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Abstract

In this article, we consider options written on yield spreads such as swap spreads and basis swap spreads under the DLG model developed by Kijima et al. (2009). For this purpose, we extend the DLG model, by shifting the short rates with deterministic functions of time, so that the initial yield curves implied by the DLG model are consistent with the observed curves in the market. This is important not only for risk management purposes, but also because the drift term of short rates affects the price of spread options. Some numerical examples are given to discuss the impact of model parameters, in particular of correlations, on option prices for the quadratic Gaussian model and the Hull-White model.

1 Introduction

Interest-rate products such as bonds, swaps and basis swaps are frequently traded in the financial market. However, because of frictions existing in the market, yield spreads such as swap spreads and basis swap spreads are observed, and market participants need to watch them carefully in order to manage their exposures to liquidity and credit.

A swap spread is a swap rate minus a bond yield (or a par yield) with the same maturity, whereas a basis swap spread is a fixed spread to be added on either leg of a basis swap to exchange LIBORs (London Interbank Offered Rates) of two currencies with principal exchange. From the pricing point of view, we can regard such a spread contract as a swap contract between two parties to exchange cash flows that depend on a yield spread.

These spreads are fluctuating to reflect several frictions and/or conditions of demand and supply in the market. For example, a swap spread is likely to get wider when 'flight to quality' takes place in the bond market. A USD/JPY basis swap often moves when a company issues a JPY denominated bond and swaps the raised fund into USD.[1] Such a yield curve spread implies a different quality of the two yield curves. The DLG model[2] developed by Kijima et al. (2009) allows one to formulate many yield curves with different quality under the no-arbitrage framework.

For risk management purposes, option contracts written on such spreads are beneficial to the market participants who pay attention to the movement of yield curve spreads. Hence, the pricing of spread options becomes important for them, in particular, due

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[1]The literature and the background of basis swaps are found in Kijima et al. (2009) and references therein.

[2]"D" stands for "Discount", "L" for "LIBOR", and "G" for "Government", respectively.

to the growing concern on credit risk and liquidity risk. See Carmona and Durrleman (2003) for a survey of the pricing of spread options when the dynamics of underlying assets follows a geometric Brownian motion.

A spread we focus in this article follows a more complicated dynamics than the geometric Brownian motion, because swaps and basis swaps involve many cash flows during their lives. However, the spread contract can be seen as a portfolio of zero-coupon bonds, and an option on a spread as a swaption with a floating strike rate. Employing this idea, this article considers an option on a spread contract in *one* currency under the DLG model, and derives the option value that is "naturally" quoted in one currency. Our result is distinct from Brigo and Mercurio (2007) who discussed several quanto derivatives on *two* currency curves.

One of crucial issues on interest-rate models for practitioners is the fitting of the initial yield curves. It is useful for risk management purposes if the initial yield curves implied by a model are consistent with the yield curves observed in the market. Hull and White (1994) first showed that it is possible by adding a time dependent function in the drift term of the short rate in the Vasicek model (1977).[3] This is equivalent to shifting the short rate by a deterministic function. Brigo and Mercurio (2001) applied this idea to several short rate models.

Note that the drift term of the short-rate processes *does* affect the price of spread options, while it does not for bond options in general. This is so, because bonds are traded assets while spreads are not. Under the risk-neutral measure, any traded asset has the risk-free short rate as the drift term, and this term will disappear after the change of measure to the forward measure. See, e.g., Brigo and Mercurio (2007) for details.

In this article, we extend the DLG model so that the initial yield curves implied by the model are consistent with the observed curves in the market. The idea is based on the deterministic shift of short rates as in Brigo and Mercurio (2001). To this end, we carry out bootstrapping of the discount factors and the forward rates from observed yield curves. Other model parameters can be calibrated from the option prices.

This article is organized as follows. In the next section, we briefly describe the DLG model and study options written on a swap spread and a basis swap spread. The issue of initial curve fitting is discussed in Section 3 for two short-rate models, the quadratic Gaussian model and the Hull-White model. In Section 4, we show some numerical examples of yield curves and option prices under the two short rate models. Section 5 concludes this article.

2 The DLG Model and Spread Options

In this section, we provide a brief summary of the DLG model developed by Kijima et al. (2009), and study options written on a swap spread and a basis swap spread. The DLG model was constructed in order to treat multi-quality of yield curves under multiple currencies within the no-arbitrage framework.

2.1 The DLG model

Consider a market in which interest-rate swaps, basis swaps and government bonds are traded among market participants. It is assumed that there exist the three yield curves, D-curve, L-curve and G-curve, in each currency. The D-curve is used to discount cash

[3] According to Inui and Kijima (1998), the Hull-White model is a special case of the Heath, Jarrow and Morton (HJM) model (1992) with Markovian state variables.

flows. The L-curve determines the LIBORs so that it is related to swap rates and basis swap rates. The G-curve determines the government bond ("Govt") rates. Each curve in each currency is associated with the short rate $r_k(t)$ and the zero-coupon bond prices $P_k(t,T)$, $k = D, L, G$.

Roughly speaking, the spread between the D-curve and the L-curve affects the basis swap spreads, while the spread between the L-curve and the G-curve determines the term structure of the swap spread that is equal to a swap rate minus a bond yield. Although we concentrate on the three curves in this article, it is straightforward to extend the model so as to include other curves such as corporate bonds.

Among currencies, the currency "USD" is supposed to support enough liquidity of the fund to all market participants, so that USD has no friction between the D-curve and the L-curve. This assumption is equivalent to saying that any floating rate note with coupons of USD LIBOR is worth a par.[4] Under this setting, it is sufficient to focus on the three yield curves of the other currency, say "JPY".

The uncertainty is represented by a probability space $(\Omega, \mathcal{F}, Q_D)$ on which the short rates $r_k(t)$, $k = D, L, G$, are defined by using a three-dimensional standard Brownian motion $W(t) = (W_D(t), W_L(t), W_G(t))^\top$. Here, $\top$ stands for transposition of vectors and matrices. The filtration generated by the Brownian motion is denoted by $\{\mathcal{F}_t\}$. The probability measure Q_D is the risk-neutral measure, since we are interested in the pricing of financial products. Hence, the short rate $r_D(t)$ is regarded as the risk-free interest rate. The expectation of random variable X conditional on $\mathcal{F}_t$ with respect to probability measure P is denoted by $E_t^P[X]$.

The forward rates on the L-curve and the G-curve are important ingredients for the pricing of swaps and government bonds. The zero-coupon bond price of the k-curve, $k = D, L, G$, is defined by

$$P_k(t,T) = E_t^{Q_D}\left[\exp\left\{-\int_t^T \left(r_k(s) - \frac{1}{2}\|\lambda_k(s)\|^2\right)\mathrm{d}s - \int_t^T \lambda_k(s)^\top \mathrm{d}W(s)\right\}\right], \qquad (1)$$

where $\lambda_k(t) = (\lambda_k^D(t), \lambda_k^L(t), \lambda_k^G(t))^\top$ represents the market prices of risk of the k-curve relative to the D-curve. Hence, we assume $\lambda_D(t) = 0$ for the D-curve.

For the period $[T_1, T_2]$, the time-t forward LIBOR $L(t, T_1, T_2)$ and forward Govt rate $G(t, T_1, T_2)$ are calculated, respectively, by making use of the forward measure $Q_D^{T_2}$ as

$$L(t, T_1, T_2) = \frac{1}{T_2 - T_1}\left(E_t^{Q_D^{T_2}}\left[P_L(T_1, T_2)^{-1}\right] - 1\right) \qquad (2)$$

and

$$G(t, T_1, T_2) = \frac{1}{T_2 - T_1}\left(E_t^{Q_D^{T_2}}\left[P_G(T_1, T_2)^{-1}\right] - 1\right). \qquad (3)$$

See Kijima et al. (2009) for details.

A swap rate $S(t, T_0, T_N)$ is a fixed rate at time t to be exchanged with the floating rate $L(T_{i-1}, T_{i-1}, T_i)$ at time T_i, $i = 1, 2, \cdots, N$, for the period $[T_0, T_N]$. On the other hand, for a basis swap contract entered at time t, the USD LIBOR is exchanged by the JPY LIBOR $L(T_{i-1}, T_{i-1}, T_i)$ plus a basis swap spread $bs(t, T_0, T_N)$ at time T_i, $i = 1, 2, \cdots, N$, for the period $[T_0, T_N]$, with a principal exchange at both the starting date and the maturity. A government bond with maturity T_N is supposed to pay a coupon $C(T_N)$ for the period $[T_0, T_N]$, and the time-t bond price is denoted by $V(t, T_N)$.

[4] One can remove the assumption if the price of the floating rate note with any maturity is known and the dynamics of the foreign exchange rate is formulated in the model.

In what follows, we assume that the same day-count convention is applied to all the products, and the relevant dates $T_0 < T_1 < \cdots < T = T_N$ are set at regularly spaced time intervals with $\delta = T_i - T_{i-1}$ for all i. The current time t is any time on or prior to T_0.

Kijima et al. (2009) obtained the following three fundamental equations:

$$
\begin{aligned}
S(t, T_0, T_N) &= \frac{\sum_{i=1}^{N} L(t, T_{i-1}, T_i) P_D(t, T_i)}{\sum_{i=1}^{N} P_D(t, T_i)}, && (4)\\
bs(t, T_0, T_N) &= \frac{P_D(t, T_0) - P_D(t, T_N)}{\delta \sum_{i=1}^{N} P_D(t, T_i)} - S(t, T_0, T_N), && (5)
\end{aligned}
$$

and

$$
V(t, T_N) = P_D(t, T_0) + \delta \sum_{i=1}^{N} \left(C(T_N) - G(t, T_{i-1}, T_i)\right) P_D(t, T_i). \tag{6}
$$

The swap rate (4) is an average of the forward LIBOR's with the weights of zero-coupon bond prices implied by the D-curve. The basis swap spread (5) can be viewed as a difference between the swap rate *without* frictions implied by the D-curve, i.e.

$$
S_D(t, T_0, T_N) = \frac{P_D(t, T_0) - P_D(t, T_N)}{\delta \sum_{i=1}^{N} P_D(t, T_i)}, \tag{7}
$$

and the swap rate $S(t, T_0, T_N)$ *with* frictions implied by both the D-curve and the L-curve. If the L-curve coincides with the D-curve completely, the difference between the two swap rates diminishes. Hence, the basis swap spread appears due to such frictions that are observed as a spread between the D-curve and the L-curve in the market.

The bond price (6) is derived by regarding the bond transaction as a swap contract to exchange the fixed coupon $C(T_N)$ with the floating Govt rates $G(T_{i-1}, T_{i-1}, T_i)$. It follows that, in the DLG model, the par yield of a government bond is given by

$$
Y(t, T_0, T_N) = \frac{\sum_{i=1}^{N} G(t, T_{i-1}, T_i) P_D(t, T_i)}{\sum_{i=1}^{N} P_D(t, T_i)}, \tag{8}
$$

in the form of a swap rate as if $G(T_{i-1}, T_{i-1}, T_i)$ were the floating rates. See Kijima et al. (2009) for details.

2.2 Bootstrapping

Bootstrapping is a method to obtain the unobservable variables P_D, L, G recursively from observed rates and prices in (4), (5) and (6). Denoting the observed or implied prices in the market by superscript M, one can carry out bootstrapping to obtain

$$
\begin{aligned}
P_D^M(0, T_i) &= \frac{P_D^M(0, T_0) - \delta \left(S^M(0, T_0, T_i) + bs^M(0, T_0, T_i)\right) \sum_{j=1}^{i-1} P_D^M(0, T_j)}{1 + \delta \left(S^M(0, T_0, T_i) + bs^M(0, T_0, T_i)\right)}, && (9)\\
L^M(0, T_{i-1}, T_i) &= \frac{S^M(0, T_0, T_i) \sum_{j=1}^{i} P_D^M(0, T_j) - S^M(0, T_0, T_{i-1}) \sum_{j=1}^{i-1} P_D^M(0, T_j)}{P_D^M(0, T_i)}, && (10)\\
G^M(0, T_{i-1}, T_i) &= C(T_i) - \frac{1}{\delta P_D^M(0, T_i)} \Big[V^M(0, T_i) - P_D^M(0, T_0) \\
&\qquad - \delta \sum_{j=1}^{i-1} \left(C(T_i) - G^M(0, T_{j-1}, T_j)\right) P_D^M(0, T_j) \Big], && (11)
\end{aligned}
$$

where $S^M(0,T_0,T_i), bs^M(0,T_0,T_i),\ V^M(0,T_i),\ i=1,2,\cdots,N$ are the observed yield curves. See Kijima et al. (2009) for details.

Let us compare the above results with the classical bootstrapping of swap rates in which basis swaps are not taken into consideration. By the classical bootstrapping, one obtains the ordinary discount factors $\tilde{P}(T_i)$ and forward LIBOR's $\tilde{L}(T_{i-1},T_i)$ as

$$\tilde{P}(T_i) = \frac{\tilde{P}(T_0) - \delta S^M(0,T_0,T_i)\sum_{j=1}^{i-1}\tilde{P}(T_j)}{1+\delta S^M(0,T_0,T_i)} \tag{12}$$

and

$$\tilde{L}(T_{i-1},T_i) = \frac{S^M(0,T_0,T_i)\sum_{j=1}^{i}\tilde{P}(T_j) - S^M(0,T_0,T_{i-1})\sum_{j=1}^{i-1}\tilde{P}(T_j)}{\tilde{P}(0,T_i)}, \tag{13}$$

respectively. If one regards a swap as an exchange of a fixed coupon bond and a floating coupon bond, the values of the fixed leg and the floating leg of the swap are both zero, i.e.

$$\begin{aligned} 0 &= -\tilde{P}(T_0) + \delta\sum_{j=1}^{i}\tilde{L}(T_{j-1},T_j)\tilde{P}(T_j) + \tilde{P}(T_i) \\ &= -\tilde{P}(T_0) + \delta S^M(0,T_0,T_i)\sum_{j=1}^{i}\tilde{P}(T_j) + \tilde{P}(T_i). \end{aligned} \tag{14}$$

Also, under the classical bootstrapping, the classical forward LIBOR is calculated as

$$\tilde{L}(T_{i-1},T_i) = \delta\left(\frac{\tilde{P}(T_{i-1})}{\tilde{P}(T_i)} - 1\right). \tag{15}$$

Hence, the classical model (14) cannot express a non-zero basis swap spread, because

$$0 \neq -\tilde{P}(T_0) + \delta\sum_{j=1}^{i}\left(\tilde{L}(T_{j-1},T_j) + bs^M(0,T_0,T_i)\right)\tilde{P}(T_j) + \tilde{P}(T_i)$$

in general.

On the other hand, bootstrapping of the DLG model (9)–(10) implies that the legs are not worth zero but the minus of the basis swap multiplied by the annuity, i.e.

$$\begin{aligned} & -bs^M(0,T_0,T_i)\sum_{j=1}^{i}P_D^M(0,T_j) \\ &= -P_D^M(0,T_0) + \delta\sum_{j=1}^{i}L^M(0,T_{j-1},T_j)P_D^M(0,T_j) + P_D^M(0,T_i) \\ &= -P_D^M(0,T_0) + \delta S^M(0,T_0,T_i)\sum_{j=1}^{i}P_D^M(0,T_j) + P_D^M(0,T_i). \end{aligned} \tag{16}$$

It follows that

$$0 = -P_D^M(0,T_0) + \delta\sum_{j=1}^{i}\left(L^M(0,T_{j-1},T_j) + bs^M(0,T_0,T_i)\right)P_D^M(0,T_j) + P_D^M(0,T_i).$$

Hence, the DLG model is consistent with a non-zero basis swap spread.

Equation (16) shows that negative basis swap spreads lead to higher discount factors than the classical ones, i.e. $P_D^M(0,T) > \tilde{P}(T)$. Nevertheless, the forward LIBOR's are similar, i.e. $L^M(0,T_{i-1},T_i) \approx \tilde{L}(T_{i-1},T_i)$, as we shall see later in numerical examples. These findings have important implications in the DLG model. As Kijima et al. (2009) stated, on-the-market swap values are always zero, independent of the bootstrapping method, while off-the-market swap values may be distinct over the two bootstrapping methods due to the existence of basis swap spreads.

2.3 Options on spread

Let us consider options on a swap spread and a basis swap spread by making use of the multi-quality curves. Needless to say, an option on a spread is an insurance against undesired movements of the spread.

In this article, a call option on a swap spread is defined as a contract to give the option buyer the right to enter into a swap to receive coupons of a swap spread $S(T_0,T_0,T_N) - Y(T_0,T_0,T_N)$ prevailing at the option expiry T_0 by paying coupons of a fixed spread k, called the *strike spread*, for the same period $[T_0,T_N]$. The option buyer will see a benefit when the swap spread gets wider than the strike spread at the option expiry. Since the period for the observation to determine the spread at the expiry is matched with the period of the underlying swap, the call option on a swap spread is clearly equivalent to a payer swaption to pay a fixed rate of $Y(T_0,T_0,T_N)+k$ versus to receive LIBOR. This fact is also confirmed by the following option price formula at time t:

$$\begin{aligned}
& P_D(t,T_0)E_t^{Q_D^{T_0}}\left[\left(\sum_{i=1}^N \delta\left(S(T_0,T_0,T_N) - Y(T_0,T_0,T_N) - k\right)P_D(T_0,T_i)\right)_+\right] \\
= \; & P_D(t,T_0)E_t^{Q_D^{T_0}}\left[\left(\sum_{i=1}^N \delta\left(L(T_0,T_{i-1},T_i) - (Y(T_0,T_0,T_N) + k)\right)P_D(T_0,T_i)\right)_+\right],
\end{aligned}$$

where $(x)_+ = \max\{x,0\}$. Here, equality follows from (4). Recall from (8) that we can calculate the bond par yield as a swap rate constructed by the D-curve and G-curve. Thus, in the DLG model, the call option can be calculated as a payer swaption with the floating strike rate of Govt swap rate plus the strike spread against LIBOR. A put option on a swap spread is parallel to the call option and equivalent to a receiver swaption with a floating strike rate.[5]

A call option on a basis swap spread gives the buyer the right to enter into a swap to receive coupons of a basis swap spread $bs(T_0,T_0,T_N)$ prevailing at the option expiry T_0 by paying coupons of a fixed spread k for the same period $[T_0,T_N]$. Since the basis swap spread is a spread between two swap rates by (5), i.e. $S_D(T_0,T_0,T_N) - S(T_0,T_0,T_N)$, the option price is written as

$$\begin{aligned}
& P_D(t,T_0)E_t^{Q_D^{T_0}}\left[\left(\sum_{i=1}^N \delta\left(bs(T_0,T_0,T_N) - k\right)P_D(T_0,T_i)\right)_+\right] \\
= \; & P_D(t,T_0)E_t^{Q_D^{T_0}}\left[\left(\sum_{i=1}^N \delta\left((S_D(T_0,T_0,T_N) - k) - L(T_0,T_{i-1},T_i)\right)P_D(T_0,T_i)\right)_+\right].
\end{aligned}$$

[5] An option to enter into an asset swap can be formulated similarly.

This implies that the call option is equivalent to a receiver swaption to receive a fixed rate of $S_D(T_0, T_0, T_N) - k$ versus to pay LIBOR. The same arguments apply to a put option on a basis swap.

Remark 2.1. The above discussions can be generalized to options with maturity T_m of the swap period to exchange cash flows that is different from the maturity T_N of swaps and/or bonds. That is, we then have

$$P_D(t, T_0) E_t^{Q_D^{T_0}} \left[\left(\sum_{i=1}^{m} \delta \left(S(T_0, T_0, T_N) - Y(T_0, T_0, T_N) - k \right) P_D(T_0, T_i) \right)_+ \right].$$

Typical examples are caps and floors on these spreads. The analytical expressions of the above option prices reveal that it reduces to a calculation of an option on a rate with convexity adjustment, which has been well studied for CMS (constant maturity swaps). A Monte Carlo simulation is a simple means to evaluate such options, although it is often very time-consuming. However, as demonstrated in Tanaka et al. (2007), the usage of bond moments is efficient when pricing options for a certain class of underlying interest rate models, including affine term structure models and quadratic Gaussian models. In Section 4, we present some numerical examples of standard spread options with $m = N$.

3 Yield Curve Fitting

Following the spirit of Hull and White (1994) and Brigo and Mercurio (2001), this section extends the results of Kijima et al. (2009) so that the initial curves implied by the DLG model are consistent with observed rates in the market by using deterministic shifting functions. Simpler versions of the short rate models studied in this section have been discussed in Kijima et al. (2009) without the initial curve fitting.

3.1 Quadratic Gaussian model

The first example is a quadratic Gaussian model where the D-curve is constructed by a quadratic Gaussian model of Pelsser (1997) while each of the L-curve and the G-curve has a Gaussian spread over the D-curve. Namely, we assume that

$$\begin{aligned} r_D(t) &= (x_D(t) + \alpha + \beta t)^2 + \varphi_D(t), \\ r_L(t) &= r_D(t) + h_L(t), \quad h_L(t) = x_L(t) + \varphi_L(t), \\ r_G(t) &= r_D(t) + h_G(t), \quad h_G(t) = x_G(t) + \varphi_G(t), \end{aligned}$$

where $x_k(t)$ are the Ornstein-Uhlenbeck processes give by

$$\mathrm{d}x_k(t) = -a_k x_k(t)\mathrm{d}t + \sigma_k \mathrm{d}W_k(t), \quad x_k(0) = 0, \quad k = D, L, G,$$

and where $\varphi_k(t)$ are deterministic functions of time t that are to be jointly determined from observed initial curves.[6] The Brownian motions $W_D(t)$, $W_L(t)$, $W_G(t)$ are independent of each other under Q_D. The market prices of risk are assumed to be given by

$$\begin{pmatrix} \lambda_D^D(t) & \lambda_D^L(t) & \lambda_D^G(t) \\ \lambda_L^D(t) & \lambda_L^L(t) & \lambda_L^G(t) \\ \lambda_G^D(t) & \lambda_G^L(t) & \lambda_G^G(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_L & 0 \\ 0 & 0 & \lambda_G \end{pmatrix} \tag{17}$$

[6]The shifting function $\varphi_D(t)$ can be negative, so that the short rate $r_D(t)$ may become negative, although $r_D(t) - \varphi_D(t)$ must be nonnegative.

with some constants λ_L, λ_G. For the implementation purpose, it is enough to consider the integral of φ_k, $k = D, L, G$, given by

$$\Phi_k(T) \equiv \int_0^T \varphi_k(t)\mathrm{d}t. \tag{18}$$

According to Pelsser (1997), the zero-coupon bond price of the D-curve is an exponential of the quadratic function of x_D. More precisely, we have

$$P_D(t,T) = \exp\left\{-\int_t^T \varphi_D(s)\mathrm{d}s + A_D(t,T) - B_D(t,T)x_D(t) - C_D(t,T)x_D(t)^2\right\}, \tag{19}$$

where

$$\begin{aligned}
\gamma &= \sqrt{a_D^2 + 2\sigma_D^2}, \\
F_D(t,T) &= 2\gamma \mathrm{e}^{\gamma(T-t)} \left((\gamma + a_D)\mathrm{e}^{2\gamma(T-t)} + \gamma - a_D\right)^{-1}, \\
C_D(t,T) &= \left(\mathrm{e}^{2\gamma(T-t)} - 1\right) \left((\gamma + a_D)\mathrm{e}^{2\gamma(T-t)} + \gamma - a_D\right)^{-1}, \\
B_D(t,T) &= 2F_D(t,T) \int_t^T \frac{\alpha + \beta s}{F_D(s,T)}\mathrm{d}s
\end{aligned}$$

and

$$A_D(t,T) = \int_t^T \left(\frac{1}{2}\sigma_D^2 B_D(s,T)^2 - \sigma_D^2 C_D(s,T) - (\alpha + \beta s)^2\right)\mathrm{d}s.$$

Explicit formulas for B_D and A_D are obtained in Kijima et al. (2008). For the reader's convenience, they are shown in Appendix. Note that, by setting $t = 0$ in (19), Φ_D must be given by

$$\Phi_D(T) \quad = \quad -\ln P_D^M(0,T) + A_D(0,T). \tag{20}$$

As to the other curves, we need to calculate the forward rates (2) and (3). To this end, define for $j, k = L, G$

$$B_k(t,T) = -\frac{1}{a_k}\left(1 - \mathrm{e}^{-a_k(T-t)}\right), \quad B_{jk}(t,T) = -\frac{1 - \mathrm{e}^{-(a_j+a_k)(T-t)}}{a_j + a_k}.$$

From the dynamics of the Ornstein-Uhlenbeck processes $x_k(t)$, $k = L, G$, we have

$$\int_t^T x_k(s)\mathrm{d}s = -B_k(t,T)x_k(t) - \sigma_k \int_t^T B_k(s,T)\mathrm{d}W_k(s).$$

Hence, the integral $\int_t^T x_k(s)\mathrm{d}s$ is normally distributed with

$$\begin{aligned}
E_t^{Q_D}\left[\int_t^T x_k(s)\mathrm{d}s\right] &= -B_k(t,T)x_k(t), \\
Var_t^{Q_D}\left[\int_t^T x_k(s)\mathrm{d}s\right] &= \frac{\sigma_k^2}{a_k^2}\left(2B_k(t,T) - B_{kk}(t,T) + T - t\right)
\end{aligned}$$

and

$$Cov_t^{Q_D}\left[\int_t^T x_k(s)\mathrm{d}s, W_k(T) - W_k(t)\right] = \frac{\sigma_k}{a_k}\left(B_k(t,T) + T - t\right).$$

By making use of the independence of Brownian motions, we then obtain from (1) with $k = L$ that

$$P_L(t,T) = P_D(t,T)\exp\left\{B_L(t,T)x_L(t) + A_L(t,T)\right\},$$

where

$$\begin{aligned} A_L(t,T) &= -\Phi_L(T) + \Phi_L(t) + \frac{\sigma_L^2}{2a_L^2}\left(2B_L(t,T) - B_{LL}(t,T) + T - t\right) \\ &\quad - \lambda_L \frac{\sigma_L}{a_L}\left(B_L(t,T) + T - t\right). \end{aligned} \tag{21}$$

The forward LIBOR (2) is given by

$$L(t,T_{i-1},T_i) = \frac{1}{\delta}\left(\frac{P_D(t,T_{i-1})}{P_D(t,T_i)}K_L(t,T_{i-1},T_i) - 1\right), \tag{22}$$

where

$$\begin{aligned} K_L(t,T_{i-1},T_i) &= \exp\Big\{-A_L(T_{i-1},T_i) - \frac{\sigma_L^2}{2}B_L(T_{i-1},T_i)^2 B_{LL}(t,T_{i-1}) \\ &\qquad - \mathrm{e}^{-a_L(T_{i-1}-t)}B_L(T_{i-1},T_i)x_L(t)\Big\}. \end{aligned} \tag{23}$$

By comparing (22) with (15), we find that the function K_L represents the effect of basis swap spreads. On the other hand, by the results of bootstrapping (9)–(10), we know the initial value of K_L implied by the observed rates in the market as

$$K_L^M(0,T_{i-1},T_i) = \frac{P_D^M(0,T_i)}{P_D^M(0,T_{i-1})}\left(1 + \delta L^M(0,T_{i-1},T_i)\right). \tag{24}$$

Therefore, by plugging (21) into (23) with $t = 0$, Φ_L must satisfy

$$\begin{aligned} \Phi_L(T_i) &= \Phi_L(T_{i-1}) + \ln K_L^M(0,T_{i-1},T_i) \\ &\quad + \frac{\sigma_L^2}{2a_L^2}\left(2B_L(T_{i-1},T_i) - B_{LL}(T_{i-1},T_i) + T_i - T_{i-1}\right) \\ &\quad - \lambda_L\frac{\sigma_L}{a_L}\left(B_L(T_{i-1},T_i) + T_i - T_{i-1}\right) + \frac{\sigma_L^2}{2}B_L(T_{i-1},T_i)^2 B_{LL}(0,T_{i-1}), \end{aligned} \tag{25}$$

which is a formula to construct $\Phi_L(T_i)$ with $\Phi_L(0) = 0$.

Note that some interpolation method, such as the cubic spline method, should be applied to calculate $\Phi_L(t)$ for $t \in (T_i, T_{i+1})$. However, by construction, Φ_D and Φ_L defined in this way can generate initial curves of swap rates and basis swap spreads which are consistent with the observed rates in the market.

For the evaluation of swaps and swaptions, the following representation may be useful:

$$\begin{aligned} P_D(t,T) &= \frac{P_D^M(0,T)}{P_D^M(0,t)}\exp\Big\{A_D(t,T) - A_D(0,T) + A_D(0,t) \\ &\qquad - B_D(t,T)x_D(t) - C_D(t,T)x_D(t)^2\Big\}, \end{aligned} \tag{26}$$

$$\begin{aligned} L(t,T_1,T_2) &= \frac{1}{\delta}\Big(\frac{P_D^M(0,T_1)}{P_D^M(0,T_2)}K_L^M(0,T_1,T_2)\exp\Big\{-(B_D(t,T_1) - B_D(t,T_2))x_D(t) \\ &\quad - (C_D(t,T_1) - C_D(t,T_2))x_D(t)^2 - \mathrm{e}^{-a_L(T_1-t)}B_L(T_1,T_2)x_L(t) \\ &\quad + M_L(t,T_1,T_2)\Big\} - 1\Big), \end{aligned} \tag{27}$$

$$\begin{aligned} M_L(t,T_1,T_2) &= A_D(t,T_1) - A_D(0,T_1) - A_D(t,T_2) + A_D(0,T_2) \\ &\quad - \frac{\sigma_L^2}{2}B_L(T_1,T_2)^2\left(B_{LL}(t,T_1) - B_{LL}(0,T_1)\right). \end{aligned}$$

One may apply Monte Calro simulation to (26) and (27) under the T_0-forward measure $Q_D^{T_0}$. Since the vector process $(W_D^{T_0}(t), W_L^{T_0}(t), W_G^{T_0}(t))^\top$ defined by

$$\begin{aligned} \mathrm{d}W_D^{T_0}(t) &= \mathrm{d}W_D(t) - \sigma_D\left(B_D(t,T_0) + 2C_D(t,T_0)x_D(t)\right)\mathrm{d}t, \\ \mathrm{d}W_L^{T_0}(t) &= \mathrm{d}W_L(t), \\ \mathrm{d}W_G^{T_0}(t) &= \mathrm{d}W_G(t), \end{aligned}$$

follows the three-dimensional standard Brownian motion under T_0-forward measure $Q_D^{T_0}$ by the Girsanov theorem, the dynamics of $x_k(t)$ under $Q_D^{T_0}$ are given by

$$\begin{aligned} \mathrm{d}x_D(t) &= \left(\sigma_D B_D(t,T_0) + \left(2\sigma_D C_D(t,T_0) - a_D\right)x_D(t)\right)\mathrm{d}t + \sigma_D \mathrm{d}W_D^{T_0}(t), \\ \mathrm{d}x_L(t) &= -a_L x_L(t)\mathrm{d}t + \sigma_L \mathrm{d}W_L^{T_0}(t), \\ \mathrm{d}x_G(t) &= -a_G x_G(t)\mathrm{d}t + \sigma_G \mathrm{d}W_G^{T_0}(t). \end{aligned}$$

Hence, $x_k(T_0)$, $k = D, L, G$, are again normally distributed under the T_0-forward measure.

Finally, using the same arguments, we can derive an explicit formula for the G-curve in a completely parallel form. In fact, it is enough to replace the notation L with G. The only difference is whether the initial curve is calculated in rates in (10) or bond prices in (11). If the initial yield curve of bond yields is given as the par yields rather than bond prices, the formula of Φ_G is completely the same as Φ_L due to the form of (8).

3.2 The Hull-White model

The second example is a correlated Gaussian model at the sacrifice of non-negativity in the short rate. That is, following the idea of Hull and White (1994), suppose that

$$\begin{aligned} r_D(t) &= x_D(t) + \varphi_D(t), \\ r_L(t) &= r_D(t) + h_L(t), \quad h_L(t) = x_L(t) + \varphi_L(t), \\ r_G(t) &= r_D(t) + h_G(t), \quad h_G(t) = x_G(t) + \varphi_G(t), \end{aligned}$$

where $x_k(t)$ are the Ornstein-Uhlenbeck processes given by

$$\mathrm{d}x_k(t) = -a_k x_k(t)\mathrm{d}t + \sigma_k \mathrm{d}W_k(t), \quad x_k(0) = 0, \quad k = D, L, G,$$

and where $\varphi_k(t)$ are deterministic functions of time t. Again, it is sufficient to specify the integral

$$\Phi_k(T) \equiv \int_0^T \varphi_k(t)\mathrm{d}t, \quad k = D, L, G. \tag{28}$$

In order to introduce correlations among $x_k(t)$ $k = D, L, G$, we assume that the Brownian motions $W_k(t)$ are correlated as

$$\mathrm{d}W_D(t)\mathrm{d}W_L(t) = \rho_{DL}\mathrm{d}t, \quad \mathrm{d}W_D(t)\mathrm{d}W_G(t) = \rho_{DG}\mathrm{d}t, \quad \mathrm{d}W_L(t)\mathrm{d}W_G(t) = \rho_{LG}\mathrm{d}t.$$

The market prices of risk are assumed to be given by (17).

Define the functions

$$B_k(t,T) = -\frac{1}{a_k}\left(1 - \mathrm{e}^{-a_k(T-t)}\right), \quad B_{jk}(t,T) = -\frac{1 - \mathrm{e}^{-(a_j+a_k)(T-t)}}{a_j + a_k}$$

for $j,k=D,L,G$. Then, we obtain

$$\begin{aligned} P_D(t,T) &= \exp\Big\{ -\Phi_D(T)+\Phi_D(t)+B_D(t,T)x_D(t) \\ &\qquad + \frac{\sigma_D^2}{2a_D^2}\left(2B_D(t,T)-B_{DD}(t,T)+T-t\right)\Big\}. \end{aligned}$$

Thus, Φ_D is given by

$$\Phi_D(T) = -\ln P_D^M(0,T)+\frac{\sigma_D^2}{2a_D^2}\left(2B_D(0,T)-B_{DD}(0,T)+T\right). \qquad (29)$$

Next, we consider the forward LIBOR $L(t,T_{i-1},T_i)$. Write Equation (1) for $k=L$ as $P_L(t,T)=E_t^{Q_D}[\mathrm{e}^X]$, where X is a Gaussian random variable given by

$$\begin{aligned} X &= -\int_t^T \left(x_D(s)+\varphi_D(s)+x_L(s)+\varphi_L(s)+\frac{1}{2}\lambda_L^2\right)\mathrm{d}s-\int_t^T \lambda_L \mathrm{d}W_L(s) \\ &= B_D(t,T)x_D(t)+\sigma_D\int_t^T B_D(s,T)dW_D(s)+B_L(t,T)x_L(t)+\sigma_L\int_t^T B_L(s,T)dW_L(s) \\ &\quad -\Phi_D(T)+\Phi_D(t)-\Phi_L(T)+\Phi_L(t)-\frac{1}{2}\lambda_L^2(T-t)-\lambda_L(W_L(T)-W_L(t)). \end{aligned}$$

Since the random variable X has the mean

$$\begin{aligned} E_t^{Q_D}[X] &= B_D(t,T)x_D(t)+B_L(t,T)x_L(t)-\Phi_D(T)+\Phi_D(t) \\ &\qquad -\Phi_L(T)+\Phi_L(t)-\frac{1}{2}\lambda_L^2(T-t) \end{aligned}$$

and the variance

$$\begin{aligned} Var_t^{Q_D}[X] &= \frac{\sigma_D^2}{a_D^2}\left(2B_D(t,T)-B_{DD}(t,T)+T-t\right)+\frac{\sigma_L^2}{a_L^2}\left(2B_L(t,T)-B_{LL}(t,T)+T-t\right) \\ &\quad +\lambda_L^2(T-t)-2\lambda_L\frac{\rho_{DL}\sigma_D}{a_D}\left(B_D(t,T)+T-t\right)-2\lambda_L\frac{\sigma_L}{a_L}\left(B_L(t,T)+T-t\right) \\ &\quad +2\frac{\rho_{DL}\sigma_D\sigma_L}{a_Da_L}\left(B_D(t,T)+B_L(t,T)-B_{DL}(t,T)+T-t\right), \end{aligned}$$

we can calculate the expectation to yield

$$P_L(t,T) = P_D(t,T)\exp\left\{B_L(t,T)x_L(t)+A_L(t,T)+A_{DL}(t,T)\right\},$$

where A_L and A_{DL} are defined as

$$\begin{aligned} A_L(t,T) &= -\Phi_L(T)+\Phi_L(t)+\frac{\sigma_L^2}{2a_L^2}\left(2B_L(t,T)-B_{LL}(t,T)+T-t\right) \\ &\qquad -\lambda_L\frac{\sigma_L}{a_L}(B_L(t,T)+T-t) \end{aligned}$$

and

$$\begin{aligned} A_{DL}(t,T) &= \frac{\sigma_D^2}{2a_D^2}\left(2B_D(t,T)-B_{DD}(t,T)\right)-\lambda_L\frac{\rho_{DL}\sigma_D}{a_D}(B_D(t,T)+T-t) \\ &\qquad +\frac{\rho_{DL}\sigma_D\sigma_L}{a_Da_L}\left(B_D(t,T)+B_L(t,T)-B_{DL}(t,T)+T-t\right), \end{aligned}$$

respectively. Note that A_L is the term related to the L-curve only while A_{DL} is the term related to an interaction between the L-curve and the D-curve.

The forward LIBOR is calculated as

$$L(t,T_{i-1},T_i) = \frac{1}{\delta}\left(\frac{P_D(t,T_{i-1})}{P_D(t,T_i)}K_L(t,T_{i-1},T_i) - 1\right),$$

where

$$\begin{aligned}K_L(t,T_{i-1},T_i) &= \exp\Big\{-A_L(T_{i-1},T_i) - A_{DL}(T_{i-1},T_i) - \frac{1}{2}B_L(T_{i-1},T_i)^2\sigma_L^2 B_{LL}(t,T_{i-1}) \\ &\quad - B_L(T_{i-1},T_i)\left(x_L(t)\mathrm{e}^{-a_L(T_{i-1}-t)} - \frac{\rho_{DL}\sigma_D\sigma_L}{a_D}\left(B_L(t,T_{i-1}) - B_{DL}(t,T_{i-1})\right)\right)\Big\}.\end{aligned}$$

It follows that Φ_L must satisfy

$$\begin{aligned}\Phi_L(T_i) &= \Phi_L(T_{i-1}) + \ln K_L^M(0,T_{i-1},T_i) \\ &\quad + \frac{\sigma_L^2}{2a_L^2}\left(2B_L(T_{i-1},T_i) - B_{LL}(T_{i-1},T_i) + T_i - T_{i-1}\right) \\ &\quad - \lambda_L\frac{\sigma_L}{a_L}\left(B_L(T_{i-1},T_i) + T_i - T_{i-1}\right) \\ &\quad + A_{DL}(T_{i-1},T_i) + \frac{1}{2}B_L(T_{i-1},T_i)^2\sigma_L^2 B_{LL}(0,T_{i-1}) \\ &\quad - \frac{\rho_{DL}\sigma_D\sigma_L}{a_D}B_L(T_{i-1},T_i)\left(B_L(0,T_{i-1}) - B_{DL}(0,T_{i-1})\right), \qquad (30)\end{aligned}$$

where K_L^M is given by (24). The remaining calculations are carried out in the same way as the quadratic Gaussian case.

At last, similar to the quadratic Gaussian case, the following expressions will be useful for the evaluation of swaps and swaptions:

$$\begin{aligned}P_D(t,T) &= \frac{P_D^M(0,T)}{P_D^M(0,t)}\exp\left\{A_D(t,T) + B_D(t,T)x_D(t)\right\}, \\ A_D(t,T) &= \frac{\sigma_D^2}{2a_D^2}\Big(2B_D(t,T) - B_{DD}(t,T) - \left(2B_D(0,T) - B_{DD}(0,T)\right) \\ &\quad + 2B_D(0,t) - B_{DD}(0,t)\Big). \\ L(t,T_1,T_2) &= \frac{1}{\delta}\Big(\frac{P_D^M(0,T_1)}{P_D^M(0,T_2)}K_L^M(0,T_1,T_2)\exp\Big\{-(B_D(t,T_1) - B_D(t,T_2))x_D(t) \\ &\quad - \mathrm{e}^{-a_L(T_1-t)}B_L(T_1,T_2)x_L(t) + M_L(t,T_1,T_2)\Big\} - 1\Big) \\ M_L(t,T_1,T_2) &= A_D(t,T_1) - A_D(t,T_2) - \frac{\sigma_L^2}{2}B_L(T_1,T_2)^2\left(B_{LL}(t,T_1) - B_{LL}(0,T_1)\right) \\ &\quad + \frac{\rho_{DL}\sigma_D\sigma_L}{a_D}B_L(T_1,T_2)\left(B_L(t,T_1) - B_{DL}(t,T_1) - \left(B_L(0,T_1) - B_{DL}(0,T_1)\right)\right).\end{aligned}$$

4 Numerical examples

In this section, we present some numerical examples of option evaluation implied by the DLG model whose initial curve is consistent with observed market rates.

Suppose that at time $t = 0$ we observe swap rates $S(T)$, basis swap spreads $bs(T)$ and bond par yields $Y(T)$ for several maturities T in the market as indicated in Table 1.

Throughout the numerical examples, swaps and bonds are assumed to have semi-annual coupon payments. Using the classical bootstrapping (12)–(13), we calculate the zero rate $\tilde{Z}(T) = -\ln \tilde{P}(T)/T$, the forward LIBOR $\tilde{L}(T) = \tilde{L}(T-0.5, T)$, and the forward Govt rate $\tilde{G}(T) = \tilde{G}(T-0.5, T)$ by assuming that the bond par yield is a swap rate. For a comparison with the DLG bootstrapping (9)–(11), we give the adjusted zero rate $\tilde{Z}(T) + bs(T)$ that reflects the basis swap spread in the last column of Table 1.

The model parameters of the quadratic Gaussian model (QG model) and the Hull-White model (HW model) discussed in the previous section are set as shown in Table 2. σ_L is set higher than σ_G in each model. The correlations of the HW model $\rho = (\rho_{DL}, \rho_{DG}, \rho_{LG})$ will be specified later in each example.

Given these information, we can calculate the shifting functions Φ_D, Φ_L and Φ_G in each model. Results for the QG model are shown in Table 3. Using the DLG bootstrapping (9)–(11), we obtain the initial zero rate $Z(T) = -\ln P_D^M(0,T)/T$, the forward LIBOR $L(T) = L_D^M(0, T-0.5, T)$ and the forward Govt rate $G(T) = G_D^M(0, T-0.5, T)$. It is interesting to note that $Z(T) \approx \tilde{Z}(T) + bs(T)$, $L(T) \approx \tilde{L}(T)$, $G(T) \approx \tilde{G}(T)$. In particular, the differences in the forward rates are within 0.2 basis points. Roughly speaking, the zero rate $Z(T)$ is lower than the classical zero rate $\tilde{Z}(T)$ by the basis swap spread, although the forward rates are kept to be the same. Therefore, as explained in Section 2.2, there exists a difference in the evaluation of off-the-market swaps by the annuity of the basis swap spreads between the two bootstrapping methods. For example, the value of annuity for 10 years is $\frac{1}{2}\sum_{i=1}^{20} P_D(0, i/2) = 9.3407$ in bootstrapping of the QG model, while it is $\frac{1}{2}\sum_{i=1}^{20} \tilde{P}(i/2) = 9.2630$ in the classical bootstrapping.

The shifting functions in the HW model are shown in Table 4. Note that Φ_D is independent of the correlations by (29), while Φ_L and Φ_G depend on the correlation ρ_{DL} and ρ_{DG}, respectively, due to (30); but they are independent of ρ_{LG}. The bootstrapped rates $Z(T), L(T), G(T)$ in these models are very close to the results obtained in the QG model (see Table 3) and are omitted.

Prices of swaptions and bond options struck at the ATMF (at-the-money-forward) rate are calculated by Monte Carlo simulation with 100,000 runs. In these numerical examples of bond options, we consider options on fictitious bonds. Namely, an ATMF bond option means an option on a bond whose coupon is equal to the ATMF yield with a strike price of a par, not an option on a bond with a strike price of the ATMF price of the bond. It is a benefit of the DLG model that such a bond option is equivalent to a swaption against Govt rates constructed by the G-curve.

There are two types of the implied volatility for a price of swaption or bond option; the yield volatility and the absolute volatility. The absolute volatility is the annual standard deviation of movements of a particular forward swap rate (or bond yield). The yield volatility is the Black-Scholes type volatility or the relative volatility that equals the absolute volatility divided by the ATMF rate. Thus, the absolute volatility assumes a normal distribution for the underlying rates, while the yield volatility assumes a log-normal distribution.

Prices and volatilities of ATMF receiver's swaptions and ATMF call bond options are shown in Table 5 for the QG model and Tables 6–8 for the HW model. At the first glance of the volatility term structure in the QG model, we observe a decreasing volatility along the underlying maturities. Options on short-dated underlyings have high volatilities compared with options on long-dated underlyings with the same expiry. The HW model exhibits flatter term structures of volatility than the QG model.

Let us compare the prices of swaptions with those of bond options, where the correlations ρ_{DL} and ρ_{DG} play an important role for option prices but ρ_{LG} does not. In the case of non-negative correlations (see Tables 6 and 7), the prices of swaptions are slightly higher than those of bond options with the same expiry and underlying maturity, since the volatility σ_L is set higher than σ_G. Therefore, in this case, the absolute volatilities of swaptions are slightly higher than those of bond options, while the yield volatilities of swaptions are lower than those of bond options due to relatively higher ATMF rates of swaps than ATMF yields of bonds. On the other hand, negative correlations produce lower absolute volatilities of swaptions than those of bond options (see Table 8). By comparing Table 6 with Tables 7 and 8 in the HW model, higher correlation ρ_{DL} (or ρ_{DG}) between the L-curve (or G-curve) and the D-curve yields higher volatilities.

In Tables 9–13, we show prices of spread options and the absolute volatilities.[7] Regardless of different levels of volatilities of the D-curves between the QG model and the HW model, Tables 9 (QG model) and 10 (HW model) show very close prices of spread options, since the dynamics of x_L and x_G are the same. By comparing Tables 11 and 12, it is evident that the prices of options on basis swaps are the same because of the same correlation ρ_{DL}, while the prices of swap spread options are quite different due to different correlation ρ_{LG}. Positive ρ_{LG} yields low prices of swap spread options. Positive (or negative) ρ_{DL} makes high (or low) prices of options on basis swap as shown in Tables 10–13.

5 Conclusion

The DLG model allows us to formulate many yield curves with different quality under the no-arbitrage setting. This article demonstrates the usefulness of our model, especially in the pricing of spread options, and how to construct a short rate model whose initial curves are consistent with the observed curves in the market. The construction of these yield curves is carried out by deterministic shifting functions of short rates and bootstrapping of the discount factors of the cash flows and the forward rates. Other model parameters can be calibrated from the option prices. It becomes clear that correlations between curves play a crucial role for the pricing of spread options.

[7] The yield volatilities do not bring meaningful information due to the small numbers of the ATMF rates in these spread options.

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Table 1: Initial yield curve (percent)

Maturity	Swap	Basis swap	Bond	Swap spread	Classical bootstrapping			
T (years)	$S(T)$	$bs(T)$	$Y(T)$	$S(T)-Y(T)$	$\tilde{Z}(T)$	$\tilde{L}(T)$	$\tilde{G}(T)$	$\tilde{Z}(T)+bs(T)$
1	1.000	-0.400	0.700	0.300	0.998	1.050	0.770	0.598
2	1.100	-0.350	0.850	0.250	1.098	1.252	1.093	0.748
3	1.190	-0.300	0.960	0.230	1.188	1.419	1.240	0.888
4	1.280	-0.250	1.060	0.220	1.279	1.604	1.420	1.029
5	1.355	-0.210	1.145	0.210	1.355	1.706	1.542	1.145
6	1.430	-0.170	1.230	0.200	1.432	1.863	1.721	1.262
7	1.500	-0.140	1.310	0.190	1.504	1.983	1.862	1.364
8	1.570	-0.120	1.390	0.180	1.577	2.134	2.035	1.457
9	1.635	-0.100	1.465	0.170	1.645	2.237	2.159	1.545
10	1.690	-0.090	1.530	0.160	1.702	2.267	2.211	1.612
12	1.785	-0.070	1.635	0.150	1.804	2.404	2.319	1.734
15	1.905	-0.060	1.765	0.140	1.934	2.592	2.509	1.874
20	2.060	-0.050	1.930	0.130	2.109	2.834	2.754	2.059
25	2.160	-0.040	2.040	0.120	2.226	2.839	2.787	2.186
30	2.220	-0.030	2.110	0.110	2.296	2.750	2.728	2.266

Table 2: Model parameters

Quadratic Gaussian (QG) model			Hull-White (HW) model		
$a_D = 0.07,$	$\sigma_D = 0.0750,$	$\alpha = \beta = 0$	$a_D = 0.07,$	$\sigma_D = 0.0090,$	
$a_L = 0.04,$	$\sigma_L = 0.0020,$	$\lambda_L = 0$	$a_L = 0.04,$	$\sigma_L = 0.0020,$	$\lambda_L = 0$
$a_G = 0.04,$	$\sigma_G = 0.0010,$	$\lambda_G = 0$	$a_G = 0.04,$	$\sigma_G = 0.0010,$	$\lambda_G = 0$

Table 3: Bootstrapping of the QG model (percent)

Maturity	Shifting function			Bootstrapping		
T	$\Phi_D(T)$	$\Phi_L(T)$	$\Phi_G(T)$	$Z(T)$	$L(T)$	$G(T)$
1	0.331	0.398	0.100	0.599	1.050	0.770
2	0.478	0.696	0.199	0.749	1.251	1.092
3	0.489	0.893	0.208	0.890	1.419	1.239
4	0.444	0.989	0.116	1.031	1.603	1.419
5	0.280	1.032	-0.008	1.148	1.704	1.541
6	0.130	0.993	-0.193	1.265	1.862	1.720
7	-0.069	0.942	-0.369	1.368	1.982	1.861
8	-0.283	0.912	-0.502	1.462	2.134	2.034
9	-0.463	0.838	-0.658	1.551	2.237	2.159
10	-0.763	0.828	-0.727	1.619	2.267	2.211
12	-1.321	0.741	-0.998	1.741	2.405	2.320
15	-2.189	0.777	-1.236	1.882	2.592	2.509
20	-3.118	0.831	-1.631	2.067	2.834	2.754
25	-3.787	0.751	-2.027	2.195	2.840	2.788
30	-4.573	0.527	-2.424	2.278	2.751	2.730

Table 4: Shifting functions of the HW model (percent)

Maturity T	$\rho = (0, 0, 0)$ $\Phi_D(T)$	$\Phi_L(T)$	$\Phi_G(T)$	$\rho = (0.4, 0.4, -0.4)$ $\Phi_D(T)$	$\Phi_L(T)$	$\Phi_G(T)$
1	0.600	-2.906	-3.205	0.600	-2.906	-3.205
2	1.508	-5.913	-6.410	1.508	-5.914	-6.411
3	2.701	-9.021	-9.706	2.701	-9.023	-9.707
4	4.195	-12.231	-13.104	4.195	-12.236	-13.106
5	5.870	-15.492	-16.532	5.870	-15.502	-16.537
6	7.808	-18.836	-20.022	7.808	-18.854	-20.031
7	9.904	-22.192	-23.502	9.904	-22.220	-23.516
8	12.157	-25.527	-26.941	12.157	-25.568	-26.962
9	14.585	-28.905	-30.401	14.585	-28.963	-30.430
10	17.012	-32.220	-33.775	17.012	-32.297	-33.814
12	22.193	-38.916	-40.655	22.193	-39.043	-40.719
15	30.451	-48.795	-50.808	30.451	-49.023	-50.921
20	45.620	-65.265	-67.728	45.620	-65.734	-67.962
25	61.751	-81.869	-84.648	61.751	-82.664	-85.045
30	78.228	-98.618	-101.568	78.228	-99.817	-102.167

Table 5: Yields and volatilities in the QG model

Option Expiry (years)	Swaption: Swap Maturity (years) 1	3	5	10	Bond option: Bond Maturity (years) 1	3	5	10
ATMF rate (percent)								
1	1.201	1.375	1.519	1.817	1.001	1.182	1.340	1.678
3	1.557	1.680	1.812	2.038	1.367	1.511	1.664	1.919
5	1.822	1.952	2.055	2.222	1.675	1.826	1.950	2.122
10	2.268	2.340	2.409	2.521	2.169	2.244	2.316	2.428
Option price (basis point)								
1	24.2	61.6	87.2	121.3	23.4	59.0	82.4	109.3
3	58.6	146.1	202.7	269.6	57.8	143.5	198.0	257.9
5	79.6	196.2	270.2	355.1	78.8	193.5	265.2	343.2
10	96.9	236.1	322.7	420.7	96.0	233.1	317.3	407.4
Yield volatility (percent)								
1	51.6	38.6	30.1	18.2	60.1	43.0	32.2	17.8
3	59.0	45.4	35.4	20.8	67.1	49.9	37.7	22.2
5	56.0	42.7	33.8	21.4	61.0	45.2	35.0	21.6
10	43.7	34.2	27.5	17.9	45.6	35.4	28.1	18.0
Absolute volatility (basis point)								
1	62.0	53.1	45.7	33.1	60.2	50.9	43.1	29.9
3	91.9	73.3	64.1	44.4	91.7	75.3	62.7	42.5
5	102.1	83.4	69.4	47.5	102.2	82.6	68.2	45.9
10	99.2	80.1	66.2	45.0	98.9	79.4	65.2	43.6

Table 6: Volatilities in the HW model with $\rho = (0, 0, 0)$

	Swaption				Bond option			
Option	Swap Maturity				Bond Maturity			
Expiry	1	3	5	10	1	3	5	10
Yield volatility (percent)								
1	73.6	59.7	50.4	36.0	87.4	68.3	56.1	38.0
3	54.0	46.4	40.0	30.4	60.7	50.7	42.7	31.4
5	43.7	37.8	33.5	26.4	46.7	39.5	34.4	26.9
10	30.5	27.5	24.9	20.4	31.1	27.9	25.2	20.5
Absolute volatility (basis point)								
1	88.4	82.1	76.6	65.4	87.5	80.8	75.1	63.8
3	84.0	77.9	72.5	61.9	83.0	76.6	71.0	60.3
5	79.6	73.8	68.8	58.7	78.1	72.1	67.1	57.1
10	69.2	64.3	60.1	51.4	67.5	62.6	58.3	49.7

Table 7: Volatilities in the HW model with $\rho = (0.4, 0.4, -0.4)$

	Swaption				Bond option			
Option	Swap Maturity				Bond Maturity			
Expiry	1	3	5	10	1	3	5	10
Yield volatility (percent)								
1	80.2	65.1	55.2	39.7	91.7	71.7	59.0	40.1
3	59.4	51.2	44.3	33.9	63.9	53.4	45.1	33.3
5	48.4	42.1	37.4	29.9	49.3	41.8	36.5	28.7
10	34.5	31.3	28.5	23.6	33.2	29.9	27.1	22.2
Absolute volatility (basis point)								
1	96.3	89.6	83.9	72.1	91.8	84.8	79.0	67.4
3	92.4	86.0	80.3	69.1	87.4	80.7	75.0	64.0
5	88.3	82.1	76.9	66.3	82.6	76.4	71.2	60.9
10	78.2	73.2	68.7	59.5	72.1	67.1	62.7	53.9

Table 8: Volatilities in the HW model with $\rho = (-0.4, -0.4, 0)$

	Swaption				Bond option			
Option	Swap Maturity				Bond Maturity			
Expiry	1	3	5	10	1	3	5	10
Yield volatility (percent)								
1	66.9	54.0	45.4	32.1	83.5	65.2	53.4	36.0
3	48.4	41.4	35.6	26.6	57.7	48.0	40.4	29.5
5	38.5	33.1	29.2	22.7	44.0	37.2	32.3	25.0
10	26.5	23.7	21.3	17.1	28.9	25.8	23.2	18.7
Absolute volatility (basis point)								
1	80.3	74.3	69.0	58.3	83.6	77.1	71.6	60.4
3	75.4	69.5	64.4	54.3	78.9	72.6	67.2	56.6
5	70.2	64.7	59.9	50.4	73.7	67.9	65.3	53.1
10	60.0	55.4	51.3	43.1	62.6	57.9	53.7	45.4

Table 9: Spread options in the QG model

	Option on swap spread				Option on basis swap			
Option Expiry	Maturity 1	3	5	10	Maturity 1	3	5	10
ATMF rate (basis point)								
1	20.0	19.3	17.9	13.9	-30.0	-19.9	-12.2	-4.5
3	18.9	16.9	14.8	11.9	-9.6	-3.5	-0.5	1.2
5	14.8	12.7	10.6	10.0	3.9	4.0	4.1	2.6
10	9.9	9.6	9.3	9.3	3.2	2.2	1.0	0.0
ATMF Option price (basis point)								
1	8.5	24.3	38.5	67.2	7.6	21.8	34.5	60.3
3	13.9	39.4	62.0	107.8	12.4	35.1	55.3	96.0
5	16.6	47.0	73.9	128.1	14.9	42.1	66.2	114.6
10	19.4	54.6	85.6	147.5	17.3	48.8	76.5	131.8
Absolute volatility (basis point)								
1	21.6	20.8	20.1	18.4	19.4	18.7	18.0	16.5
3	20.8	20.0	19.3	17.7	18.6	17.9	17.2	15.7
5	20.0	19.2	18.5	17.0	17.9	17.2	16.6	15.2
10	18.4	17.7	17.0	15.6	16.4	15.8	15.2	13.9

Table 10: Spread options in the HW model with $\rho = (0, 0, 0)$

	Option on swap spread				Option on basis swap			
Option Expiry	Maturity 1	3	5	10	Maturity 1	3	5	10
ATMF Option price (basis point)								
1	8.6	24.6	38.8	67.9	7.6	21.6	34.1	59.7
3	14.0	39.8	62.8	109.1	12.4	35.2	55.4	96.4
5	16.9	47.7	75.0	129.8	14.9	42.3	66.5	115.1
10	19.5	55.0	86.3	148.7	17.3	48.8	76.5	131.9
Absolute volatility (basis point)								
1	21.8	21.0	20.3	18.5	19.2	18.5	17.8	16.3
3	21.1	20.3	19.5	17.9	18.6	17.9	17.3	15.8
5	20.3	19.5	18.8	17.2	18.0	17.3	16.7	15.3
10	18.5	17.8	17.2	15.7	16.4	15.8	15.2	13.9

Table 11: Spread options in the HW model with $\rho = (0.4, 0.4, -0.4)$

	Option on swap spread				Option on basis swap			
Option	Maturity				Maturity			
Expiry	1	3	5	10	1	3	5	10
ATMF Option price (basis point)								
1	9.9	28.1	44.3	77.0	7.6	21.9	35.0	62.0
3	16.0	45.1	70.8	122.1	12.7	36.6	58.3	103.6
5	19.0	53.5	83.7	143.5	15.7	45.1	71.8	127.5
10	21.4	60.0	93.5	159.2	19.4	55.6	88.5	157.2
Absolute volatility (basis point)								
1	25.1	24.0	23.1	21.0	19.4	18.8	18.2	16.9
3	24.0	23.0	22.0	20.0	19.1	18.6	18.1	17.0
5	22.9	21.9	21.0	19.0	18.9	18.5	18.0	16.9
10	20.3	19.4	18.6	16.8	18.4	18.0	17.6	16.6

Table 12: Spread options in the HW model with $\rho = (0.4, 0.4, 0.4)$

	Option on swap spread				Option on basis swap			
Option	Maturity				Maturity			
Expiry	1	3	5	10	1	3	5	10
ATMF Option price (basis point)								
1	7.0	19.8	31.3	54.3	7.6	21.9	35.0	62.0
3	11.3	32.0	50.1	86.1	12.7	36.6	58.3	103.6
5	13.4	37.7	58.9	100.6	15.7	45.1	71.8	127.5
10	15.1	42.1	65.4	110.7	19.4	55.6	88.5	157.2
Absolute volatility (basis point)								
1	17.7	17.0	16.3	14.8	19.4	18.8	18.2	16.9
3	17.0	16.3	15.6	14.1	19.1	18.6	18.1	17.0
5	16.2	15.4	14.8	13.3	18.9	18.5	18.0	16.9
10	14.3	13.6	13.0	11.7	18.4	18.0	17.6	16.6

Table 13: Spread options in the HW model with $\rho = (-0.4, -0.4, 0)$

	Option on swap spread				Option on basis swap			
Option	Maturity				Maturity			
Expiry	1	3	5	10	1	3	5	10
ATMF Option price (basis point)								
1	8.6	24.5	38.9	68.4	7.5	21.3	33.4	57.6
3	14.0	39.9	63.3	111.0	12.1	33.9	52.8	89.8
5	17.0	48.5	76.7	134.3	14.2	39.5	61.3	103.3
10	20.4	57.9	91.4	159.8	15.4	42.6	65.7	109.5
Absolute volatility (basis point)								
1	21.7	21.0	20.3	18.7	19.1	18.2	17.4	15.7
3	21.0	20.3	19.7	18.2	18.2	17.3	16.4	14.7
5	20.5	19.8	19.2	17.8	17.1	16.2	15.4	13.7
10	19.3	18.7	18.2	16.9	14.6	13.8	13.1	11.6

A Formulas

We have

$$B_D(t,T) = 2F_D(t,T)\int_t^T \frac{\alpha+\beta s}{F_D(s,T)}\mathrm{d}s = \frac{2B_1(t,T)}{\gamma^2 A_5(t,T)}$$

and

$$\begin{aligned} A_D(t,T) &= \int_t^T \left(\frac{1}{2}\sigma_D^2 B_D(s,t)^2 - \sigma_D^2 C_D(s,T) - (\alpha+\beta s)^2\right)\mathrm{d}s \\ &= -\sigma_D^2\left(\frac{A_4(t,T)}{\gamma^5 A_5(t,T)} + A_6(t,T)\right) - \alpha^2(T-t) - \alpha\beta(T^2-t^2) - \frac{1}{3}\beta^2(T^3-t^3), \end{aligned}$$

where

$$\begin{aligned} \Gamma_a &= \gamma - a_D, \\ \Gamma_b &= \gamma + a_D, \\ A_{1a}(t,T) &= -\mathrm{e}^{\gamma(T-t)} + 4 - \mathrm{e}^{-\gamma(T-t)}(3+2\gamma(T-t)), \\ A_{1b}(t,T) &= \mathrm{e}^{-\gamma(T-t)} - 4 + \mathrm{e}^{\gamma(T-t)}(3-2\gamma(T-t)), \\ A_{2a}(t,T) &= \mathrm{e}^{\gamma(T-t)}(1-\gamma T) - 2(1-\gamma(t+T)) + \mathrm{e}^{-\gamma(T-t)}(1-\gamma(2t+T)+\gamma^2(t^2-T^2)), \\ A_{2b}(t,T) &= \mathrm{e}^{-\gamma(T-t)}(1+\gamma T) - 2(1+\gamma(t+T)) + \mathrm{e}^{\gamma(T-t)}(1+\gamma(2t+T)+\gamma^2(t^2-T^2)), \\ A_{3a}(t,T) &= -4\gamma t(1-\gamma T) - \mathrm{e}^{\gamma(T-t)}(1-\gamma T)^2 \\ &\quad + \mathrm{e}^{-\gamma(T-t)}\left(1+2\gamma t - \gamma^2(2t^2+T^2) + \frac{2}{3}\gamma^3(t^3-T^3)\right), \\ A_{3b}(t,T) &= -4\gamma t(1+\gamma T) + \mathrm{e}^{-\gamma(T-t)}(1+\gamma T)^2 \\ &\quad + \mathrm{e}^{\gamma(T-t)}\left(-1+2\gamma t + \gamma^2(2t^2+T^2) + \frac{2}{3}\gamma^3(t^3-T^3)\right), \\ A_4(t,T) &= \Gamma_a\left(\alpha^2\gamma^2 A_{1a}(t,T) + 2\alpha\beta\gamma A_{2a}(t,T) + \beta^2 A_{3a}(t,T)\right) \\ &\quad + \Gamma_b\left(\alpha^2\gamma^2 A_{1b}(t,T) + 2\alpha\beta\gamma A_{2b}(t,T) + \beta^2 A_{3b}(t,T)\right), \\ A_5(t,T) &= \Gamma_a \mathrm{e}^{-\gamma(T-t)} + \Gamma_b \mathrm{e}^{\gamma(T-t)}, \\ A_6(t,T) &= -\frac{1}{2}(T-t)\left(\Gamma_a^{-1} - \Gamma_b^{-1}\right) + \frac{1}{2\gamma}\left(\Gamma_a^{-1} + \Gamma_b^{-1}\right)\ln\frac{A_5(t,T)}{2\gamma} \end{aligned}$$

and

$$\begin{aligned} B_1(t,T) &= -\alpha\gamma\left(\mathrm{e}^{-\gamma T} - \mathrm{e}^{-\gamma t}\right)\left(\Gamma_a \mathrm{e}^{\gamma t} + \Gamma_b \mathrm{e}^{\gamma T}\right) \\ &\quad + \beta\left(\Gamma_a \mathrm{e}^{-\gamma(T-t)}(1-\gamma t) + \Gamma_b \mathrm{e}^{\gamma(T-t)}(1+\gamma t) - \Gamma_a(1-\gamma T) - \Gamma_b(1+\gamma T)\right). \end{aligned}$$