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Existence of a Pure Strategy Equilibrium in Markov Games with Strategic Complementarities for Finite Actions and States

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Abstract

In this paper, we provide the sufficient conditions for a Markov perfect equilibrium in pure strategies to exist for a class of stochastic games with finite horizon, in which any stage game has strategic complementarities. In contrast to previous studies (e.g., Curtat (1996) and Amir (2002)), we assume herein that the sets of actions and the set of states is finite and do not assume dominant diagonal conditions for payoffs and the transition probability, which yield the uniqueness of equilibria. The greatest equilibrium is monotonically increasing in the state. The main result is applied to a model of a Bertrand competition with investment.

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1 Introduction

A stochastic game is a collection of strategic form games indexed by a state variable. The players choose their actions relying on a state, which changes from one period to the next according to a stochastic process that depends on the actions of all of the players. Stochastic games were introduced in Shapley (1953) and Markov games, which provide the most typical and practical models in stochastic games, have been developed in the field of economics and operations research. A number of models of Markov games have been applied to a number of interesting problems, such as an oligopolistic industry with investment (Ericson and Pakes (1995)), political economics (Acemoglu and Robinson (2001)), and competition with inventory control or supply chain management (Hong, McAfee, and Nayyar (2002)).

In this paper, we establish the existence of a Markov perfect equilibrium in pure strategies for a class of Markov games with finite horizons, in which each stage game has strategic complementarities. In general, the existence of stationary equilibria in Markov games with infinite horizons is complex, as indicated by several studies. For example, Shapley (1953) proved the existence of a stationary Markov equilibrium for each zero-sum discounted game with an infinite horizon. Also, Federgruen (1978) extended the existence result for non-zero-sum discounted games with a finite or countable state space. Finally, Nowak (1985) presented a condition for the existence of $\epsilon-$equilibrium with an uncountable state space and compact metric action spaces.

The Markov games with finite horizons and finite states are included in the class of finite games in extensive form which always have equilibria if we allow the mixed strategies or assume that the sets of actions are compact metric spaces.

However, the existence of Markov equilibria in pure strategies for the finite sets of actions still remains to be solved. Several of the applications described above considered the pure strategy equilibrium in their models, and numerical results were computed for the data set in discretized actions and states. Thus, the proof of the existence of Markov equilibria in pure strategies with finite actions and states is a critical problem.

In general, even a one-shot game in strategic form may not have a pure strategy equilibrium. Hence, additional conditions are required in order to ensure the existence of equilibria in pure strategies. Games with strategic complementarities, which have been investigated by Topkis (1978, 1979, 1998) Vives (1990), Milgrom and Roberts (1990), and Milgrom and Shannon

(1994), provide a class of games in strategic form that guarantees the existence of pure strategy equilibria[1]. However, Vives (2007) showed that multi-stage games, including Markov games, may not preserve strategic complementarities, even if each stage game has a strategic complementarity. Hence, additional assumptions are necessary for the existence of a Markov perfect equilibrium in pure strategies even if a horizon of the Markov game is finite. Echenique (2004) also reported that the extension of strategic complementarities to dynamic games in extensive form is very restrictive.

The remarkable conditions for the existence of a pure strategy equilibrium were investigated by Curtat (1996) and Amir (2002). Curtat (1996) considered games with strategic complementarities that have infinite horizons and the compact metric spaces of actions and states and showed a sufficient condition for the existence of a stationary equilibrium in pure strategies. Amir (2002) presented another proof for the results of Curtat (1996) and showed that the condition also ensures the equilibrium in pure strategies for a Markov game with a finite horizon.

The purpose of this paper is to extend the results of Amir (2002) to discrete action spaces. Curtat (1996) and Amir (2002) assumed *dominant diagonal conditions* for the payoffs and the transition probability. However, the dominant diagonal conditions requires twice differentiability for the payoff functions and the transition probability and so cannot be applied to a game with finite actions.

In Curtat (1996) and Amir (2002), the dominant diagonal conditions ensure the uniqueness of the equilibrium and induce Lipschitz continuity of the Bellman mappings. This implies that the Bellman mappings satisfy the conditions for Schauder's fixed point theorem, which derives the existence of the stationary equilibrium.

This paper examines that the greatest equilibrium stile exists in the game with a finite horizon even if the uniqueness of the equilibrium may not hold. This is established by showing that the monotonicities and supermodularities for the payoffs and the transition probability are sufficient for the existence of the greatest equilibrium of reduced games.

Recently, Amir (2009) independently obtains the similar results to this paper for the Markov games in which the set of actions are compact metric spaces. Amir (2009) shows the existence of the equilibrium in pure strategies with both finite and infinite horizons if the payoff functions and the transition probability are continuous and they satisfy the conditions for strategic

[1] Elementary surveys are given by Amir (2005) and Vives (2005).

complementarities and monotonicities.

In contrast to the results of Amir (2009), the purpose of this paper is to establish the existence of the equilibria for games in which the sets of both actions and states are finite. As we stated above, a number of applications of Markov games analyze the numerical examples for the data set in discretized actions and states. Our motivation is to ensure the existence of the equilibrium for the computation of the Markov equilibria for such applications. In addition, we examine multi-dimensional states taking such applications into account while Amir (2009) treats one dimensional states. Since Amir (2009) assumes that the set of states and the sets of actions are compact metric spaces, several mathematical tools for both functional analysis (for example, Schauder's fixed point theorem, the compactness and convergence of the set of continuous functions, the Arzera-Ascoli theorem, etc.) and lattice theory are required. On the other hand, our proof is simple and requires only knowledge of lattice theory by restricting the sets to be finite. To provide these simple and elementary proofs of the results with this discretized settings is another contirbution of this paper.

The remainder of the paper is organized as follows. In Section 2, Markov games and the equilibrium concepts are defined. In Section 3, we show the sufficient conditions for the existence and monotonicity of the equilibrium. In Section 4, we apply the condition to a model of a Bertrand oligopoly with investment. In Appendix, we summarize the definitions and properties of lattice theory.

2 Model

2.1 Markov Games with Finite Actions and States

We consider a stochastic game in discrete time with a finite horizon indexed by parameter t in $\{0, 1, 2, \cdots, T\}$. The set of players is denoted by $N = \{1, 2, \cdots, n\}$. In the following, for any n-dimensional vectors $x = (x_1, \ldots, x_n)$, we denote $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ as x_{-i}. The set of actions for player i, denoted by A_i, is assumed to be an integer interval of a Euclidean space R^m:

$$A_i = \{a_i \in Z^m | \underline{a_i} \leq a_i \leq \overline{a_i}\}$$

for some $\underline{a_i}, \overline{a_i} \in Z^m$. The set of profiles of actions is given by $A = A_1 \times \cdots \times A_n$.

The set of states, denoted by $S \subset Z^k$ is also an integer interval of a Euclidean space R^k:

$$S = \{s \in Z^k | \underline{s} \le s \le \overline{s}\}.$$

A transition probability from state $s \in S$ to $s' \in S$ when all players take actions $a \in A$ is denoted by $f(s'|s, a)$. A state of actions at time t is denoted by s^t. Also, $\delta \in (0, 1)$ denotes a discount factor. A single-period payoff function of player i is denoted by $u_i : S \times A \to R$. We assume that u_i is bounded on $S \times A$, so there exists M such that $|u_i(s, a)| \le M$ for any $i \in N$, $s \in S$ and $a \in A$.

In stochastic games with observable actions, the action of each player at time t can generally depend on time t, state s^t, and the history of action profiles and states until time $t - 1$. In this paper, we restrict the strategies of any player to *Markovian strategies*, in which the action of each player at time t depends only on time t and the state at time t. Here, $\sigma_i^t : S \to A_i$ is referred to as the strategy of player i at time t, where $\sigma_i^t(s)$ specifies the action of player i at time t and state s. The strategy of player i is denoted by $\boldsymbol{\sigma}_i = (\sigma_i^0, \sigma_i^1, \dots, \sigma_i^T)$. We use the notation $\sigma^t = (\sigma_1^t, \sigma_2^t, \dots, \sigma_n^t)$ which is a profile of all players at time t. Let $\boldsymbol{\sigma} = (\sigma^0, \sigma^1, \dots, \sigma^T)$ be a strategy profile. $\boldsymbol{\sigma}^{>t}$ and $\boldsymbol{\sigma}_i^{>t}$ denote the subsequence from t of $\boldsymbol{\sigma}$ and that of $\boldsymbol{\sigma}_i$, respectively, i.e., $\boldsymbol{\sigma}^{>t} = (\sigma^t, \sigma^{t+1}, \dots, \sigma^T)$, and $\boldsymbol{\sigma}_i^{>t} = (\sigma_i^t, \sigma_i^{t+1}, \dots, \sigma_i^T)$.

2.2 Definitions of Payoffs and Equilibria

For a Markov game with finite horizon and any strategy profile $\boldsymbol{\sigma}$, the payoff of player i at time t and the realized state $s^t = s$, $U_i^t(\boldsymbol{\sigma}^{>t})(s)$ is given by

$$U_i^t(\boldsymbol{\sigma}^{>t})(s) = E[\sum_{\tau=t}^{T} \delta^{\tau-t} u_i(s^\tau, \sigma^\tau(s^\tau))|s^t = s].$$

The payoff of player i in a Markov game with a finite horizon is calculated recursively. In the following, we describe the recursive formula and define the notion of equilibria using the recursive formula.

For a given strategy profile, $U_i^t(\boldsymbol{\sigma}^{>t})$ is regarded as a function from S to R. We denote $U_i^t(\boldsymbol{\sigma}^{>t})$ by $v_i(s)$, which is referred to as the continuation value of player i at state s. The set of these functions is denoted by $\mathcal{V}$.

Let v be the profile of continuation values, which is a function from S to R^n, as denoted by $v(s) = (v_1(s), \dots, v_n(s))$.

For any $s \in S$ and $v_i \in \mathcal{V}$, let $\bar{\phi}_i(s, v_i)$ be the function from A to R defined as follows:

$$\bar{\phi}_i(s, v_i)(a) = u_i(s, a) + \delta \sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a).$$

Then, payoff at time t for strategy profile $\boldsymbol{\sigma}$ is calculated recursively as follows:

$$\begin{array}{lll} t = T & \forall s \in S & U_i^T(\boldsymbol{\sigma}^{>T})(s) = u_i(s, \sigma^T(s)), \\ 0 \leq t \leq T-1 & \forall s \in S & U_i^t(\boldsymbol{\sigma}^{>t})(s) = \bar{\phi}_i(s, U_i^{t+1}(\boldsymbol{\sigma}^{>t+1}))(\sigma^t(s)). \end{array}$$

Thus, if the profile of continuation values $v \in \mathcal{V}^n$ and any state $s \in S$ are given, a game for players can be regarded as a one-shot game in which the payoff of player i is $\bar{\phi}_i(s, v_i)$. The n-person game $(N, (A_i)_{i=1}^n, (\bar{\phi}_i(s, v_i))_{i=1}^n)$, denoted by $\bar{\Gamma}(s, v)$, is referred to as a reduced game for (s, v).

A Markov perfect equilibrium is defined as follows.

Definition 2.1 (Markov perfect equilibrium with a finite horizon). *A strategy profile* $\boldsymbol{\sigma}^*$ *is said to be a Markov perfect equilibrium if for any* $t \leq T$, $i \in N$ *and* $s \in S$,

$$\forall \boldsymbol{\sigma}_i' \quad U_i^t(\boldsymbol{\sigma}^{*>t})(s) \geq U_i^t(\boldsymbol{\sigma}_i'^{>t}, \boldsymbol{\sigma}_{-i}^{*>t})(s).$$

By the definition of a reduced game, the strategy profile $\boldsymbol{\sigma}^*$ is a Markov perfect equilibrium, if and only if for any $t \leq T$, $i \in N$ and $s \in S$,

$$\begin{array}{ll} t = T & U_i^T(\boldsymbol{\sigma}^{*>T})(s), = \max_{a_i \in A_i} u_i(s, a_i, \sigma_{-i}^{*T}(s)) \\ 0 \leq t \leq T-1 & U_i^t(\boldsymbol{\sigma}^{*>t})(s) = \max_{a_i \in A_i} \bar{\phi}_i(s, U_i^{t+1}(\boldsymbol{\sigma}^{*>t+1}))(a_i, \sigma_{-i}^{*t}(s)). \end{array} \tag{1}$$

3 Results

In this section, we investigate the sufficient conditions for the existence and monotonicity of the equilibrium. To obtain the results of this study, we use some of the properties and definitions of supermodular games on lattice theory, which were established by Milgrom and Roberts (1990), Milgrom and Shannon (1994), and Topkis (1998). The definitions and properties are stated in the Appendix.

First, we consider the following four conditions.

(U1) $u_i(s, a_i, a_{-i})$ is supermodular in a_i for any $s \in S$ and $a_{-i} \in A_{-i}$:

$$\forall a_i', a_i \in A_i \quad u_i(s, a_i' \vee a_i, a_{-i}) + u_i(s, a_i' \wedge a_i, a_{-i}) \geq u_i(s, a_i', a_{-i}) + u_i(s, a_i, a_{-i}).$$

(U2) $u_i(s, a_i, a_{-i})$ has increasing differences in (a_i, a_{-i}) for any $s \in S$:

$$\forall a'_i \geq a_i \quad \forall a'_{-i} \geq a_{-i} \quad u_i(s, a'_i, a'_{-i}) - u_i(s, a_i, a'_{-i}) \geq u_i(s, a'_i, a_{-i}) - u_i(s, a_i, a_{-i}).$$

(C1) $\sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a)$ is supermodular in a_i for any $s \in S$ and $a_{-i} \in A_{-i}$:

$$\begin{aligned} \forall a'_i, a_i \in A_i \quad & \textstyle\sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a'_i \vee a_i, a_{-i}) + \sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a'_i \wedge a_i, a_{-i}) \\ & \geq \textstyle\sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a'_i, a_{-i}) + \sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a_i, a_{-i}). \end{aligned}$$

(C2) $\sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a)$ has increasing differences in (a_i, a_{-i}) for any $s \in S$:

$$\begin{aligned} \forall a'_i \geq a_i \quad \forall a'_{-i} \geq a_{-i} \quad & \textstyle\sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a'_i, a'_{-i}) - \sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a_i, a'_{-i}) \\ & \geq \textstyle\sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a'_i, a_{-i}) - \sum_{\hat{s} \in S} v_i(\hat{s}) f(\hat{s}|s, a_i, a_{-i}) \end{aligned}$$

Now we can observe that the above four conditions yield a supermodularity of each reduced game.

Proposition 3.1. *If $\bar{\phi}_i(s, v_i)$ for any $i \in N$ satisfies (U1), (U2), (C1), and (C2), then the game $\bar{\Gamma}(s, v)$ is a supermodular game for any $s \in S$.*

Proof. (U1) and (C1) imply that

$$\begin{aligned} \forall a'_i, a_i \in A_i \quad & \bar{\phi}_i(s, v_i)(s, a'_i \vee a_i, a_{-i}) + \bar{\phi}_i(s, v_i)(s, a'_i \wedge a_i, a_{-i}) \\ & \geq \bar{\phi}_i(s, v_i)(s, a'_i, a_{-i}) + \bar{\phi}_i(s, v_i)(s, a_i, a_{-i}) \end{aligned}$$

and (U2) and (C2) imply that

$$\begin{aligned} \forall a'_i \geq a_i \quad \forall a'_{-i} \geq a_{-i} \quad & \bar{\phi}_i(s, v_i)(s, a'_i, a'_{-i}) - \bar{\phi}_i(s, v_i)(s, a_i, a'_{-i}) \\ & \geq \bar{\phi}_i(s, v_i)(s, a'_i, a_{-i}) - \bar{\phi}_i(s, v_i)(s, a_i, a_{-i}) \end{aligned}$$

Since A_i is a finite lattice, we conclude that $\bar{\Gamma}(s, v)$ is a supermodular game. □

Many applications in economics and operations research, such as price competition games, search games, and investment games, are supermodular games (see Vives (2005) and Amir (2005)). Since a Markov game, in which each stage game is a supermodular game satisfies (U1) and (U2), Proposition 3.1 shows that an extension of the application to a Markov game with a finite horizon has a pure strategy of equilibrium, if the Markov game satisfies (C1) and (C2). However, since (C1) and (C2) include continuation values that are obtained only by recursive calculation, the condition cannot be confirmed by the initial settings. Then, the goal is to replace (C1) and (C2) with some conditions for payoff functions and the transition probability.

By Topkis (1998) (shown as Proposition A.3 in Appendix), stochastic supermodularity and stochastically increasing differences of the transition probability imply (C1) and (C2), respectively, if continuation values v_i are increasing in s.

Next, we define stochastic supermodularity and stochastically increasing differences.

For fixed $y \in Y$, let $P_f(\hat{S}|s,a)$ be the probability of the set $\hat{S} \subseteq S$ occurring with respect to $f(s'|s,a)$, i.e.,

$$P_f(\hat{S}|s,a) = \sum_{z \in \hat{S}} f(z|s,a).$$

Here, $\hat{S} \subseteq S$ is said to be an increasing set if $s' \in \hat{S}$ and $s'' \geq s'$ imply $s'' \in \hat{S}$.

(T1) $f(s'|s,a)$ is stochastically supermodular in a_i for any $s \in S$ and $a_{-i} \in A_{-i}$:

$$\begin{aligned} \forall a_i', a_i \in A_i \quad & P_f(\hat{S}|s, a_i' \vee a_i, a_{-i}) + P_f(\hat{S}|s, a_i' \wedge a_i, a_{-i}) \\ & \geq P_f(\hat{S}|s, a_i', a_{-i}) + P_f(\hat{S}|s, a_i, a_{-i}), \end{aligned}$$

for any increasing set $\hat{S} \subseteq S$.

(T2) $f(s'|s,a)$ has stochastically increasing differences in (a_i, a_{-i}) for any $s \in S$:

$$\begin{aligned} \forall a_i' \geq a_i \quad \forall a_{-i}' \geq a_{-i} \quad & P_f(\hat{S}|s, a_i', a_{-i}') - P_f(\hat{S}|s, a_i, a_{-i}') \\ & \geq P_f(\hat{S}|s, a_i', a_{-i}) - P_f(\hat{S}|s, a_i, a_{-i}), \end{aligned}$$

for any increasing set $\hat{S} \subseteq S$.

Lemma 3.2. *Suppose that f satisfies (T1) and (T2) for given $s \in S$ and $v_i \in \mathcal{V}$. If v_i is increasing in s, then $\bar{\phi}_i(s, v_i)$ satisfies (C1) and (C2).*

Proof. This lemma is directly implied by Proposition A.3. □

Lemma 3.2 means that if (U1), (U2), (T1), and (T2) hold and v_i is increasing in s, then $\bar{\Gamma}(s,v)$ is a supermodular game and so has the greatest equilibrium. Moreover, if the following two conditions hold, then the greatest equilibrium is increasing in s by a property of monotone comparative statics for supermodular games by Milgrom and Roberts (1990) (shown as Theorem A.1).

(U3) $u_i(s, a_i, a_{-i})$ has increasing differences in (a_i, s) for any $a_{-i} \in A_{-i}$:

$$\forall a_i' \geq a_i \quad \forall s' \geq s \quad u_i(s', a_i', a_{-i}) - u_i(s', a_i, a_{-i}) \geq u_i(s, a_i', a_{-i}) - u_i(s, a_i, a_{-i}).$$

(T3) $f(s'|s, a)$ has stochastically increasing differences in (a_i, s) for any $a_{-i} \in A_{-i}$:

$$\begin{aligned}\forall a_i' \geq a_i \quad \forall s' \geq s \quad & P_f(\hat{S}|s', a_i', a_{-i}) - P_f(\hat{S}|s', a_i, a_{-i}) \\ & \geq P_f(\hat{S}|s, a_i', a_{-i}) - P_f(\hat{S}|s, a_i, a_{-i}),\end{aligned}$$

for any increasing set $\hat{S} \subseteq S$.

Lemma 3.3. *Suppose that u_i satisfies (U1)–(U3) and f satisfies (T1)–(T3) for any $i \in N$. If v_i is increasing in s, then the game $\bar{\Gamma}(s, v)$ has the greatest equilibrium $a^*(s, v)$ and $a^*(s, v)$ is increasing in s.*

Proof. By Lemma 3.2, $\bar{\Gamma}(s, v)$ is a supermodular game and so has the greatest equilibrium $a^*(s, v)$. Since $\bar{\phi}_i(s, v_i)(a)$ has increasing differences in (s, a_i) by (U3) and (T3), we find that $a^*(s, v)$ is increasing in s by Theorem A.1. □

Lemma 3.2 and Lemma 3.3 also require the condition for the increase of continuation values. Hence, we need conditions that imply the increase of the continuation values. The following two conditions leads the value of the equilibrium is increasing in state (Lemma 3.4), which inductively implies the increase of the continuation values (Theorem 3.5).

(U4) For any $i \in N$ and $a_i \in A_i$, $u_i(s, a)$ is increasing in (s, a_{-i}):

$$\forall s' \geq s \quad \forall a_{-i}' \geq a_{-i} \quad u_i(s', a_i, a_{-i}') \geq u_i(s, a_i, a_{-i}).$$

(T4) $f(\cdot|s, a)$ is stochastically increasing in (s, a_{-i}):

$$\forall s' \geq s \quad \forall a_{-i}' \geq a_{-i} \quad P_f(\hat{S}|s', a_i, a_{-i}') \geq P_f(\hat{S}|s, a_i, a_{-i})$$

for any increasing set $\hat{S} \subseteq S$.

Lemma 3.4. *Suppose that u_i satisfies (U1)–(U4) and f satisfies (T1)–(T4) for any $i \in N$. If v_i is increasing in s, then*

(1) *the game $\bar{\Gamma}(s, v)$ has the greatest equilibrium $a^*(s, v) = (a_1^*(s, v), \ldots a_n^*(s, v))$,*

(2) *$a^*(s,v)$ is increasing in s, and*

(3) *$\bar{\phi}_i(s,v_i)(a^*(s,v))$ is increasing in s.*

Proof. By Lemma 3.3, the game $\bar{\Gamma}(s,v)$ has the greatest equilibrium $a^*(s,v)$ that is increasing in s. Suppose that $s' \geq s$. To simplify the notation, let $a'_i = a^*_i(s',v)$, $a^*_i = a^*_i(s,v)$

$$\begin{aligned} a^*_{-i} &= (a^*_1(s,v),\ldots,a^*_{i-1}(s,v),a^*_{i+1}(s,v),\ldots,a^*_n(s,v)), \qquad \text{and} \\ a'_{-i} &= (a^*_1(s',v),\ldots,a^*_{i-1}(s',v),a^*_{i+1}(s',v),\ldots,a^*_n(s',v)). \end{aligned}$$

Since a'_i is an equilibrium strategy of player i for game $\bar{\Gamma}(s',v)$, we have

$$\begin{aligned} \bar{\phi}_i(s',v_i)(a^*(s',v)) &= u_i(s',a'_i,a'_{-i}) + \delta \textstyle\sum_{\hat{s}\in S} v_i(\hat{s}) f(\hat{s}|s',a'_i,a'_{-i}) \\ &\geq u_i(s',a^*_i,a'_{-i}) + \delta \textstyle\sum_{\hat{s}\in S} v_i(\hat{s}) f(\hat{s}|s',a^*_i,a'_{-i}) \end{aligned}$$

$a^*(s',v) \geq a^*(s,v)$ yields $a'_{-i} \geq a^*_{-i}$. In conjunction with (U4), this imply that $u_i(s',a^*_i,a'_{-i}) \geq u_i(s,a^*_i,a^*_{-i})$. Similarly, $a'_{-i} \geq a^*_{-i}$ and (T4) implies

$$P_f(\hat{S}|s',a^*_i,a'_{-i}) \leq P_f(\hat{S}|s,a^*_i,a^*_{-i})$$

for any increasing set $\hat{S} \subseteq S$. Then, for any increasing function v_i,

$$\sum_{\hat{s}\in S} v_i(\hat{s}) f(\hat{s}|s',a^*_i,a'_{-i}) \geq \sum_{\hat{s}\in S} v_i(\hat{s}) f(\hat{s}|s,a^*_i,a^*_{-i})$$

because of a property of the first-order stochastic dominance.

This implies that

$$\begin{aligned} u_i(s',a^*_i,a'_{-i}) + \delta \textstyle\sum_{\hat{s}\in S} v_i(\hat{s}) f(\hat{s}|s',a^*_i,a'_{-i}) &\geq u_i(s,a^*_i,a^*_{-i}) + \delta \textstyle\sum_{\hat{s}\in S} v_i(\hat{s}) f(\hat{s}|s,a^*_i,a^*_{-i}) \\ &= \bar{\phi}_i(s,v_i)(a^*(s,v)). \end{aligned}$$

Hence, we conclude that $\bar{\phi}_i(s',v_i)(a^*(s',v)) \geq \bar{\phi}_i(s,v_i)(a^*(s,v))$.

□

Theorem 3.5. *If a Markov game with a finite horizon satisfies (U1)–(U4) and (T1)–(T4), then*

(1) *the game has a Markov perfect equilibrium $\boldsymbol{\sigma}^*$,*

(2) *$\sigma^t(s)$ is increasing in state s, for any $0 \leq t \leq T$, and*

(3) *$U^t_i(\boldsymbol{\sigma}^{*>t})(s)$ is also increasing in state s for any $i \in N$ and for any $0 \leq t \leq T$.*

Proof. The proof is given by induction based on the terminal period T.

First, suppose $T = 0$ and fix a state s. Then, the game is a one-shot n-person game in which the payoff of a player is given by $\bar{\phi}_i(s, \mathbf{0})(a) = u_i(s, a)$, where $\mathbf{0}$ means that $v_i(s') = 0$ for any state $s' \in S$. Then, noting that $\mathbf{0}$ is increasing in s, Lemma 3.4 implies that there exists an equilibrium $\sigma^{*0}(s) = a^*(s)$, which is increasing in s. Since $\boldsymbol{\sigma}^* = (\sigma^{*0}(s))$, (1) and (2) hold for $T = 0$. Moreover, by Lemma 3.4, $U_i^t(\boldsymbol{\sigma}^{*>0})(s) = \bar{\phi}_i(s, \mathbf{0})(a^*(s))$ is increasing in s. Hence, (1)–(3) hold for $T = 0$.

Next, suppose that $T \geq 1$ and (1)–(3) hold for the terminal period $T - 1$. Since a subgame following to $t = 1$ is a Markov game with $T-1$ periods, there exists a Markov perfect equilibrium, according to an induction hypothesis of (1). Hence, we denote the Markov perfect equilibrium of the subgame following to $t = 1$ by $\boldsymbol{\sigma}^{*>1} = (\sigma^{1*}, \sigma^{2*}, \ldots, \sigma^{T*})$. Induction hypotheses of (2) and (3) also imply that $\boldsymbol{\sigma}^{*>1}$ and $U_i^1(\boldsymbol{\sigma}^{*>1})(s)$ are increasing in s. Let $v_i(s) = U_i^1(\boldsymbol{\sigma}^{*>1})(s)$ for any $i \in N$ and consider the reduced game $\bar{\Gamma}(s, v) = (N, (A_i)_{i=1}^n, (\bar{\phi}_i(s, v_i))_{i=1}^n)$. Lemma 3.4 implies that

(r1) $\bar{\Gamma}(s, v)$ has the greatest equilibrium $a^*(s, v)$,

(r2) $a^*(s, v)$ is increasing in state s, and

(r3) $\bar{\phi}_i(s, v_i)(a^*(s, v))$ is increasing in state s for any $i \in N$.

Let $\sigma^{0*}(s) = a^*(s, v)$. Then, (r1) and induction hypothesis (1) imply that $\boldsymbol{\sigma}^* = (\sigma^{0*}, \sigma^{1*}, \sigma^{2*}, \ldots, \sigma^{T*})$ is a Markov perfect equilibrium. Since $\sigma^{0*}(s)$ is increasing in s by (r2), $\boldsymbol{\sigma}^*$ is increasing in s. Finally, $U_i^1(\boldsymbol{\sigma}^{*>1})(s) = \bar{\phi}_i(s, v_i)(a^*(s, v))$ is increasing in s by (r3). Hence (1)–(3) hold for T. This concludes the proof. □

4 An Application: Bertrand Oligopoly with Investment

One of the applications satisfying (U1)–(U4) and (T1)–(T4) is a Bertrand oligopoly game with investments, which is stated as follows. Firm i ($i = 1, \cdots, n$) sells product i with price $p_i \in [0, \bar{p}]_Z$. The demand of product i is decided by the prices of all types of products, $(p_1, \ldots, p_n)$ and the accumulation of the investment s_i. At time t, firm i decides the price of product i and the amount of the investment $I_i \in [0, \bar{I}]_Z$. The accumulation of the investment increases the appeal of product i, thereby increasing the demand of the product. The accumulation of the investment of player i at time t is denoted by s_i^t and is given by $s_i^{t+1} = s_i^t + h_i(I_i)$, where $h_i(I_i)$ is an increment of the accumulation of the investment. Thus, we assume that the accumulation of the investment of firm i depends only on the associated amount of investment, I_i. We also assume that $h_i(I_i)$ is a random variable according to distribution function F_i defined by

$$F_i(x|I_i) = Prob[h_i(I_i) \leq x_i].$$

The payoff of player i is defined by

$$u_i(s, a) = (p_i - c_i)D_i(s_i, p_1, \ldots, p_n) - k_i I_i$$

where $c_i \geq 0$ is the marginal cost of product i and $k_i \geq 0$ is the marginal cost of the investment of i. $D_i(s_i, p_1, \ldots, p_n)$ is the demand of product i for an accumulation of the investment s_i and prices $(p_1, \ldots, p_n)$.

If D_i is a linear function of the prices, i.e.,

$$\textbf{(D1)} \quad \begin{array}{lll} D_i(s_i, p_1, \ldots, p_n) = \alpha(s_i) - \beta_{ii} p_i + \sum_{j \neq i} \beta_{ij} p_j & & \\ s'_i \geq s_i \quad \rightarrow & \alpha(s'_i) \geq \alpha(s_i), & \\ \beta_{ii} \geq 0, \quad \beta_{ij} \geq 0, & \alpha(s_i) > \bar{p}\beta_{ii} & \text{for any } s_i, \end{array}$$

then we find that $u_i(s, a)$ satisfies (U1)–(U4).

Moreover, if the probability distribution of the increment of the investment of each player i satisfies the following:

(F1) $F_i(x|I'_i) \leq F_i(x|I_i)$ for any x and $I'_i \geq I_i$, and

(F2) $F_i(x'|I'_i) - F_i(x'|I_i) \leq F_i(x|I'_i) - F_i(x|I_i)$ for any $I'_i \geq I_i$ and $x \geq x'$,

then the transition probability satisfies (T1)–(T4).

To demonstrate this, we introduce the following notation. An action of player i, a_i, consists of two components, the price of product i, p_i, and the amount of investment, I_i. Therefore, let $a_i = (p_i, I_i)$ and $a'_i = (p'_i, I'_i)$.

If $\hat{S} \subseteq S$ is an increasing set, then there exists $\hat{s}(\hat{S}) = (\hat{s}_1(\hat{S}) \ldots \hat{s}_n(\hat{S}))$ such that

$$\hat{S} = [\hat{s}_1(\hat{S}), \overline{s}_1] \times \cdots \times [\hat{s}_n(\hat{S}), \overline{s}_n].$$

Let $\bar{F}_j(\hat{s}_j|s_j, I_j) = 1 - F_j(\hat{s}_j - s_j|I_j)$. Then, for any increasing set $\hat{S}$, we have

$$P_f(\hat{S}|s, a) = \Pi_{j \in N} \bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j)$$

, which is the probability that the subsequent state is greater than $\hat{s}$. For any increasing set $\hat{S}$, $j \in N$ and $a'_j \geq a_j$, (F1) implies that

$$\bar{F}_j(\hat{s}_j(\hat{S})|s_j, I'_j) \geq \bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j), \tag{2}$$

and (F2) implies that

$$\bar{F}_j(\hat{s}_j(\hat{S})|s'_j, I'_j) - \bar{F}_j(\hat{s}_j(\hat{S})|s'_j, I_j) \geq \bar{F}_j(\hat{s}_j(\hat{S})|s_j, I'_j) - \bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j), \tag{3}$$

for any $s'_j \geq s_j$. Moreover, if $s'_j \geq s_j$, then

$$\bar{F}_j(\hat{s}_j(\hat{S})|s'_j, I_j) \geq \bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j). \tag{4}$$

for any increasing set $\hat{S}$, $j \in N$ and I_j because $F_j(\hat{s}_j(\hat{S}) - s'_j|I_j) \leq F_j(\hat{s}_j(\hat{S}) - s_j|I_j)$.

Next, we show that (T1)–(T4) hold if the probability distribution of the increment of the investment accumulation of each player satisfies (F1) and (F2). (T1) holds in the equality because, for any $a'_i, a_i \in A_i$, s and increasing set $\hat{S}$,

$$\begin{aligned}
&P_f(\hat{S}|s, a'_i \vee a_i, a_{-i}) + P_f(\hat{S}|s, a'_i \wedge a_i, a_{-i}) \\
&= \bar{F}_i(\hat{s}_i(\hat{S})|s_i, \max\{I'_i, I_i\})\Pi_{j\neq N}\bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j) + \bar{F}_j(\hat{s}_i(\hat{S})|s_i, \min\{I'_i, I_i\})\Pi_{j\neq N}\bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j) \\
&= \bar{F}_i(\hat{s}_i(\hat{S})|s_i, I'_i)\Pi_{j\neq N}\bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j) + \bar{F}_j(\hat{s}_i(\hat{S})|s_i, I_i)\Pi_{j\neq N}\bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j) \\
&= P_f(\hat{S}|s, a'_i, a_{-i}) + P_f(\hat{S}|s, a_i, a_{-i}).
\end{aligned}$$

Next, we will show that (T2) is satisfied. For any s, $a'_i \geq a_i$, $a'_{-i} \geq a_{-i}$, and any increasing set $\hat{S}$,

$$\begin{aligned}
&P_f(\hat{S}|s, a'_i, a'_{-i}) - P_f(\hat{S}|s, a_i, a'_{-i}) \\
&= \bar{F}_i(\hat{s}_i(\hat{S})|s_i, I'_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S})|s_j, I'_j) - \bar{F}_i(\hat{s}_i(\hat{S})|s_i, I_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S})|s_j, I'_j) \\
&\geq \bar{F}_i(\hat{s}_i(\hat{S})|s_i, I'_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j) - \bar{F}_i(\hat{s}_i(\hat{S})|s_i, I_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j) \\
&= P_f(\hat{S}|s, a'_i, a_{-i}) - P_f(\hat{S}|s, a_i, a_{-i}),
\end{aligned}$$

where the second inequality is obtained by (2).

Similarly, for any $s' \geq s$, $a'_i \geq a_i$, a_{-i} and any increasing set $\hat{S}$,

$$\begin{aligned}
&P_f(\hat{S}|s', a'_i, a_{-i}) - P_f(\hat{S}|s', a_i, a_{-i}) \\
&= \bar{F}_i(\hat{s}_i(\hat{S})|s'_i, I'_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S})|s'_j, I_j) - \bar{F}_i(\hat{s}_i(\hat{S})|s'_i, I_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S})|s'_j, I_j) \\
&= (\bar{F}_i(\hat{s}_i(\hat{S})|s'_i, I'_i) - \bar{F}_i(\hat{s}_i(\hat{S})|s'_i, I_i))\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S})|s'_j, I_j) \\
&\geq (\bar{F}_i(\hat{s}_i(\hat{S})|s_i, I'_i) - \bar{F}_i(\hat{s}_i(\hat{S})|s_i, I_i))\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S})|s'_j, I_j) \\
&\geq (\bar{F}_i(\hat{s}_i(\hat{S})|s_i, I'_i) - \bar{F}_i(\hat{s}_i(\hat{S})|s_i, I_i))\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j) \\
&= P_f(\hat{S}|s, a'_i, a_{-i}) - P_f(\hat{S}|s, a_i, a_{-i}),
\end{aligned}$$

where the first and second inequality are implied by (3) and (4), respectively. Hence, (T3) is satisfied.

Finally, it is easily shown that (T4) is true. For any $s' \geq s$, $a' \geq a$ and any increasing set $\hat{S}$,

$$\begin{aligned} P_f(\hat{S}|s', a_i') &= \Pi_{j \in N} \bar{F}_j(\hat{s}_j(\hat{S})|s_j', I_j') \\ &\geq \Pi_{j \in N} \bar{F}_j(\hat{s}_j(\hat{S})|s_j, I_j) \\ &= P_f(\hat{S}|s', a_i') \end{aligned}$$

where the inequality is implied by (4).

5 Conclusions

The present paper gives the sufficient conditions for a Markov perfect equilibrium in pure strategies to exist for a class of stochastic games with a finite horizon and finite states, in which any stage game has strategic complementarities for finite actions. We show the class of a Bertrand competition with investment, which satisfies the conditions.

The results for finite states and actions have advantages for numerical computation. Indeed, computation of equilibrium of Markov games has attracted a great deal of attention in the modeling of industrial dynamics, e.g., Ericson and Pakes (1995) and Doraszelski and Satterthwaite (2008). It would be useful to extend the results of this study to these models.

Appendix: Summary of Lattice Theory: Definitions and Results

In this appendix, we summarize some of the properties of lattice theory that were used herein.

A.1 A Partially Ordered Set and a Lattice

A binary relation $\geq$ on a nonempty set X is a *partial order* if $\geq$ is reflexive, transitive, and anti-symmetric.

For a subset Y of X, $\hat{y} \in X$ is referred to as an upper (lower) bound on Y if $\hat{y} \geq y$ ($\hat{y} \leq y$) for all $y \in Y$. $\hat{y} \in Y$ is the greatest (least) element of Y if $\hat{y} \geq y$ ($\hat{y} \leq y$) for any $y \in Y$. A supremum (infimum) of $Y \subseteq X$ is a least upper bound (greatest lower bound) and is denoted by $\sup_X Y$ ($\inf_X Y$). If a supremum (infimum) of Y belongs to Y, then it is the greatest element of Y.

For any two elements x' and x in a partially ordered set $(X, \geq)$, $x' \vee x$ and $x' \wedge x$ is defined by $\sup\{x', x\}$ and $\inf\{x', x\}$, respectively. A partially ordered set $(X, \geq)$ is said to be a *lattice* if for any $x', x \in X$, both $x' \vee x$ and $x' \wedge x$ also belong to X.

A.2 Supermodularity and Increasing Differences

A function $g : X \to R$ on a lattice X is said to be *supermodular* if for any $x, x' \in X$, $g(x' \vee x) + g(x' \wedge x) \geq g(x') + g(x)$.

Let $(X, \geq_X)$ be a lattice, and let $(Y, \geq_Y)$ be a partially ordered set. The function $g : X \times Y \to R$ has *increasing differences* in (x, y) if $g(x', y) - g(x, y)$ is increasing in y for $x' >_X x$, namely:

$$g(x', y') - g(x, y') \geq g(x', y) - g(x, y) \quad \forall x' >_X x \quad \text{and} \quad \forall y' >_Y y. \tag{5}$$

A.3 Supermodular Games

An *n-person game* Γ is a three tuple $\Gamma = (N, \{A_i\}_{i=1}^n, \{\gamma_i\}_{i=1}^n)$, where

- $N = \{1, \cdots, n\}$ is the set of players,
- A_i is the set of actions for player i, and
- $\gamma_i : A \to R$ is the payoff function of player i, where $A = A_1 \times \cdots \times A_n$.

An n-person game is said to be supermodular if the set of actions for each player i is a compact lattice and the payoff function $\gamma_i(a_i, a_{-i})$ is supermodular in $a_i \in A_i$ for fixed $a_{-i} \in A_{-i}$ and satisfies increasing differences in (a_i, a_{-i}), where $A_{-i} = A_1 \times \cdots A_{i-1} \times A_{i+1} \cdots \times A_n$.

Theorem A.1 (Milgrom and Roberts (1990), Topkis (1998) and Echenique (2002)). *Let Y be a partially ordered set, and let $\{\Gamma(x) = (N, (A_i)_{i=1}^n, (\gamma_i(y)\}_{i=1}^n))|y \in Y\}$ be a family of a supermodular game in which payoff functions $\{\gamma_i(y)\}_{i=1}^n)$ are parameterized by $y \in Y$. If $\gamma_i(y)(a_i, a_{-i})$ has increasing differences in (y, a_i) for any $i \in N$ and $a_{-i} \in A_{-i}$, then there exists the greatest equilibrium $a^*(y)$ of $\Gamma(y)$ for any $y \in X$ and $a^*(y)$ is increasing in y.*

A.4 Monotonicity of Probability Functions

Let $X = \{x \in Z^k | \underline{x} \leq s \leq \overline{x}\}$ be a k-dimensional integer interval, and let Y be a partially ordered finite set. Suppose that $f(x, y)$ is a probability function on X parameterized by $y \in Y$: f that satisfies

(1) $f(x, y) \geq 0$ for any $x \in X$,

(2) $\sum_{x \in X} f(x, y) = 1$,

for any $y \in Y$.

For fixed $y \in Y$, let $P_f(X', y)$ be a probability of the set $X' \subseteq X$ occurring with respect to $f(x, y)$, which is defined by

$$P_f(X', y) = \sum_{z \in X'} f(z, y).$$

$X' \subseteq X$ is said to be an increasing set if $x \in X'$ and $x' \geq x$ imply $x' \in X'$.

$f(x, y)$ is said to be *stochastically increasing* in y, if for any $y' \geq y$, $P_f(X', y') \geq P_f(X', y)$ for any increasing set X'.

The following proposition is known as the theory of multivariative stochastic orders (Topkis (1998) and Müller and Stoyan (2002)).

Proposition A.2 (Multivariative Stochastic Dominance). *Let $v(x)$ be an increasing function in x, and suppose that $f(x, y)$ is stochastically increasing in y. Then, the expectation $\sum_{x \in X} v(x) f(x, y)$ is increasing in y.*

Topkis (1998) showed that Proposition A.2 implies that, if the function is increasing, supermodularity and increasing differences of an expectation of a function for some parameters are derived from submodularity and decreasing differences of a distribution function for the parameters.

$f(x, y)$ is said to be *stochastically supermodular* in y, if for any increasing set $X' \subset X$ and $y' \geq y$,

$$P_f(X', y' \vee y) + P_f(X', y' \wedge y) \geq P_f(X', y') + P_f(X', y).$$

Consider a probability function $f(x, y, z)$ on X parameterized by $(y, z) \in Y \times Z$ where y and Z are partially ordered sets. Here, $f(x, y, z)$ is said to represent *stochastically increasing differences* in (y, z), if for any increasing set $X' \subset X$, $y' \geq y$ and $z' \geq z$,

$$P_f(X', y', z') - P_f(X', y, z') \geq P_f(X', y', z) - P_f(X', y, z).$$

Proposition A.3 (Topkis (1998)). *Let $v : X \to R$ be an increasing function of $x \in X$, and let $f(x, y, z)$ be a probability function on X parameterized by $(y, z) \in Y \times Z$. Then, the following three properties hold:*

(1) *If f is stochastically supermodular in $y \in Y$ for any $(x, z) \in X \times Z$, then the expectation* $\sum_{x \in X} v(x) f(x, y, z)$ *is supermodular in $y \in Y$, and*

(2) *If f has stochastically increasing differences in (y, z) for any $x \in X$, then the expectation* $\sum_{x \in X} v(x) f(x, y, z)$ *has increasing differences in (y, z).*

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