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Abstract

This article investigates an optimal allocation problem of assets under Gaussian state variables consisting of the risk-free rate and the multi dimensional market price of risk. The optimal portfolio consisting of a bond and several stocks is derived. The idea is that the indirect utility as a solution of a partial differential equation is expressed as a zero-coupon bond price in a different economy having a quadratic Gaussian short rate by the Feynman-Kac formula, then the bond price is obtained with an exponentially quadratic state-price density process which is closely related to some of quadratic term structure models.

Key words: optimal portfolio, quadratic Gaussian interest rate model

1 Introduction

We investigate an optimal allocation problem of assets under Gaussian state variables consisting of the risk-free rate and the multi dimensional market price of risk. It is well known that the indirect utility can be expressed as a zero-coupon bond price in a different economy having a short rate. The difficulty is that a square of the market price of risk appears in the short rate. Therefore, the problem is naturally related to quadratic term structure models (QTSMs). Our contribution is to overcome the optimal allocation problem with an exponentially quadratic state-price density process and several findings on QTSMs. This technique is worth discussing in a paper since it will be useful in related analysis.

Among many interest rate models in the literature, affine term structure models (ATSMs) have been widely discussed from theoretical, empirical and practical points of view. The Vasicek model [14] is the simplest ATSM but it has a drawback that the risk-free rate becomes negative with positive probability. The CIR model [6] is a popular model yielding positive interest rates in the class of ATSMs. On the other hand, QTSMs have received less attention than ATSMs due to the nonlinearity. Nevertheless, the quadratic form easily leads to positive interest rates, and as a result, the study of QTSMs has progressed gradually. Several specific models in QTSMs are investigated in Longstaff [8], Beaglehole and Tenny [2][3], Constantinides [5] and others.

All assets are priced by the state-price density process in a complete market. The state-price density processes in QTSMs have special features in the functional form. Rogers [13] discusses several term structure models with a focus on positive interest rates via the potential approach. As an example he derives a QTSM by a state-price density process given in an exponentially quadratic form of state variables. In the SAINTS model of Constantinides [5], the state-price density process is also an exponentially quadratic function of the state variables. Ahn et al. [1] characterizes QTSMs with three assumptions; (a) the state variables follow a multi-dimensional Ornstein-Uhlenbeck process, (b) the risk-free rate is a quadratic function of the state variables, and (c) the market prices of risk are affine functions of the state variables. In the analysis it is shown that the SAINTS model is a special form of the QTSMs and a special form of CIR model belongs to QTSMs. Pelsser [12] obtains a zero-coupon bond price under a setting of a squared-Gaussian risk-free rate by solving a system of ordinary differential equations. Although the market price of risk is not mentioned in the paper, we find that Pelsser's model is implied by a state-price density which is an exponentially quadratic function of the state variables, similar to Constantinides [5]. Thus, both of Constantinides [5] and Pelsser [12] are built with exponentially quadratic state-price density processes.

An optimal portfolio problem of Merton [9] has been studied extensively and extended in several directions. Recently, Liu [7] analyzes the bond portfolio problem under a QTSM. Chiarella et al. [4] study investment strategies under inflation risk. Munk [10] studies the problem under habit formation. Seminal papers close to our analysis are Wachter [15] and Munk et al. [11]. Both papers analyze the optimal portfolio consisting of a stock and a money market account under a setting of a mean-reverting market price of stock market risk and Gaussian risk-free rate. Therefore, quadratic functions of the market price of risk are observed in the indirect utility and the optimal consumption-wealth ratio. Due to the single dimensionality, the derivation is relatively straightforward.

We extend Wachter [15] and Munk et al. [11] to a case including a bond and several stocks and derive the optimal allocation. Our idea is that the indirect utility as a solution of a partial differential equation (PDE) is expressed as a zero-coupon bond price in a different economy having a quadratic Gaussian short rate via the Feynman-Kac formula, then we obtain the bond price with an exponentially quadratic state-price density process which is closely related to Constantinides [5] and Pelsser [12].

This article is organized as follows. In Section 2 we describe the model setup and obtain the optimal allocation with unknown function to be solved. Section 3 is devoted to solve the function by making use of the Feynman-Kac formula and techniques related to quadratic Gaussian model. Then the optimal allocation is shown as a function of the market prices of risk only. Section 4 concludes this paper.

2 Setup

Let $z(t)$ denote the N-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ where P is an objective measure. There are one risk-free asset $B(t)$ and N risky assets $S(t) = (S_1(t), \cdots, S_N(t))^\top$. $S_1(t)$ is a bond price with maturity $U \geq T$. $r(t)$ is the risk-free rate and $\lambda(t) = (\lambda_1(t), \cdots, \lambda_N(t))^\top$ is called the vector of the market price of risk. These security prices obey the stochastic differential equations (SDEs)

$$\begin{aligned} dB(t) &= r(t)B(t)dt, \\ dS(t) &= \mathrm{diag}[S(t)]_N(r(t)\mathbf{1}_N + \sigma_S(t)^\top \lambda(t))dt + \mathrm{diag}[S(t)]_N \sigma_S(t)^\top dz(t), \end{aligned} \tag{1}$$

where $\sigma_S(t) \in \mathbb{R}^{N\times N}$ is an invertible deterministic matrix, and $\mathbf{1}_N = (1, \cdots, 1)^\top$. We denote by $\mathrm{diag}[x]_N$ or $\mathrm{diag}[x_i]_N$ an $N \times N$-diagonal matrix with the element x_i of an N-dimensional vector $x = \{x_i\}$ on the diagonal. Furthermore, $[m_{ij}]_{n\times m}$ is meant to be an $n \times m$ matrix with elements of m_{ij}. We assume that $r(t)$ and $\lambda(t)$ are Ornstein-Uhlenbeck processes

$$\begin{aligned} dr(t) &= a_r(b_r - r(t))dt + \sigma_r^\top dz(t), \\ d\lambda(t) &= a_\lambda(b_\lambda - \lambda(t))dt + \sigma_\lambda^\top dz(t), \end{aligned}$$

where $a_r, b_r \in \mathbb{R}$ are constants, $\sigma_r, b_\lambda \in \mathbb{R}^N$ are constant vectors, and $a_\lambda, \sigma_\lambda \in \mathbb{R}^{N\times N}$ are constant matrices, given by

$$\sigma_r = \begin{pmatrix} \sigma_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad a_\lambda = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_N \end{pmatrix}, \quad \sigma_\lambda^\top = \begin{pmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_N \end{pmatrix}.$$

Due to the form of (1), $r(t)$ and $\lambda(t)$ jointly describes the structure of the asset prices. Thus, by getting together them, we call the state variables the $(N+1)$-dimensional vector

$$X(t) = (X_0(t), X_1(t), \cdots, X_N(t))^\top \equiv (r(t), \lambda(t)^\top)^\top.$$

Obviously the state variables X satisfy a SDE

$$dX(t) = a_X(b_X - X(t))dt + \sigma_X^\top dz(t), \tag{2}$$

where

$$a_X = \begin{pmatrix} a_r & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_N \end{pmatrix}, \quad \sigma_X^\top = \begin{pmatrix} \sigma_r & 0 & 0 & \cdots & 0 \\ \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_N \end{pmatrix}.$$

Note that $r(t)$ and $\lambda_1(t)$ are correlated though other pairs of the state variables are independent. As notations throughout the paper, elements of $(N+1)$-dimensional matrices and vectors are indexed with $r, 1, 2, \cdots, N$ rather than $1, 2, \cdots, N, N+1$ so as to make it easier to figure out the effects of $r(t)$ and $\lambda_i(t)$ $(i = 1, 2, \cdots, N)$.

At each time, the investor allocates wealth $W(t)$ between the risk-free asset and the risky assets. Let $\alpha(t)$ denote the vector of the allocation ratio to the risky securities.

The investor faces the maximization problem of the expected utility from the terminal wealth

$$
\begin{aligned}
&\max_{\{\alpha(t)\}} E\left[e^{-\delta T}\frac{1}{1-\gamma}W(T)^{1-\gamma}\right], && (3)\\
\text{s.t.}\quad &\frac{dW(t)}{W(t)} = \left(r(t)+\alpha(t)^\top\sigma_S(t)^\top\lambda(t)\right)dt + \alpha(t)^\top\sigma_S(t)^\top dz(t).
\end{aligned}
$$

Throughout the paper we assume that

$$
\begin{aligned}
&\gamma > 1, \quad \sigma_i \neq 0, \quad (i = 1, 2, \cdots, N),\\
&a_r > 0, \quad a_r \neq \frac{1-\gamma}{\gamma}\sigma_1 + \sqrt{\left(a_1 - \frac{1-\gamma}{\gamma}\sigma_1\right)^2 - \frac{\sigma_1^2}{\gamma}}, && (4)\\
&a_i > \frac{1-\gamma}{\gamma}\sigma_i + \frac{\sqrt{1-\gamma+\gamma^2}}{\gamma}|\sigma_i|, \quad (i = 1, 2, \cdots, N). && (5)
\end{aligned}
$$

Let $J(w, x, t)$ denote the indirect utility associated with (3) when $W(t) = w$, $X(t) = x \equiv (r, \lambda^\top)^\top$. Then the Hamilton-Jacobi-Bellman (HJB) equation is given by

$$
\begin{aligned}
0 = &\max_{\{\alpha(t)\}}\Big[J_t + \left(r + \alpha^\top\sigma_S^\top\lambda\right)J_w w + \frac{1}{2}\alpha^\top\sigma_S^\top\sigma_S\alpha J_{ww}w^2\\
&+ \alpha^\top\sigma_S^\top\sigma_X J_{wx}w + (a_X(b_X - x))^\top J_x + \frac{1}{2}\mathrm{tr}\left(\sigma_X^\top\sigma_X J_{xx}\right)\Big], && (6)\\
J(w, x, T) = &\ e^{-\delta T}\frac{1}{1-\gamma}w^{1-\gamma},
\end{aligned}
$$

where the function J with the subscript of a variable means the partial derivative with respect to the variable, and the variables are suppressed to save space. The first order condition

$$\sigma_S^\top\lambda J_w w + \sigma_S^\top\sigma_S\alpha^* J_{ww}w^2 + \sigma_S^\top\sigma_X J_{wx}w = 0$$

implies the optimal allocation

$$\alpha^*(t) = \frac{-J_w(w, x, t)}{wJ_{ww}(w, x, t)}\sigma_S(t)^{-1}\lambda(t) + \sigma_S(t)^{-1}\sigma_X\frac{-J_{wx}(w, x, t)}{wJ_{ww}(w, x, t)}. \tag{7}$$

The first term $\frac{-J_w(w,x,t)}{wJ_{ww}(w,x,t)}\sigma_S(t)^{-1}\lambda(t)$ is called the myopic portfolio or the mean-variance portfolio while the second term $\sigma_S(t)^{-1}\sigma_X\frac{-J_{wx}(w,x,t)}{wJ_{ww}(w,x,t)}$ is the intertemporal hedging portfolio for the risk of the state variables.

For further investigation, we conjecture that the indirect utility function is in the form of

$$J(w, x, t) = e^{-\delta t}\frac{1}{1-\gamma}w^{1-\gamma}F(x, t)^{1-\gamma}$$

with some function F satisfying $F(x, T) = 1$ for all x. Then the optimal allocation (7) is reduced to

$$\alpha^*(t) = \frac{1}{\gamma}\sigma_S(t)^{-1}\lambda(t) + \frac{1-\gamma}{\gamma}\sigma_S(t)^{-1}\sigma_X\frac{F_x(x, t)}{F(x, t)}. \tag{8}$$

It follows that we need to know the function F_x/F in the intertemporal hedging portfolio in order to characterize the optimal allocation more explicitly.

3 Optimal allocation

By plugging (8) into the HJB equation (6), we obtain a PDE for F

$$
\begin{aligned}
&F_t + \left(a_X(b_X - x) + \frac{1-\gamma}{\gamma}\sigma_X^\top \lambda\right)^\top F_x + \frac{1}{2}\mathrm{tr}\left(\sigma_X^\top \sigma_X F_{xx}\right) \\
&\quad - \left(\frac{\delta}{1-\gamma} - r - \frac{1}{2\gamma}\lambda^\top \lambda\right) F = 0, \qquad (9) \\
&F(x,T) = 1.
\end{aligned}
$$

In order to solve the PDE (9), we can apply the Feynman-Kac formula by following Chiarella et al. [4]. The Feynman-Kac formula tells us that the solution, if it exists, of the partial differential equation (9) is given by

$$
F(x,t) = E^{\tilde{Q}}\left[\exp\left(-\int_t^T R(Y(u))du\right) \Big| Y(t) = x\right], \qquad (10)
$$

where $\tilde{Q}$ is a reference pricing measure equipped with an N-dimensional standard Brownian motion $z^{\tilde{Q}}$, Y is an $(N+1)$-dimensional vector of the state variables

$$
Y(t) = (r^Y(t), \lambda^Y(t)^\top)^\top, \quad r^Y(t) \in \mathbb{R}, \quad \lambda^Y(t) \in \mathbb{R}^N,
$$

whose SDE is implied by the coefficients of F_x and F_{xx} in (9) as

$$
\begin{aligned}
dY(t) &= (a_Y(b_Y - Y(t))dt + \sigma_Y^\top dz^{\tilde{Q}}(t), \qquad (11) \\
a_Y &= a_X - \frac{1-\gamma}{\gamma}\left(\ \mathbf{0}_{N+1}, \quad \sigma_X^\top\ \right), \quad b_Y = a_Y^{-1} a_X b_X, \quad \sigma_Y = \sigma_X,
\end{aligned}
$$

where $\mathbf{0}_{N+1} = (0, \cdots, 0)^\top$ is the $(N+1)$-dimensional zero vector, $\left(\ \mathbf{0}_{N+1}, \quad \sigma_X^\top\ \right)$ is the $(N+1)\times(N+1)$ matrix stacked with $\mathbf{0}_{N+1}$ and $\sigma_X^\top$, and R is given by a quadratic function of the state variable

$$
R(Y(t)) = \frac{\delta}{1-\gamma} - r^Y(t) - \frac{1}{2\gamma}\lambda^Y(t)^\top \lambda^Y(t), \qquad (12)
$$

implied by the coefficient of F in (9). Note that the reference measure $\tilde{Q}$ is different from the so-called risk-neutral measure. $\tilde{Q}$ should be understood to be the martingale measure with respect to a numéraire

$$
\tilde{B}(t) = \exp\left(\int_0^t R(Y(u))du\right).
$$

Therefore, (10) is the zero-coupon bond price when a pair $(\tilde{Q}, R)$ of martingale measure and risk-free rate is given. The risk-free rate (12) is a quadratic form of the state vector Y which is an $(N+1)$-dimensional Ornstein-Uhlenbeck process under a reference pricing measure $\tilde{Q}$ as shown in (11). Such an interest rate model is studied by Pelsser [12] who obtains bond prices by solving differential equations. In what follows, we calculate (10) by making use of the state-price density process studied by Rogers [13]. Although (11) does not involve the market prices of risk explicitly, we can derive them by assuming an exponentially quadratic state-price density process if the drift term of the state-price density process is consistent with (12) as discussed in Rogers [13].

Suppose, in a different and fictitious economy, the vector $Y(t)$ of (11) is the state vector and a process

$$\xi(t) = e^{-\beta t}\frac{f(Y(t))}{f(Y(0))} \tag{13}$$

is the state-price density in a reference objective measure $\tilde{P}$, where the function f is given by

$$f(x) = \exp\left(\frac{1}{2}x^\top Kx + L^\top x\right)$$

with some $(N+1)\times(N+1)$ symmetric matrix K and $(N+1)$-dimensional vector L. Therefore, such a measure $\tilde{P}$ is defined by

$$\frac{d\tilde{P}}{d\tilde{Q}} = \frac{1}{\xi(t)\tilde{B}(t)}. \tag{14}$$

It follows that the zero-coupon bond price is obtained as

$$E^{\tilde{Q}}\left[\frac{\tilde{B}(t)}{\tilde{B}(T)}\,\middle|\, Y(t)=x\right] = E^{\tilde{P}}\left[\frac{\xi(T)}{\xi(t)}\,\middle|\, Y(t)=x\right]. \tag{15}$$

Since the left hand side of (15) includes an integral of R which is a quadratic form of Y, the expectation is not easy to obtain. On the other hand, the right hand side of (15) is an expectation of $\xi(T)$ that is an exponentially quadratic function of $Y(T)$. The expectation may be straightforward because of the normality of Y. This point is the reason for the introduction of the measure $\tilde{P}$.

First, we shall identify the state-price density ξ such that (15) is consistent with (12). In other words, we calculate K and L such that (12) is the risk-free rate implied by the state-price density ξ in (13). By Ito's formula we know that the SDE for $\xi(t)$ under $\tilde{Q}$ is

$$\frac{d\xi(t)}{\xi(t)} = \mu_\xi(t)dt + (\sigma_Y(L+KY(t)))^\top dz^{\tilde{Q}}(t), \tag{16}$$

where

$$\begin{aligned}\mu_\xi(t) = {}& \frac{1}{2}(L+KY(t))^\top\sigma_Y^\top\sigma_Y(L+KY(t)) + \frac{1}{2}\mathrm{tr}\left(\sigma_Y K\sigma_Y^\top\right) \\ &+ (a_Y(b_Y - Y(t)))^\top(L+KY(t)) - \beta,\end{aligned} \tag{17}$$

which is a quadratic function of $Y(t)$. By Girsanov theorem for (14) and (16), the process

$$z^{\tilde{P}}(t) = z^{\tilde{Q}}(t) + \int_0^t \sigma_Y(L+KY(u))du$$

is a Brownian motion under $\tilde{P}$. Therefore, the state-price density obeys under $\tilde{P}$

$$\frac{d\xi(t)}{\xi(t)} = \left(\mu_\xi(t) - (\sigma_Y(L+KY(t)))^\top\sigma_Y(L+KY(t))\right)dt + (\sigma_Y(L+KY(t)))^\top dz^{\tilde{P}}(t).$$

Then the coefficient of the drift term of the state-price density under the reference objective measure $\tilde{P}$ should be equal to the minus of the risk-free rate times $\xi(t)$,

$$\mu_\xi(t) - (\sigma_Y(L + KY(t)))^\top \sigma_Y(L + KY(t)) = -R(Y(t)). \tag{18}$$

By replacing $Y(t)$ in (18) with $x = (r^x, (\lambda^x)^\top)^\top$, we have an equation for x from (17) and (18) as

$$\begin{aligned} 0 &= \frac{1}{2}(L + Kx)^\top \sigma_Y^\top \sigma_Y (L + Kx) + \frac{1}{2}\mathrm{tr}\left(\sigma_Y K \sigma_Y^\top\right) + (a_Y(b_Y - x))^\top (L + Kx) \\ &\quad - \beta - (\sigma_Y(L + Kx))^\top \sigma_Y(L + Kx) + \frac{\delta}{1-\gamma} - r^x - \frac{1}{2\gamma}(\lambda^x)^\top \lambda^x. \end{aligned} \tag{19}$$

If (19) holds for all x with some K, L, β, it follows that R given by (12) is actually a risk-free rate associated with ξ. Namely, matrices K, L and a constant β must satisfy

$$\begin{aligned} &K^\top \sigma_Y^\top \sigma_Y K + 2a_Y^\top K = -\frac{1}{\gamma}\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \\ &-L^\top \sigma_Y^\top \sigma_Y K - L^\top a_Y + b_Y^\top a_Y^\top K = (1, 0, \cdots, 0), \\ &-\frac{1}{2}L^\top \sigma_Y^\top \sigma_Y L + \frac{1}{2}\mathrm{tr}\left(\sigma_Y K \sigma_Y^\top\right) + b_Y^\top a_Y^\top L - \beta + \frac{\delta}{1-\gamma} = 0. \end{aligned} \tag{20}$$

In addition to the above conditions, we need to impose conditions

$$a_i + k_i \sigma_i^2 > 0 \quad \text{for all } i = 1, 2, \cdots, N \tag{21}$$

to make a matrix positive definite in the following analysis. By a tedious calculation it is found that the solution for (20) exists[1] and is given by

$$K = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & k_1 & 0 & \cdots & 0 \\ 0 & 0 & k_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k_N \end{pmatrix}, \tag{22}$$

$$k_i = \frac{-a_i + \frac{1-\gamma}{\gamma}\sigma_i + \sqrt{\left(a_i - \frac{1-\gamma}{\gamma}\sigma_i\right)^2 - \frac{\sigma_i^2}{\gamma}}}{\sigma_i^2}, \quad (i = 1, \cdots, N), \tag{23}$$

[1] Due to the quadratic form of (20) one can find another solution for k_i as

$$\frac{-a_i + \frac{1-\gamma}{\gamma}\sigma_i - \sqrt{\left(a_i - \frac{1-\gamma}{\gamma}\sigma_i\right)^2 - \frac{\sigma_i^2}{\gamma}}}{\sigma_i^2}.$$

However, this solution doesn't satisfy the condition (21). Thus, we disregard this solution.

$$
\begin{aligned}
L &= (l_r, l_1, \cdots, l_N)^\top, \\
& \quad l_r = -\frac{1}{a_r}, \quad l_1 = \frac{\frac{\sigma_r}{a_r}(-k_1\sigma_1 + \frac{1-\gamma}{\gamma}) + k_1 a_1 b_1}{\sqrt{\left(a_1 - \frac{1-\gamma}{\gamma}\sigma_1\right)^2 - \frac{\sigma_1^2}{\gamma}}}, \\
& \quad l_i = \frac{k_i a_i b_i}{\sqrt{\left(a_i - \frac{1-\gamma}{\gamma}\sigma_i\right)^2 - \frac{\sigma_i^2}{\gamma}}}, \quad (i = 2, \cdots, N), \\
\beta &= -\frac{1}{2}(\sigma_r l_r + \sigma_i l_i)^2 - \frac{1}{2}\sum_{i=2}^{N}(\sigma_i l_i)^2 + \frac{1}{2}\sum_{i=1}^{N} k_i \sigma_i^2 \\
& \quad + b_r l_r + \sum_{i=1}^{N} b_i l_i + \frac{\delta}{1-\gamma}.
\end{aligned} \tag{24}
$$

The k_i in (23) are real numbers due to the assumption (5) and satisfy $a_i + k_i\sigma_i^2 > 0$ for all i.

Once the state-price density price is identified, the SDE of the state vector $Y(t)$ under $\tilde{P}$ becomes

$$
dY(t) = \tilde{a}_Y(\tilde{b}_Y - Y(t))dt + \sigma_Y^\top dz^{\tilde{P}}(t),
$$

where $\tilde{a}_Y = a_Y + \sigma_Y^\top \sigma_Y K$ and $\tilde{a}_Y \tilde{b}_Y = a_Y b_Y - \sigma_Y^\top \sigma_Y L$. We observe that the matrix $\tilde{a}_Y$ is diagonalizable to $\Lambda = U^{-1}\tilde{a}_Y U$ by a matrix U, where

$$
\begin{aligned}
\Lambda &= \begin{pmatrix}
a_r & 0 & 0 & \cdots & 0 \\
0 & a_1 + k_1\sigma_1^2 & 0 & \cdots & 0 \\
0 & 0 & a_2 + k_2\sigma_2^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_N + k_N\sigma_N^2
\end{pmatrix}, \\
U &= \begin{pmatrix}
1 & -\varepsilon & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}, \quad \varepsilon = \frac{k_1\sigma_1\sigma_r}{a_1 + k_1\sigma_1^2 - a_r}.
\end{aligned}
$$

The denominator of ε is not zero due to the assumption (4).
Define $\tau = T - t$, $D(x) = (1 - e^{-x\tau})/x$, $\kappa_r = a_r$, $\kappa_i = a_i + k_i\sigma_i^2$, and

$$
\begin{aligned}
& \Phi_\tau = \exp(-\Lambda\tau), \\
& [v_{ij}]_{(N+1)\times(N+1)} = U^{-1}\sigma_Y^\top \sigma_Y \left(U^\top\right)^{-1}, \\
& M_\tau = [v_{ij} D(\kappa_i + \kappa_j)]_{(N+1)\times(N+1)}.
\end{aligned}
$$

Ahn et al. [1] show that the increment $Y(T) - Y(t)$ is normally distributed with the mean vector μ_τ and the covariance matrix V_τ such that

$$
\mu_\tau = U\Lambda^{-1}(I_{N+1} - \Phi_\tau)U^{-1}\tilde{a}_Y\tilde{b}_Y + U\Phi_\tau U^{-1}Y(t), \quad V_\tau = UM_\tau U^\top.
$$

It follows that the mean is $\mu_\tau = \mu_{0\tau} + \mu_{1\tau} Y(t)$, where $d(x) = e^{-\tau x}$,

$$\mu_{0\tau} = \begin{pmatrix} \phi \\ (1-d(\kappa_1))\left(b_1 - \frac{\sigma_r\sigma_1 l_r + \sigma_1^2 l_1}{a_1 + k_1\sigma_1^2}\right) \\ (1-d(\kappa_2))\left(b_2 - \frac{\sigma_2^2 l_2}{a_2 + k_2\sigma_2^2}\right) \\ \vdots \\ (1-d(\kappa_N))\left(b_N - \frac{\sigma_N^2 l_N}{a_N + k_N\sigma_N^2}\right) \end{pmatrix}, \tag{25}$$

$$\mu_{1\tau} = \begin{pmatrix} d(\kappa_r) & \varepsilon\,(d(\kappa_r) - d(\kappa_1)) & 0 & \cdots & 0 \\ 0 & d(\kappa_1) & 0 & \cdots & 0 \\ 0 & 0 & d(\kappa_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d(\kappa_N) \end{pmatrix}, \tag{26}$$

and

$$\begin{aligned} \phi &= (1-d(\kappa_r))\left(b_r - \frac{\sigma_r^2 l_r + \sigma_r\sigma_1 l_r}{a_r} - \frac{\sigma_r k_1 \sigma_1}{a_r}\left(b_1 - \frac{\sigma_r\sigma_1 l_r + \sigma_1^2 l_1}{a_1 + k_1\sigma_1^2}\right)\right) \\ &\quad + (1 - \eta\,(d(\kappa_r) - d(\kappa_1)))\left(b_1 - \frac{\sigma_r\sigma_1 l_r + \sigma_1^2 l_1}{a_1 + k_1\sigma_1^2}\right), \\ \eta &= \frac{a_r\varepsilon - k_1\sigma_1\sigma_r}{a_1 + k_1\sigma_1^2}. \end{aligned}$$

The covariance matrix V_τ is given by

$$V_\tau = \begin{pmatrix} m_r - 2\varepsilon m_{r1} + \varepsilon^2 m_1 & m_{r1} - \varepsilon m_1 & 0 & \cdots & 0 \\ m_{r1} - \varepsilon m_1 & m_1 & 0 & \cdots & 0 \\ 0 & 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_N \end{pmatrix},$$

where

$$\begin{aligned} m_r &= (\sigma_r + \varepsilon\sigma_1)^2 D(2a_r), \\ m_{r1} &= \sigma_1(\sigma_r + \varepsilon\sigma_1) D(a_r + a_1 + k_1\sigma_1^2), \\ m_i &= \sigma_i^2 D(2(a_i + k_i\sigma_i^2)), \quad (i = 1, 2, \cdots, N). \end{aligned}$$

Since $m_i > 0$ for all $i = r, 1, 2, \cdots, N$ and the function D satisfies $D(x)D(y) > D\left(\frac{x+y}{2}\right)^2$ for $x, y > 0$, it holds that

$$\det V_\tau = (m_r m_1 - m_{r1}^2) m_2 \cdots m_N > 0,$$

thanks to (5) and (21). Hence, all of the eigen values of the symmetric matrix V_τ are positive and V_τ is positive definite.

For the calculation of an expectation of an exponentially quadratic function of a normal distribution, the next lemma is useful.

Lemma 1. *If an n-dimensional random variable y is normally distributed with mean μ and the covariance matrix Σ, then it holds that*

$$E\left[\exp\left(\frac{1}{2}y^\top ay+b^\top y\right)\right]=\left(det\left(I_n-\Sigma\frac{a+a^\top}{2}\right)\right)^{-1/2}$$
$$\times\exp\left(\frac{1}{2}\left(\mu+\Sigma b\right)^\top\left(\Sigma-\Sigma\frac{a+a^\top}{2}\Sigma\right)^{-1}\left(\mu+\Sigma b\right)-\frac{1}{2}\mu^\top\Sigma^{-1}\mu\right)$$

for any $b\in\mathbb{R}^n, a\in\mathbb{R}^{n\times n}$ such that a symmetric matrix $\Sigma^{-1}-\frac{1}{2}(a+a^\top)$ is positive definite.

We need to confirm the positive definiteness of $V_\tau^{-1}-\frac{1}{2}(K+K^\top)=V_\tau^{-1}-K$ for the application of the above lemma. Since the matrix is given by

$$V_\tau^{-1}-K=\begin{pmatrix}\frac{m_1}{m_rm_1-m_{r1}^2} & -\frac{m_{r1}-\varepsilon m_1}{m_rm_1-m_{r1}^2} & 0 & \cdots & 0\\ -\frac{m_{r1}-\varepsilon m_1}{m_rm_1-m_{r1}^2} & \frac{m_r-2\varepsilon m_{r1}+\varepsilon^2m_1}{m_rm_1-m_{r1}^2}-k_1 & 0 & \cdots & 0\\ 0 & 0 & m_2^{-1}-k_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & m_N^{-1}-k_N\end{pmatrix},$$

we observe that the determinant is $\det\left(V_\tau^{-1}-K\right)=\frac{m_1}{m_rm_1-m_{r1}^2}\prod_{i=1}^N\frac{1-k_im_i}{m_i}$ and the trace of the first block diagonal matrix is positive

$$\mathrm{tr}\begin{pmatrix}\frac{m_1}{m_rm_1-m_{r1}^2} & -\frac{m_{r1}-\varepsilon m_1}{m_rm_1-m_{r1}^2}\\ -\frac{m_{r1}-\varepsilon m_1}{m_rm_1-m_{r1}^2} & \frac{m_r-2\varepsilon m_{r1}+\varepsilon^2m_1}{m_rm_1-m_{r1}^2}-k_1\end{pmatrix}=\frac{m_1\left(1+\left(\varepsilon-\frac{m_{r1}}{m_1}\right)^2\right)}{m_rm_1-m_{r1}^2}+\frac{1-k_1m_1}{m_1}>0.$$

Then for the positive definiteness of $V_\tau^{-1}-K$, it is enough to show that $1-k_im_i>0$ for all $i=1,2,\cdots,N$. When $k_i\le 0$, then $1-k_im_i\ge 1$ because of $1-k_im_i=1-k_i\sigma_i^2D(2(a_i+k_i\sigma_i^2))$ and (21). When $k_i>0$ it holds that

$$\begin{aligned}1-k_im_i &= \frac{2a_i+k_i\sigma_i^2}{2(a_i+k_i\sigma_i^2)}+\frac{k_i\sigma_i^2d(2(a_i+k_i\sigma_i^2))}{2(a_i+k_i\sigma_i^2)}\\ &> \frac{2a_i+k_i\sigma_i^2}{2(a_i+k_i\sigma_i^2)}\\ &= \frac{a_i+\frac{1-\gamma}{\gamma}\sigma_i+\sqrt{\left(a_i-\frac{1-\gamma}{\gamma}\sigma_i\right)^2-\frac{\sigma_i^2}{\gamma}}}{2(a_i+k_i\sigma_i^2)}\\ &> \frac{2\frac{1-\gamma}{\gamma}\sigma_i+\frac{\sqrt{1-\gamma+\gamma^2}}{\gamma}|\sigma_i|+\left|\frac{1-\gamma}{\gamma}\sigma_i\right|}{2(a_i+k_i\sigma_i^2)}\\ &> 0,\end{aligned}$$

by (5) and (21) regardless of the sign of σ_i. It follows that $V_\tau^{-1}-K$ is positive definite as desired. The determinant of $I_{N+1}-V_\tau K$ is given by $\det\left(I_{N+1}-V_\tau K\right)=(1-k_1m_1)\cdots(1-k_Nm_N)>0$.

With these preliminary calculations we can obtain the zero-coupon bond price (15) as

$$\begin{aligned}
E^{\tilde{P}}\left[\frac{\xi(T)}{\xi(t)}\,\middle|\,Y(t)=x\right] &= E^{\tilde{P}}\left[e^{-\beta(T-t)}\frac{\exp\left(\frac{1}{2}Y(T)^\top KY(T)+L^\top Y(T)\right)}{\exp\left(\frac{1}{2}Y(t)^\top KY(t)+L^\top Y(t)\right)}\,\middle|\,Y(t)=x\right] \\
&= \left(\det\left(I_{N+1}-V_\tau K\right)\right)^{-\frac{1}{2}}\exp\Big(-\beta\tau-\frac{1}{2}\mu_\tau^\top V_\tau^{-1}\mu_\tau \\
&\quad +\frac{1}{2}\left(\mu_\tau+V_\tau(L+Kx)\right)^\top H_\tau\left(\mu_\tau+V_\tau(L+Kx)\right)\Big),
\end{aligned}$$

with $H_\tau = V_\tau^{-1}\left(I_{N+1}-V_\tau K\right)^{-1}$. Namely, we reach to the solution of PDE as

$$\begin{aligned}
F(x,t) &= \left((1-k_1m_1)\cdots(1-k_Nm_N)\right)^{-1/2} \\
&\quad\times\exp\Big(\frac{1}{2}x^\top\left((V_\tau K+\mu_{1\tau})^\top H_\tau(V_\tau K+\mu_{1\tau})-\mu_{1\tau}^\top V_\tau^{-1}\mu_{1\tau}\right)x \\
&\quad+\frac{1}{2}\left((V_\tau L+\mu_{0\tau})^\top H_\tau(V_\tau K+\mu_{1\tau})-\mu_{0\tau}^\top V_\tau^{-1}\mu_{1\tau}\right)x \\
&\quad+\frac{1}{2}x^\top\left((V_\tau K+\mu_{1\tau})^\top H_\tau(V_\tau L+\mu_{0\tau})-\mu_{1\tau}^\top V_\tau^{-1}\mu_{0\tau}\right) \\
&\quad+\frac{1}{2}(V_\tau L+\mu_{0\tau})^\top H_\tau(V_\tau L+\mu_{0\tau})-\frac{1}{2}\mu_{0\tau}^\top V_\tau^{-1}\mu_{0\tau}-\beta\tau\Big).
\end{aligned} \tag{27}$$

Therefore, the part of the intertemporal hedging portfolio in (8) is given by

$$\frac{F_x(x,t)}{F(x,t)} = A_\tau(L+Kx)+B_\tau(\mu_{0\tau}+\mu_{1\tau}x), \tag{28}$$

where

$$\begin{aligned}
A_\tau &= (V_\tau K+\mu_{1\tau})^\top H_\tau V_\tau \\
&= \begin{pmatrix}
d(\kappa_r) & 0 & 0 & \cdots & 0 \\
A_{r1} & \frac{k_1m_1+d(\kappa_1)}{1-k_1m_1} & 0 & \cdots & 0 \\
0 & 0 & \frac{k_2m_2+d(\kappa_2)}{1-k_2m_2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{k_Nm_N+d(\kappa_N)}{1-k_Nm_N}
\end{pmatrix},
\end{aligned} \tag{29}$$

$$A_{r1} = \varepsilon\left(d(\kappa_r)-d(\kappa_1)\right)+k_1\frac{1+d(\kappa_1)}{1-k_1m_1}(m_{r1}-\varepsilon m_1),$$

$$\begin{aligned}
B_\tau &= (V_\tau K+\mu_{1\tau})^\top H_\tau-\mu_{1\tau}^\top V_\tau^{-1} \\
&= \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & k_1\frac{1+d(\kappa_1)}{1-k_1m_1} & 0 & \cdots & 0 \\
0 & 0 & k_2\frac{1+d(\kappa_2)}{1-k_2m_2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & k_N\frac{1+d(\kappa_N)}{1-k_Nm_N}
\end{pmatrix}.
\end{aligned} \tag{30}$$

Finally, we obtain the following result from (8).

Theorem 1. *The optimal portfolio is given by*

$$\alpha^*(t) = \frac{1}{\gamma}\sigma_S(t)^{-1}\lambda(t) + \frac{1-\gamma}{\gamma}\sigma_S(t)^{-1}\sigma_X\left[A_\tau(L+KX(t)) + B_\tau(\mu_{0\tau}+\mu_{1\tau}X(t))\right],$$

where $X(t) = (r(t), \lambda(t)^\top)^\top$, $\tau = T - t$, the matrices $K, L, A_\tau, B_\tau, \mu_{0\tau}, \mu_{1\tau}$ are given by (22), (24), (29), (30), (25) and (26), respectively.

Note that the optimal portfolio is linear in X and the intertemporal portfolio consists of two portfolios; one is a (non-zero) fixed portfolio

$$\frac{1-\gamma}{\gamma}\sigma_S(t)^{-1}\sigma_X\left[A_\tau L + B_\tau\mu_{0\tau}\right]$$

regardless of the state variables. Another one is a sensitive portfolio to the state variables

$$\frac{1-\gamma}{\gamma}\sigma_S(t)^{-1}\sigma_X\left[A_\tau K + B_\tau\mu_{1\tau}\right]X(t).$$

The investor adjusts the intertemporal hedging portfolio according to the state variables via the portfolio. However, a simple calculation shows

$$\left[A_\tau K + B_\tau\mu_{1\tau}\right]X(t) = \begin{pmatrix} 0 \\ \gamma_1\lambda_1(t) \\ \gamma_2\lambda_2(t) \\ \vdots \\ \gamma_N\lambda_N(t) \end{pmatrix},$$

where

$$\gamma_i = \frac{k_i(1+k_i m_i)}{1-k_i m_i}(1+d(\kappa_i)), \quad (i = 1, 2, \cdots, N),$$

which implies that the portfolio depends on the market prices of risk only and it is not sensitive to the risk-free rate. The seeming independence on the risk-free rate may be due to our setting (1).

4 Conclusion

We obtain the optimal portfolio with the Feynman-Kac formula and the exponentially quadratic state-price density for a different fictitious economy. The portfolio is linear in the state variables and is decomposed into three portfolios. The linearity will hold under more general settings though further investigations will be required. There are several directions of the extension of our analysis. The first one is to explore the cases of correlated prices among security prices, the risk-free rate and the market prices of risk. As the second one, the extension to more general affine type state variables will be fruitful. The third one is to apply for the filtering of the unobserved market prices of risk from the observed security prices. These agendas are left for future research.

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