satisfying

$$y_j^{i*} = \operatorname*{argmax}_{y_1,\cdots,y_K} \mathbb{E}\left[u_i\left(X_i + \sum_{j=1}^{K} y_j(S_j - s_j)\right)\right], \qquad \forall i,\ j, \tag{3.2}$$

$$\sum_{i=1}^{N} y_j^{i*} = 0, \qquad j = 1, 2, \ldots, K. \tag{3.3}$$

The optimal $\{y^{i*} = (y_1^{i*}, \ldots, y_K^{i*}),\ i = 1, \ldots, N\}$ is an *equilibrium risk exchange* and $\{X_i + Y_i^*,\ i = 1, 2, \ldots, N\}$ an *equilibrium risk allocation*. The condition (3.3) is the *market clearing* condition for each traded asset. As before, we do not consider budget constraints and initial wealths for simplicity. Also, the riskfree interest rate is assumed to be zero, unless stated otherwise.

The FOC of (3.2) with respect to y_j is given by

$$\mathbb{E}\left[S_j u_i'\left(X_i + \sum_{j=1}^{K} y_j^i(S_j - s_j)\right)\right] = s_j \mathbb{E}\left[u_i'\left(X_i + \sum_{j=1}^{K} y_j^i(S_j - s_j)\right)\right], \qquad \forall i,\ j. \tag{3.4}$$

Conversely, since u_i is concave, it can be readily checked that this condition implies (3.2).

Suppose that all the agents have exponential utilities as in Subsection 2.1. Then, from (3.4), we derive the following system of simultaneous equations:

$$\mathbb{E}\left[S_j \mathrm{e}^{-\lambda_i\left(X_i + \sum_{j=1}^{K} y_j^i S_j\right)}\right] = s_j \mathbb{E}\left[\mathrm{e}^{-\lambda_i\left(X_i + \sum_{j=1}^{K} y_j^i S_j\right)}\right], \qquad \forall i,\ j. \tag{3.5}$$

In this section, we consider the case that the rates of return of all the risks are normally distributed in order to solve the simultaneous equations (3.5). Some other cases are discussed in Section 4.

In the following, we denote

$$R_j = \frac{S_j - s_j}{s_j}, \quad j = 1, 2, \ldots, K; \qquad R_\varepsilon^i = \frac{\varepsilon_i - \pi(\varepsilon_i)}{\pi(\varepsilon_i)}, \quad i = 1, 2, \ldots, N,$$

where ε_i are defined in (3.1). Note that the pricing operator π must be *linear* in order to preclude arbitrage opportunities.

We use the following result repeatedly. See Kijima and Muromachi (2001) for the proof.

Lemma 3.1. *Suppose that (X, Z) is normally distributed. Then,*

$$\mathbb{E}\left[f(X)\mathrm{e}^{-\lambda Z}\right] = \mathbb{E}\left[f(X - \lambda \mathrm{cov}(X, Z))\right] \mathbb{E}\left[\mathrm{e}^{-\lambda Z}\right]$$

for any $f(x)$ for which the expectations exist, where cov *denotes the covariance operator.*

3.1. **The case that the rates of return are normally distributed.** In this section, we assume that the random vector $(R_1, \ldots, R_K, R_\varepsilon^1, \ldots, R_\varepsilon^N)$ defined above is normally distributed. The normality assumption on asset returns has been frequently used in the finance literature. Indeed, the instantaneous rate of return $\mathrm{d}S(t)/S(t)$ is usually assumed to follow the stochastic differential equation

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}w(t),$$

where $\mu(t)$ and $\sigma(t)$ are stochastic processes adapted to the filtration $\{\mathcal{F}_t\}$ generated by the standard Brownian motion $w(t)$. In this setting, given the history $\mathcal{F}_t$, the instantaneous rate of return $\mathrm{d}S(t)/S(t)$ follows a normal distribution with mean $\mu(t)\mathrm{d}t$ and variance $\sigma^2(t)\mathrm{d}t$.

We note from (3.1) that the potential risk X_i can be written as

$$X_i = \pi_i\left(1 + \sum_j w_j^i R_j + W^i R_\varepsilon^i\right), \qquad i = 1, 2, \ldots, N, \tag{3.6}$$

where $\pi_i = \pi(X_i) = \sum_j x_j^i s_j + \pi(\varepsilon_i)$ and

$$w_j^i = \frac{s_j x_j^i}{\pi_i}, \quad j = 1, 2, \ldots, K; \qquad W^i = \frac{\pi(\varepsilon_i)}{\pi_i} = 1 - \sum_j w_j^i.$$

Also, we have

$$\sum_j y_j^i S_j = \sum_j y_j^i s_j (1 + R_j) = \sum_j y_j^i s_j R_j + \sum_j y_j^i s_j.$$

It follows that

$$X_i + \sum_j y_j^i S_j = \sum_j s_j (x_j^i + y_j^i) R_j + \pi(\varepsilon_i) R_\varepsilon^i + \pi_i + \sum_j y_j^i s_j.$$

Therefore, the FOC (3.5) can be written as

$$\mathbb{E}\left[R_j \mathrm{e}^{-\lambda_i \Gamma^i}\right] = 0, \qquad \Gamma^i \equiv \sum_{j=1}^K s_j (x_j^i + y_j^i) R_j + \pi(\varepsilon_i) R_\varepsilon^i, \tag{3.7}$$

for all i and j.

A direct application of Lemma 3.1 to (3.7) then yields

$$\mu_j - \lambda_i \mathrm{cov}(R_j, \Gamma^i) = 0,$$

where $\mu_j = \mathbb{E}[R_j]$ denotes the mean rate of return of asset j. It follows from the definition of Γ^i that

$$\frac{1}{\lambda_i} \mu_j = \pi(\varepsilon_i) \sigma_{ij}^\varepsilon + \sum_k s_k (x_k^i + y_k^i) \sigma_{kj},$$

where $\sigma_{ij}^\varepsilon = \mathrm{cov}(R_j, R_\varepsilon^i)$ and $\sigma_{ij} = \mathrm{cov}(R_i, R_j)$. Summing over i and utilizing the market clearing condition (3.3), we obtain

$$\frac{1}{\lambda} \mu_j = \xi_j + \sum_k s_k A_k \sigma_{kj}, \qquad j = 1, 2, \ldots, K, \tag{3.8}$$

where $\xi_j = \sum_i \pi(\varepsilon_i) \sigma_{ij}^\varepsilon$ and λ is defined in (2.11).

Now, denoting $\gamma_k = s_k A_k$, Equation (3.8) can be written in matrix form as

$$\frac{1}{\lambda} \mu - \xi = \mathbf{\Sigma} \gamma \quad \text{or, equivalently,} \quad \gamma = \frac{1}{\lambda} \mathbf{\Sigma}^{-1} (\mu - \lambda \xi),$$

where $\mu = (\mu_j)$, $\xi = (\xi_j)$ and $\gamma = (\gamma_j)$ are K-dimensional vectors, and $\mathbf{\Sigma} = (\sigma_{ij})$ is a $K \times K$ symmetric matrix. Here, we have assumed that the covariance matrix $\mathbf{\Sigma}$ is positive definite (hence, it is invertible). Therefore, we obtain the equilibrium prices as

$$s_j = \frac{1}{\lambda A_j} \left[\mathbf{\Sigma}^{-1} (\mu - \lambda \xi)\right]_j, \qquad j = 1, 2, \ldots, K, \tag{3.9}$$

where $[b]_j$ denotes the jth component of vector b. Note that the pricing formula (3.9) is not explicit, because the values ξ_j contain unknown prices $\pi(\varepsilon_i)$ and covariances σ_{ij}^ε.

It remains to show that the prices (3.9) coincide with those calculated from (2.12). To this end, first note that the aggregated risk is given by

$$Z = \sum_{i=1}^N X_i = \sum_{j=1}^K A_j S_j + \sum_{i=1}^N \varepsilon_i, \qquad A_j = \sum_{i=1}^N x_j^i, \tag{3.10}$$

and its rate of return is written as

$$R_Z = \sum_{j=1}^K \tilde{w}_j R_j + \sum_{i=1}^N \tilde{w}^i R_\varepsilon^i; \qquad \tilde{w}_j = \frac{A_j s_j}{\pi(Z)}, \quad \tilde{w}^i = \frac{\pi(\varepsilon_i)}{\pi(Z)}, \tag{3.11}$$

where $R_Z = (Z - \pi(Z))/\pi(Z)$.

Now, from (2.12), we need to calculate

$$J_j \equiv \frac{\mathbb{E}[s_j(1+R_j)\mathrm{e}^{-\lambda\pi(Z)(1+R_Z)}]}{\mathbb{E}[\mathrm{e}^{-\lambda\pi(Z)(1+R_Z)}]}, \qquad j = 1, 2, \dots, N.$$

Again, thanks to Lemma 3.1, we obtain

$$J_j = s_j \left(1 + \mu_j - \lambda\pi(Z)\mathrm{cov}(R_j, R_Z)\right).$$

It follows from (3.11) that

$$J_j = s_j \left(1 + \mu_j - \lambda \left[\sum_k A_k s_k \sigma_{kj} + \sum_i \pi(\varepsilon_i)\sigma_{ij}^{\varepsilon}\right]\right).$$

But, from (3.9) or equivalently from (3.8), the square bracket term is equal to μ_j, whence we obtain $J_j = s_j$. It follows that, when the rates of return of risks are normally distributed, the equilibrium price of traded asset j is given by (2.12).

Summarizing, we have the following result.

Theorem 3.1. *Suppose that the rates of return of all the risks are normally distributed. Then, the equilibrium price of any traded asset is given by* (2.12).

The equilibrium risk allocation is not given by (2.13), because there are residual risks ε_i in our incomplete market setting. We shall return this problem later.

3.2. **A refined CAPM.** In the previous subsection, we show that the multivariate Esscher transform (2.12) holds for the incomplete market setting. In this subsection, we discuss how the ordinary CAPM (capital asset pricing model) is modified in our setting.[3]

Suppose that the rates of return of all the risks are normally distributed. Then, from (2.12), we have

$$0 = \mathbb{E}\left[(S_j - s_j)\mathrm{e}^{-\lambda Z}\right] = \mathbb{E}\left[R_j \mathrm{e}^{-\lambda\pi(Z)(1+R_Z)}\right] = \mathbb{E}\left[R_j \mathrm{e}^{-\lambda\pi(Z)R_Z}\right].$$

It follows from Lemma 3.1 that

$$0 = \mu_j - \lambda\pi(Z)\mathrm{cov}(R_j, R_Z),$$

which coincides with (3.8). Moreover, if we consider the riskfree interest rate r_f, the above expression becomes

$$\mu_j - r_f = \lambda\pi(Z)\mathrm{cov}(R_j, R_Z). \tag{3.12}$$

Hence, the term $\lambda\pi(Z)\mathrm{cov}(R_j, R_Z)$ is considered to be a risk premium. Similarly, for the pricing of the market portfolio M, we obtain

$$\mu_M - r_f = \lambda\pi(Z)\mathrm{cov}(R_M, R_Z).$$

It follows that

$$\mu_j - r_f = \frac{\mathrm{cov}(R_j, R_Z)}{\mathrm{cov}(R_M, R_Z)}(\mu_M - r_f) \tag{3.13}$$

for each traded asset j.

However, in the expression (3.13), the role of the market portfolio M is unclear. In order to consider M explicitly, we assume that the aggregated risk Z is defined as in (3.10), and the market portfolio and its rate of return are given, respectively, by

$$M = \sum_{j=1}^{K} A_j S_j, \qquad R_M = \frac{M - \pi(M)}{\pi(M)}. \tag{3.14}$$

[3] Wang (2002) also discussed the CAPM within the framework of Wang transform.

Since we then have

$$Z = M + \bar{\varepsilon}, \qquad \bar{\varepsilon} = \sum_{i=1}^{N} \varepsilon_i,$$

it follows from (3.13) that

$$\mu_j - r_f = \frac{\sigma_{jM} + \sigma_{j\varepsilon}}{\sigma_M^2 + \sigma_{M\varepsilon}} (\mu_j - r_f),$$

where $\sigma_{jM} = \mathrm{cov}(R_j, R_M)$, etc. Therefore, denoting

$$\alpha_j = \frac{\sigma_{j\varepsilon}}{\sigma_M^2 + \sigma_{M\varepsilon}}, \qquad \beta_j = \frac{\sigma_{jM}}{\sigma_M^2 + \sigma_{M\varepsilon}},$$

we conclude that

$$\mu_j - r_f = \alpha_j + \beta_j(\mu_M - r_f). \tag{3.15}$$

Note that the ordinary CAPM is a special case of (3.15) when the residuals ε_i are all zero.

If the market portfolio M is sufficiently diversified, it is plausible to assume that $\sigma_{M\varepsilon} = 0$. In this case, we have the following result.

Theorem 3.2 (Refined CAPM)**.** *Suppose that the rates of return of all the risks are normally distributed and that the market portfolio defined in* (3.14) *is fully diversified. Then,*

$$\mu_j - r_f = \alpha_j + \beta_j(\mu_j - r_f); \qquad \alpha_j = \frac{\sigma_{j\varepsilon}}{\sigma_M^2}, \quad \beta_j = \frac{\sigma_j}{\sigma_M} \rho_{jM}, \tag{3.16}$$

for any traded asset j, where ρ_{jM} denotes the correlation between R_j and R_M.

Note that the β_j in Theorem 3.2 coincides with the beta in the ordinary CAPM. However, the intercept α_j does not vanish,[4] because the correlation (risk) between each asset and the residuals cannot be eliminated in general. Note that α_j can be either positive or negative. If it is positive (negative, respectively), i.e. asset j is positively (negatively) correlated to the residuals, more (less) premiums are required. We call (3.16) a refined CAPM.

3.3. **Pricing of derivative securities.** Suppose that derivative securities written on traded assets are introduced in the market and that the equilibrium prices of the derivatives are given by (2.12). In this subsection, we show that the risk-neutral pricing method holds true for the pricing of derivative securities.

Consider, as an example, a call option with strike price K written on asset j. That is, we denote

$$Y = (S_j - K)_+ = f(R_j), \qquad f(x) = (s_j(1+x) - K)_+,$$

where $(x)_+ = \max\{x, 0\}$, and assume that

$$R_j = \mu_j + \sigma_j w_j, \tag{3.17}$$

where w_j follows a standard normal distribution. Here, μ_j denotes the mean rate of return and σ_j the volatility of asset j. According to (2.12) with R_Z being normally distributed, the call option price is then given by

$$\begin{aligned} \pi(Y) &= \frac{\mathbb{E}[f(R_j) \mathrm{e}^{-\lambda\pi(Z)R_Z}]}{\mathbb{E}[\mathrm{e}^{-\lambda\pi(Z)R_Z}]} \\ &= \mathbb{E}\left[f(\mu_j + \sigma_j w_j - \lambda\pi(Z)\mathrm{cov}(R_j, R_Z))\right], \end{aligned} \tag{3.18}$$

where we have used Lemma 3.1 for the second equation. However, since the relation (3.12) holds in equilibrium, we obtain

$$\pi(Y) = \mathbb{E}\left[(s_j(1 + \sigma_j w_j) - K)_+\right] \tag{3.19}$$

[4] The coefficient α_j often called Jensen's alpha (1968) to measure the investment performance in the finance literature.

as the call option price. This is so, because the risk premium $\lambda\pi(Z)\mathrm{cov}(R_j, R_Z)$ in (3.12) is already reflected in the price of the underlying asset j. This result is important for practice, because we do not need to estimate the unknown (unobservable) parameters λ, $\pi(Z)$ and $\mathrm{cov}(R_j, R_Z)$ for the pricing of derivative securities.

Moreover, if we adopt the approximation

$$1 + \sigma_j w_j \;\approx\; \mathrm{e}^{\sigma_j w_j - \sigma_j^2/2}, \tag{3.20}$$

it is readily shown that the call price (3.19) becomes

$$\pi(Y) = s_j \Phi(d) - K\Phi(d - \sigma_j), \qquad d = \frac{\log(s_j/K)}{\sigma_j} + \frac{\sigma_j}{2},$$

the famous Black–Scholes formula (1973) with $r_f = 0$ and $T = 1$.

Theorem 3.3. *Suppose that the rates of return of all the risks are normally distributed. Then, the price of a call option written on a traded asset is given by the Black–Scholes formula, provided that the approximation* (3.20) *is valid.*

Next, as in Merton (1976), suppose that there is a jump in the rate of return of asset j with some probability. That is, instead of (3.17), we assume that

$$R_j = \begin{cases} R_1 \equiv \mu_j + \sigma_j w_j, & \text{probability } 1 - p, \\ R_2 \equiv \mu_j + \sigma_j w_j + \mu_J + \sigma_J w_J, & \text{probability } p, \end{cases} \tag{3.21}$$

where p stands for the probability of jump. Note that, as in Merton (1976), the jump size is assumed to be normally distributed with mean μ_J and variance σ_J^2, while the correlation between w_j and w_J may not be zero.

Suppose that the pricing formula is given by (2.12) with R_Z being normally distributed. Then, as before, we have from (3.21) that

$$\begin{aligned} 0 &= \mathbb{E}\left[R_j \mathrm{e}^{-\lambda\pi(Z)R_Z}\right] \\ &= (1-p)\mathbb{E}\left[R_1 \mathrm{e}^{-\lambda\pi(Z)R_Z}\right] + p\mathbb{E}\left[R_2 \mathrm{e}^{-\lambda\pi(Z)R_Z}\right] \\ &= (1-p)\left[\mu_j - \lambda\pi(Z)\mathrm{cov}(R_1, Z)\right] + p\left[\mu_j + \mu_J - \lambda\pi(Z)\mathrm{cov}(R_2, Z)\right]. \end{aligned}$$

After some simple algebra, we then have

$$\mu_j + p\mu_J = \lambda\pi(Z)\left(\sigma_j \mathrm{cov}(w_j, R_Z) + p\sigma_J \mathrm{cov}(w_J, R_Z)\right), \tag{3.22}$$

from which we can recover the formula (3.12) with $r_f = 0$, since $\mathbb{E}[R_j] = \mu_j + p\mu_J$.

Now, as before, consider a call option with strike price K written on the asset j with jump risk defined in (3.21). Then, similar to (3.18), we obtain

$$\begin{aligned} \pi(Y) &= (1-p)\mathbb{E}\left[f(\mu_j + \sigma_j w_j - \lambda\pi(Z)\sigma_j \mathrm{cov}(w_j, R_Z))\right] \\ &\quad + p\mathbb{E}\left[f(\mu_j + \mu_J + \sigma_j w_j + \sigma_J w_J - \lambda\pi(Z)(\sigma_j \mathrm{cov}(w_j, R_Z) + \sigma_J \mathrm{cov}(w_J, R_Z)))\right]. \end{aligned} \tag{3.23}$$

Note the difference between (3.19) and (3.23). In (3.23), we have no means to eliminate the risk premiums $\lambda\pi(Z)\sigma_j\mathrm{cov}(w_j, R_Z)$ and $\lambda\pi(Z)\sigma_J\mathrm{cov}(w_J, R_Z)$ by using (3.22). This is the significant difference in the jump model, although the asset price satisfies the relation (3.12).

When $\mathrm{cov}(w_J, R_Z) = 0$, however, we have from (3.22) that

$$\mu_j + p\mu_J = \lambda\pi(Z)\sigma_j \mathrm{cov}(w_j, R_Z),$$

from which the pricing formula (3.23) is reduced to

$$\pi(Y) = (1-p)\mathbb{E}\left[f(-p\mu_J + \sigma_j w_j)\right] + p\mathbb{E}\left[f((1-p)\mu_J + \sigma_j w_j + \sigma_J w_J)\right].$$

Hence, if the jump risk premium $\lambda\pi(Z)\sigma_J\mathrm{cov}(w_J, R_Z)$ is known, say λ_J in general, then the call option price is given by a mixture of the Black–Scholes formulas as in Merton (1976), provided that the approximation (3.20) is valid. An extension to multiple jumps is straightforward.

4. Some Discussions

In this section, we discuss some related problems. Namely, of interest are the cases that risks are themselves normally distributed and that there are no residuals ε_i in the market. The meaning of equilibrium risk allocation (2.13) becomes clear by considering these cases.

4.1. **The case that risks are normally distributed.** The simplest case to solve the problem (3.5) is to assume that the random vector $(X_i, S_1, \ldots, S_K)$ is normally distributed.[5] In this case, since $U_i \equiv X_i + \sum_{j=1}^K y_j^i S_j$ follows a normal distribution, a direct application of Lemma 3.1 to (3.5) yields

$$\nu_j - \lambda_i \left(\sigma_{ij}^X + \sum_{k=1}^K \sigma_{jk} y_k^i \right) = s_j, \qquad j = 1, 2, \ldots, K, \tag{4.1}$$

where $\nu_j = \mathbb{E}[S_j]$, $\sigma_{ij}^X = \mathrm{cov}(S_j, X_i)$ and $\sigma_{ij} = \mathrm{cov}(S_i, S_j)$. It follows that

$$\lambda_i \sum_{k=1}^K \sigma_{jk} y_k^i = \nu_j - s_j - \lambda_i \sigma_{ij}^X, \qquad j = 1, 2, \ldots, K,$$

or, in the matrix form,

$$\lambda_i \mathbf{\Sigma} y^i = \nu - s - \lambda_i \sigma_i^X,$$

where $\mathbf{\Sigma} = (\sigma_{ij})$ is a $K \times K$ matrix, $\nu = (\nu_j)$, $s = (s_j)$ and $\sigma_i^X = (\sigma_{ij}^X)$ are K-dimensional vectors. Assuming that the matrix $\mathbf{\Sigma}$ is invertible, the equilibrium risk exchange y^{i*} is obtained as

$$y^{i*} = \frac{1}{\lambda_i} \mathbf{\Sigma}^{-1} (\nu - s) - \mathbf{\Sigma}^{-1} \sigma_i^X. \tag{4.2}$$

Summing (4.2) over i, it follows from the market clearing condition (3.3) that

$$0 = \frac{1}{\lambda} \mathbf{\Sigma}^{-1} (\nu - s) - \mathbf{\Sigma}^{-1} \sigma^X, \tag{4.3}$$

where λ is defined in (2.11), 0 denotes the zero vector, and $\sigma^X = \sum_{i=1}^N \sigma_i^X$. Therefore, we obtain componentwise

$$s_j = \nu_j - \lambda \sum_{i=1}^N \sigma_{ij}^X, \qquad j = 1, 2, \ldots, K. \tag{4.4}$$

It is easy to check that (4.4) is equivalent to (2.12) for this case. To see this, using Lemma 3.1, the equilibrium price (2.12) can be written as

$$\pi(S_j) = \frac{\mathbb{E}[S_j \mathrm{e}^{-\lambda Z}]}{\mathbb{E}[\mathrm{e}^{-\lambda Z}]} = \mathbb{E}[S_j - \lambda \mathrm{cov}(S_j, Z)] = \nu_j - \lambda \mathrm{cov}(S_j, Z),$$

which coincides with (4.4), where $Z = \sum_{i=1}^N X_i$.

Next, we consider the equilibrium risk allocation in this setting. To this end, the equilibrium risk exchange is obtained from (4.2) and (4.3) as $(y^{1*}, y^{2*}, \ldots, y^{N*})$, where

$$y^{i*} = \mathbf{\Sigma}^{-1} \left(\frac{\lambda}{\lambda_i} \sigma^X - \sigma_i^X \right), \qquad i = 1, 2, \ldots, N. \tag{4.5}$$

[5] Note that lognormal distributions do not admit moment generating functions. In other words, the Esscher transform does not exist for lognormal risks. The normal assumption for asset prices seems strange; however, economists usually adopt this assumption.

Recall that each potential risk X_i is given by (3.1). Since

$$\sigma_{ij}^X = \mathrm{cov}(X_i, S_j) = \sum_{k=1}^K x_k^i \sigma_{kj} + \sigma_{ij}^\varepsilon,$$

where $\sigma_{ij}^\varepsilon = \mathrm{cov}(\varepsilon_i, S_j)$, we obtain

$$\sigma_i^X = \mathbf{\Sigma} x^i + \sigma_i^\varepsilon \quad \text{or, equivalently,} \quad \mathbf{\Sigma}^{-1}\sigma_i^X = x^i + \mathbf{\Sigma}^{-1}\sigma_i^\varepsilon \tag{4.6}$$

with $x^i = (x_j^i)$ and $\sigma_i^\varepsilon = (\sigma_{ij}^\varepsilon)$. It follows from (4.5) that

$$x^i + y^{i*} = \frac{\lambda}{\lambda_i}\mathbf{\Sigma}^{-1}\sigma^X - \mathbf{\Sigma}^{-1}\sigma_i^\varepsilon.$$

Hence, we obtain

$$X_i + y^{i*}S = (x^i + y^{i*})S + \varepsilon_i = \frac{\lambda}{\lambda_i}\sigma^X\mathbf{\Sigma}^{-1}S - \sigma_i^\varepsilon\mathbf{\Sigma}^{-1}S + \varepsilon_i,$$

where $S = (S_j)$ is the K-dimensional vector. On the other hand, from (4.6), we have

$$Z = \sum_{i=1}^N X_i = \sum_{i=1}^N (x^i S + \varepsilon_i) = \sigma^X\mathbf{\Sigma}^{-1}S - \sigma^\varepsilon\mathbf{\Sigma}^{-1}S + \sum_{i=1}^N \varepsilon_i,$$

where $\sigma^\varepsilon = \sum_i \sigma_i^\varepsilon$. Therefore, the equilibrium risk allocation is given by

$$X_i + y^{i*}S = \frac{\lambda}{\lambda_i}Z + D_i, \qquad i = 1, 2, \ldots, N, \tag{4.7}$$

where

$$D_i \equiv \left(\frac{\lambda}{\lambda_i}\sum_k \sigma_k^\varepsilon - \sigma_i^\varepsilon\right)\mathbf{\Sigma}^{-1}S - \left(\frac{\lambda}{\lambda_i}\sum_k \varepsilon_k - \varepsilon_i\right).$$

Of course, when the residuals ε_i are all zero, the equilibrium risk allocation (4.7) coincides with (2.13), because $D_i = 0$ in this case. This is so, because our market is *incomplete*, we have no means to duplicate the potential risks X_i by using the tradable assets Y_j only. Hence, the equilibrium risk allocation is affected by the residuals ε_i.

However, an interesting observation is in order. Suppose that

$$\varepsilon_i = \frac{\lambda}{\lambda_i}\sum_k \varepsilon_k, \qquad i = 1, 2, \ldots, N. \tag{4.8}$$

That is, even though there are residual risks in the economy, the residual risk allocation is proportional to the aggregated residual risks $\sum_k \varepsilon_k$ with weight $\lambda/\lambda_i > 0$. Then, since

$$\sigma_{ij}^\varepsilon = \mathrm{cov}(\varepsilon_i, S_j) = \frac{\lambda}{\lambda_i}\sum_k \mathrm{cov}(\varepsilon_k, S_j) = \frac{\lambda}{\lambda_i}\sum_k \sigma_{kj}^\varepsilon,$$

we conclude that $D_i = 0$, whence the equilibrium risk allocation (2.13) holds true.

4.2. **The case of no residual risks.** In this subsection, we assume that the potential risks are portfolios of traded assets. That is, the potential risk X_i of agent i is given by

$$X_i = \sum_{j=1}^K x_j^i S_j, \qquad i = 1, 2, \ldots, N,$$

while we impose no assumptions on asset distributions. Although this assumption seems very restrictive in actual insurance markets, we can solve the nonlinear simultaneous equations (3.5) explicitly for arbitrary S_j.

Note that, in this case, the time-0 portfolio of agent i before trading is $x^i = (x_1^i, \dots, x_K^i)$, and the portfolio of agent i after trading is given by $x^i + y^i = (x_1^i + y_1^i, \dots, x_K^i + y_K^i)$, where $y^i = (y_1^i, \dots, y_K^i)$.

Suppose that the utilities are exponential as before. In this setting, from (3.5), we derive the following system of simultaneous equations:

$$\mathbb{E}\left[S_j \mathrm{e}^{-\lambda_i \sum_{j=1}^K (x_j^i + y_j^i) S_j}\right] = s_j \mathbb{E}\left[\mathrm{e}^{-\lambda_i \sum_{j=1}^K (x_j^i + y_j^i) S_j}\right], \qquad \forall i, j. \tag{4.9}$$

For each asset j, a sufficient condition for (4.9) to hold for any $(S_1, \dots, S_K)$ is given by

$$\lambda_i (x_j^i + y_j^i) = \lambda_{i+1} (x_j^{i+1} + y_j^{i+1}), \qquad i = 1, 2, \dots, N-1, \tag{4.10}$$

together with the clearing condition (3.3), for each asset j. Since we have N equations and N unknowns y_j^i for each j, we can solve the simultaneous equations. Indeed, the solution of the simultaneous equations is given by

$$y_j^{i*} = \frac{\lambda}{\lambda_i} \sum_{i=1}^N x_j^i - x_j^i = \frac{\lambda}{\lambda_i} A_j - x_j^i, \qquad i = 1, 2, \dots, N, \tag{4.11}$$

where λ is defined in (2.11) and A_j denotes the total number of asset j issued in the market.

Now, from (4.11), we have

$$\lambda_i (x_j^i + y_j^{i*}) = \lambda A_j, \qquad i = 1, 2, \dots, N,$$

whence we obtain

$$\lambda_i \sum_{j=1}^K (x_j^i + y_j^{i*}) S_j = \lambda \sum_{j=1}^K A_j S_j = \lambda Z. \tag{4.12}$$

It follows from (4.9) that the equilibrium price of asset j is given by

$$s_j = \frac{\mathbb{E}[S_j \mathrm{e}^{-\lambda Z}]}{\mathbb{E}[\mathrm{e}^{-\lambda Z}]}, \qquad Z = \sum_{j=1}^K A_j S_j.$$

The equilibrium risk allocation is obtained from (4.12) as

$$X_i + y^{i*} S = \sum_j \left(x_j^i + y_j^{i*}\right) S_j = \frac{\lambda}{\lambda_i} Z,$$

which coincides with (2.13). Of course, this must be so, because we have no residual risks in this setting.

Next, consider the case that there are residual risks ε_i in the economy whose allocation is given by (4.8). In this case, it is expected that the equilibrium risk exchange remains the same as (4.11), because the residual risks are already proportional to the equilibrium risk allocation. In fact, the FOC (3.5) is written as

$$\mathbb{E}\left[S_j \mathrm{e}^{-\lambda_i \sum_{j=1}^K (x_j^i + y_j^i) S_j - \lambda_i \varepsilon_i}\right] = s_j \mathbb{E}\left[\mathrm{e}^{-\lambda_i \sum_{j=1}^K (x_j^i + y_j^i) S_j - \lambda_i \varepsilon_i}\right], \qquad \forall i, j,$$

and a sufficient condition for this simultaneous equations to hold for any $(S_1, \dots, S_K)$ is given by (4.10), because we have $\lambda_i \varepsilon_i = \lambda_{i+1} \varepsilon_{i+1}$ for all i due to the condition (4.8). Equation (4.12) is modified as

$$\lambda_i \sum_{j=1}^K (x_j^i + y_j^{i*}) S_j + \lambda_i \varepsilon_i = \lambda \sum_{j=1}^K A_j S_j + \lambda \sum_k \varepsilon_k = \lambda Z,$$

where we have used (4.8) for the first equality and (3.10) for the second, respectively. Therefore, the equilibrium price of asset j is given by

$$s_j = \frac{\mathbb{E}[S_j \mathrm{e}^{-\lambda Z}]}{\mathbb{E}[\mathrm{e}^{-\lambda Z}]}, \qquad Z = \sum_{j=1}^{K} A_j S_j + \sum_{i=1}^{N} \varepsilon_i,$$

and the equilibrium risk allocation is obtained as

$$X_i + y^{i*} S = \sum_j \left(x_j^i + y_j^{i*}\right) S_j + \varepsilon_i = \frac{\lambda}{\lambda_i} Z.$$

That is, the same equilibrium results as in Bühlmann (1980) hold true even in our incomplete market setting under the stated assumptions.

Summarizing, we have obtained the following result.

Theorem 4.1. *Suppose that the residual risks ε_i in the economy are either all zero or allocated as in* (4.8)*. Then, the equilibrium price and the equilibrium risk allocation are given by* (2.12) *and* (2.13)*, respectively.*

5. Concluding Remarks

In this paper, we examine the Bühlmann's equilibrium pricing model (1980) in an incomplete market setting and derive the multivariate Esscher transform (2.12) within the equilibrium framework under some assumptions. The result reveals that the (multivariate) Esscher transform is an appropriate probability transform for the pricing of insurance risks whose market is presumably incomplete. However, we may suspect that the pricing formula (2.12) is not always true. Indeed, we can provide some counterexamples.

Consider for simplicity the case that $K = 1$ and $N = 2$, and assume that

$$X_1 = x_1 S + \varepsilon_1, \qquad X_2 = x_2 S + \varepsilon_2.$$

Then, the problem (3.5) becomes

$$\mathbb{E}\left[S \mathrm{e}^{-\lambda_1(\varepsilon_1 + (x_1+y)S)}\right] = \pi(S) \mathbb{E}\left[\mathrm{e}^{-\lambda_1(\varepsilon_1+(x_1+y)S)}\right] \tag{5.1}$$

and at the same time

$$\mathbb{E}\left[S \mathrm{e}^{-\lambda_2(\varepsilon_2 + (x_2-y)S)}\right] = \pi(S) \mathbb{E}\left[\mathrm{e}^{-\lambda_2(\varepsilon_2+(x_2-y)S)}\right], \tag{5.2}$$

where we have applied the market clearing condition for (5.2).

Suppose that S follows a standard normal distribution and, given S, $(\varepsilon_1, \varepsilon_2)$ follows a bivariate normal distribution with zero means, variances S and correlation ρ. Then, since $Z = AS + \varepsilon_1 + \varepsilon_2$ with $A = x_1 + x_2$, we have

$$\begin{aligned} \mathbb{E}\left[S \mathrm{e}^{-\lambda Z}\right] &= \mathbb{E}\left[\mathbb{E}\left[S \mathrm{e}^{-\lambda(AS+\varepsilon_1+\varepsilon_2)} | S\right]\right] \\ &= \mathbb{E}\left[S \mathrm{e}^{-\lambda AS} \mathbb{E}\left[\mathrm{e}^{-\lambda(\varepsilon_1+\varepsilon_2)} | S\right]\right] \\ &= \mathbb{E}\left[S \mathrm{e}^{-\lambda AS} \mathrm{e}^{\lambda^2(1+\rho)S}\right]. \end{aligned}$$

Therefore, from (2.12) and Lemma 3.1, we obtain

$$\pi(S) = \lambda^2(1+\rho) - \lambda(x_1 + x_2), \qquad \lambda = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}. \tag{5.3}$$

On the other hand, we have

$$\begin{aligned} \mathbb{E}\left[S \mathrm{e}^{-\lambda_1(\varepsilon_1+(x_1+y)S)}\right] &= \mathbb{E}\left[S \mathrm{e}^{-\lambda_1(x_1+y)S} \mathrm{e}^{\lambda_1^2 S/2}\right] \\ &= \left(\frac{\lambda_1^2}{2} - \lambda_1(x_1+y)\right) \mathbb{E}\left[\mathrm{e}^{\lambda_1^2 S/2}\right]. \end{aligned}$$

Hence, from (5.1), we obtain

$$\pi(S) = \frac{\lambda_1^2}{2} - \lambda_1(x_1 + y).$$

Similarly, from (5.2), we have

$$\pi(S) = \frac{\lambda_2^2}{2} - \lambda_2(x_2 - y).$$

Solving these simultaneous equations, we obtain

$$\frac{1}{\lambda}\pi(S) = \frac{\lambda_1 + \lambda_2}{2} - (x_1 + x_2).$$

It follows that

$$\pi(S) = \frac{\lambda_1 \lambda_2}{2} - \lambda(x_1 + x_2). \tag{5.4}$$

Note that the prices (5.3) and (5.4) coincide with each other when $\lambda_1 = \lambda_2$ and $\rho = 1$, i.e., the condition (4.8) holds (see Theorem 4.1).

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