Research Paper Series

No. 47

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March 2008

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(This Version: March 15, 2008)

Abstract. This paper shows that the one-factor Gaussian copula model, the standard market model for valuing CDO's, can be derived from the multivariate Wang transform, which is consistent with Bühlmann's equilibrium pricing model, whence it has a *sound* economic interpretation. The Gaussian model is then extended within the Bühlmann's framework to fit market prices of CDO tranches better. Unlike the existing modes, we calibrate the parameters associated with risk aversion index of investors, not the correlation parameter. Through numerical experiments, we show that our model provides a better fit to the market data compared with the existing models.

Keywords: Bühlmann's economic premium principle, multivariate Wang transform, One-factor Gaussian copula model, Structural model, Student t distribution

* A previous version of this paper was presented at the Third Bachelier Colloquium held at Metabief, France on January 7, 2008. The authors are grateful to participants of the workshop for helpful discussions. The first author is also grateful to the financial support by the Ministry of Education, Science, Sports and Culture (MEXT), Grand-in-Aid for Scientific Research (B) #18310104, 2006.

1 Introduction

The one-factor Gaussian copula model has become the standard market model for valuing collateralized debt obligations (CDO's) and other basket-type credit derivatives. The Gaussian copula approach is very convenient to model default time correlation given the marginal default probabilities, and allows the semi-analytical form[1] for the pricing and hedging of such products.

Despite the popularity of the model in practice, however, the copula approach is often criticized due to a couple of reasons.[2] Among them, it is claimed that the copula approach is difficult to interpret and the dependence structure is exogenously given without a theoretical justification. Also, it is well known that the model cannot explain the market prices of CDO tranches, i.e., it exhibits the so-called correlation smile.[3] As a result, it cannot price non-standard credit derivatives such as bespoke CDO's to be consistent with market quotes for tranches of standard CDO's.

The aim of this paper is twofold. First, contrary to the criticism, we show that the one-factor Gaussian copula model is consistent with Bühlmann's equilibrium pricing model (1980),[4] whence it has a *sound* economic interpretation. Second, the Gaussian model is extended within the Bühlmann's framework to fit market prices of CDO tranches better by taking the well-recognized facts in the credit derivatives market into consideration.

Namely, we introduce the risk aversion index for each tranche to be calibrated from the market prices of CDO tranches, while keeping the correlation structure as given under the actual probability measure,[5] since the CDO market is segmented into tranches according to

[1] See, e.g., Hull and White (2004) and Laurent and Gregory (2005) for details of the one-factor Gaussian copula model.

[2] Given these problems, Mortensen (2006) used a multivariate version of the intensity-based approach to develop a semi-analytical valuation method for CDO's. However, this approach involves many parameters to be estimated and/or calibrated and seems to be computationally difficult and instable to apply for actual markets, compared with the copula approach. See also Duffie and Garleanu (2001) for the intensity-based approach.

[3] If the Gaussian copula model fitted market prices well, the implied base correlation would be approximately constant across tranches. In order to overcome the deficiency, a number of researchers look for copulas that fit market prices better than the Gaussian copula. See, e.g., Hull and White (2006) and Burtschell, Gregory and Laurent (2007) for such extensions.

[4] Bühlmann's model (1980) has been developed for the pricing and hedging of insurance risk. Insurance market is *incomplete*, in the sense that risks in the market cannot be replicated by other assets in the market, and so is the CDO market.

[5] This is consistent with Girsanov's theorem for the Gaussian case that, when changing the measures from the actual measure to the risk-neutral measure, the mean is adjusted to represent the risk preference

investor's preference against risks. Also, we apply the Student t copula for the risk-adjusted model, because some empirical studies suggest to use t distributions with $\nu = 3$ to 7 degrees of freedom for return distributions of financial and insurance products.[6]

This paper is organized as follows. In the next section, we briefly describe the fact that the multivariate Wang transform is consistent with Bühlmann's economic premium principle (1980). It is then shown in Section 3 that the one-factor Gaussian copula model is derived from the multivariate Wang transform, whence it has a sound economic interpretation. Section 4 is devoted to propose an alternative to the standard Gaussian copula model within the Bühlmann's framework, which is further extended in Section 5 to accommodate the Student t copula. Numerical examples show that our model fit market quotes for tranches of standard SDO's better than the existing models in the literature. Section 6 concludes this paper.

Throughout this paper, we shall denote the actual probability measure by P and the risk-neutral probability measure by Q. The normal distribution with mean μ and variance σ^2 is denoted by $N(\mu, \sigma^2)$.

2 The Multivariate Wang Transform

In the actuarial literature, there have been developed many probability transforms for pricing financial and insurance risks. Recently, Wang (2000, 2002) proposed a pricing method based on the following transformation from $G(x)$ to $G^Q(x)$:

$$G^Q(x) = \Phi[\Phi^{-1}(G(x)) + \theta], \tag{2.1}$$

where Φ denotes the standard normal cumulative distribution function (CDF for short) and θ is a constant. The transform is now called the *Wang transform* and produces a risk-adjusted CDF $G^Q(x)$ under Q. The mean value evaluated under $G^Q(x)$ will define a risk-adjusted "fair value" of risk X with CDF $G(x)$ under P at some future time, which can be discounted to time zero using the risk-free interest rate. The parameter θ is considered to be a risk premium.

The Wang transform not only possesses various desirable properties as a pricing method, but also has a *sound* economic interpretation. For example, as Wang (2003) observed, the transform (2.1) is consistent with Bühlmann's economic premium principle.

of investors, but the variance-covariance structure is not changed. See, e.g., Kijima (2002) for details.

[6] See, e.g., Platen and Stahl (2003) and Wang (2004) for such empirical studies.

More precisely, Bühlmann (1980) considered risk exchanges among a set of agents. Each agent is characterized by his/her exponential utility function $u_i(x) = -\mathrm{e}^{-\lambda_i x}$, $i = 1, 2, \ldots, n$, and faces a risk of potential loss X_i. In a pure risk exchange model, Bühlmann (1980) derived the following equilibrium pricing formula for risk X:

$$\pi(X) = E[\eta X], \qquad \eta = \frac{\mathrm{e}^{-\lambda Z}}{E[\mathrm{e}^{-\lambda Z}]}, \tag{2.2}$$

where E denotes the expectation operator under P, $Z = \sum_{i=1}^{n} X_i$ is the aggregate risk, and λ is given by

$$\lambda^{-1} = \sum_{i=1}^{n} \lambda_i^{-1}, \qquad \lambda_i > 0.$$

The parameter λ is thought of the *risk aversion index* of the representative agent in the market.

Unfortunately, however, the actuarial pricing functional is not linear, whence admits an arbitrage opportunity.[7] In order to develop a linear pricing method while maintaining probability distortions, Kijima (2006) derived a multivariate version of the Wang transform (2.1) from the Bühlmann's equilibrium pricing formula (2.2).

Suppose that the underlying risks are described by an n-dimensional random vector, $(X_1, X_2, \ldots, X_n)$ say. Suppose further that the underlying risks are formulated by a Gaussian copula under P. That is, define

$$Z_i \equiv \Phi^{-1}[G_i(X_i)], \qquad i = 1, 2, \ldots, n,$$

where $G_i(x)$ is the marginal CDF of X_i.[8] A Gaussian copula assumes that $(Z_1, Z_2, \ldots, Z_n)$ follows an n-variate standard normal distribution with correlation matrix $\mathbf{\Sigma} = (\rho_{ij})$.

Now, suppose that the aggregate risk Z consists of many individual risks X_i so that it can be approximated by a normal distribution. Let $Z_0 = (Z - \mu_Z)/\sigma_Z$ be the standardized normal random variable, where $\mu_Z = E[Z]$ and $\sigma_Z^2 = V[Z]$ denote the mean and the variance of Z under P, respectively. Next, suppose that the standard normal variable Z_0 is related to the Gaussian copula as

$$Z_0 \overset{\mathrm{d}}{=} \xi + \sum_{i=1}^{n} w_i Z_i, \qquad Z_i = \Phi^{-1}[G_i(X_i)], \tag{2.3}$$

[7] See, e.g., Harrison and Kreps (1979) for details. The pricing functional π is said to be linear if $\pi(aX + bY) = a\pi(X) + b\pi(Y)$ for all risks X, Y and constants a, b.

[8] Throughout the paper, it is assumed for the sake of simplicity that all the CDF's under consideration are strictly increasing.

for some constants w_i and a random variable ξ, which is independent of Z_i, where $\stackrel{d}{=}$ stands for equality in law. Using the assumption (2.3), Kijima (2006) obtained the following multivariate transformation from the Bühlmann's equilibrium pricing formula (2.2):

$$G^Q(\boldsymbol{x}) = \Phi_{n:\Sigma}\left(\Phi^{-1}[G_1(x_1)] + \sum_{i=1}^{n} \lambda_i \rho_{1i}, \ldots, \Phi^{-1}[G_n(x_n)] + \sum_{i=1}^{n} \lambda_i \rho_{ni}\right), \tag{2.4}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\Phi_{n:\Sigma}$ denotes the CDF of the n-variate standard normal distribution with correlation matrix $\boldsymbol{\Sigma}$. Note that, when $n = 1$, (2.4) coincides with the Wang transform (2.1) since $\rho_{11} = 1$.

In particular, when $(X_1, X_2, \ldots, X_n)$ is normally distributed with correlation matrix $\boldsymbol{\Sigma} = (\rho_{ij})$ under P, we have $G_i(x) = \Phi((x - \mu_i)/\sigma_i)$, where $\mu_i = E[X_i]$ and $\sigma_i^2 = V[X_i]$. It follows that the multivariate Wang transform (2.4) becomes

$$G^Q(\boldsymbol{x}) = \Phi_{n:\Sigma}\left(\frac{x_1 - \mu_1}{\sigma_1} + \lambda C_1, \ldots, \frac{x_n - \mu_n}{\sigma_n} + \lambda C_n\right), \qquad C_i \equiv Cov(X_i, Z), \tag{2.5}$$

since

$$\sum_{j=1}^{n} \lambda_j \rho_{ji} = \lambda Cov(X_i, Z).$$

Here, $Cov(X, Z)$ is the covariance between X and Z, λ is the risk aversion index, and Z denotes the aggregated market risk. We note that, in the normal case, the transform (2.5) can be derived directly from the Bühlmann's formula (2.2) without the assumption (2.3).[9]

3 The One-Factor Gaussian Copula Model

Consider an asset pool consisting of n defaultable assets whose default epochs are denoted by τ_i, $i = 1, 2, \ldots, n$. Let $N_i(t) = 1_{\{\tau_i \le t\}}$ be the default indicator process and M_i the loss given default of name i. Then, the time-t cumulative loss of the asset pool is defined by

$$L(t) = \sum_{i=1}^{n} M_i N_i(t), \qquad 0 \le t \le T, \tag{3.1}$$

where T denotes the maturity of the CDO.

3.1 The multivariate Merton model

In order to model the joint distribution of $(\tau_1, \tau_2, \ldots, \tau_n)$, we follow the structural model of Merton (1974). That is, consider the firm value V_i of name i, and assume that default occurs

[9] Kijima (2006) also developed a multivariate version of the Esscher transform and showed that the multivariate Wang transform (2.5) agrees with the Esscher counterpart for the normal case.

before time t, i.e. $\tau_i \leq t$, if and only if the firm value V_i is less than some default threshold. In other words, denoting $X_i = \log V_i$, we assume that

$$\{\tau_i \leq t\} = \{X_i < x\} \tag{3.2}$$

for some x. It is easier to model the joint distribution of $(X_1, X_2, \ldots, X_n)$ rather than to model that of $(\tau_1, \tau_2, \ldots, \tau_n)$ directly. This is the basic idea adopted by CreditMetrics (1997) to evaluate the portfolio credit risk.

CreditMetrics (1997) assumes that $(X_1, X_2, \ldots, X_n)$ follows an n-variate standard normal distribution with correlation matrix $\mathbf{\Sigma} = (\rho_{ij})$. However, it is computationally very time consuming to obtain the distribution of the cumulative loss $L(t)$ given by (3.1) for the general correlation case. Hence, CreditMetrics (1997) uses the Monte Carlo simulation to approximate the distribution of $L(t)$ given the correlation structure. Note that CreditMetrics (1997) works under the actual measure P, not under the risk-neutral probability measure Q, for the purpose of credit risk management.[10]

The industry convention to model the joint distribution of $(X_1, X_2, \ldots, X_n)$ is to employ the one-factor model.[11] More specifically, let U and $U_i, i = 1, 2, \ldots, n$, be independent and follow the standard normal distribution $N(0, 1)$, and suppose

$$X_i = \rho_i U + \sqrt{1 - \rho_i^2}\, U_i, \qquad i = 1, 2, \ldots, n, \tag{3.3}$$

where $-1 < \rho_i < 1$ are constants. It is readily seen that X_i also follow $N(0, 1)$ with correlation $\rho_{ij} = \rho_i \rho_j$, $i \neq j$. In the following, we denote the correlation matrix for this special case by $\mathbf{\Sigma}_\rho$. Also, from (3.2), we assume that

$$F_i(t) = \Phi(x) \quad \Longleftrightarrow \quad x = \Phi^{-1}(F_i(t)), \qquad i = 1, 2, \ldots, n, \tag{3.4}$$

where $F_i(t)$ denotes the *marginal* CDF of τ_i.

Given the common factor U in (3.3), it is readily shown that the conditional default probability of name i is given by

$$q_i(t|U) \equiv P\{\tau_i \leq t|U\} = P\{X_i \leq x|U\} = \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho_i U}{\sqrt{1 - \rho_i^2}}\right). \tag{3.5}$$

Since τ_i are conditionally independent, the *joint* CDF of $(\tau_1, \tau_2, \ldots, \tau_n)$, denoted by $F(\boldsymbol{t})$, $\boldsymbol{t} = (t_1, \ldots, t_n)$, under the actual probability measure P is obtained as

$$F(\boldsymbol{t}) = \int_{-\infty}^{\infty} \left[\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i u}{\sqrt{1 - \rho_i^2}}\right)\right] \phi(u) \mathrm{d}u, \tag{3.6}$$

[10] While the actual measure is used for risk management, the risk-neutral measure is needed only for the pricing of financial and insurance products. See Kijima and Muromachi (2000) for details.

[11] An extension of the model to the multi-factor case is straightforward.

where $\phi(u)$ is the probability density function (PDF for short) of the standard normal distribution $N(0, 1)$.

3.2 Change of measures from P to Q

In this subsection, we apply the change of measure formula (2.5) to obtain the distribution of the cumulative loss $L(t)$ under the risk-neutral probability measure Q. Recall that, in Girsanov's theorem for the Gaussian case, the variance-covariance structure is not changed, but the mean is adjusted to represent the risk preference of investors under the change of measures from P to Q.

Consider the multivariate risks $(X_1, X_2, \ldots, X_n)$, each X_i being defined by (3.3). We denote the joint CDF of $(\tau_1, \tau_2, \ldots, \tau_n)$ under the risk-neutral probability measure Q by $F^Q(\boldsymbol{t})$, $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$. Since X_i follows $N(0, 1)$, we have from (2.5) that

$$G^Q(\boldsymbol{x}) = \Phi_{n:\Sigma_\rho}(x_1 + \lambda C_1, \ldots, x_n + \lambda C_n), \tag{3.7}$$

where $\boldsymbol{\Sigma}_\rho = (\rho_{ij})$ with $\rho_{ij} = \rho_i \rho_j$, $G^Q(\boldsymbol{x})$ is the joint CDF of $(X_1^*, X_2^*, \ldots, X_n^*)$, the transformed risks under Q, and

$$C_i \equiv Cov(X_i, Z) = 1 - \rho_i^2 + \rho_i \sum_{j=1}^{n} \rho_j.$$

Note that $(X_1^*, X_2^*, \ldots, X_n^*)$ also follows an n-variate normal distribution with means $-\lambda C_i$ and the same correlation matrix $\boldsymbol{\Sigma}_\rho$. Hence, we conclude that

$$(X_1^*, X_2^*, \ldots, X_n^*) \stackrel{\mathrm{d}}{=} (X_1 - \lambda C_1, X_2 - \lambda C_2, \ldots, X_n - \lambda C_n), \tag{3.8}$$

where X_i are given by (3.3).

Now, we have from (3.2) and (3.8) that

$$F_i^Q(t) = Q\{\tau_i \leq t\} = Q\{X_i^* \leq x\} = \Phi(x + \lambda C_i), \qquad i = 1, 2, \ldots, n, \tag{3.9}$$

or, equivalently,

$$x = \Phi^{-1}(F_i^Q(t)) - \lambda C_i, \qquad i = 1, 2, \ldots, n. \tag{3.10}$$

It follows from (3.7) and (3.8) that the joint CDF $F^Q(\boldsymbol{t})$ of $(\tau_1, \tau_2, \ldots, \tau_n)$ under the risk-neutral measure Q is given by

$$F^Q(\boldsymbol{t}) = \Phi_{n:\Sigma_\rho}\left(\Phi^{-1}(F_1^Q(t_1)), \ldots, \Phi^{-1}(F_n^Q(t_n))\right). \tag{3.11}$$

But, given the common factor U in (3.3), the conditional default probability of name i under Q is obtained as

$$q_i^Q(t|U) \equiv Q\{\tau_i \le t|U\} = Q\{X_i^* \le x|U\} = \Phi\left(\frac{\Phi^{-1}(F_i^Q(t)) - \rho_i U}{\sqrt{1-\rho_i^2}}\right), \tag{3.12}$$

where we have used (3.10) in the second equality. The joint CDF $F^Q(\boldsymbol{t})$ under Q is then given by

$$F^Q(\boldsymbol{t}) = \int_{-\infty}^{\infty} \left[\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(F_i^Q(t_i)) - \rho_i u}{\sqrt{1-\rho_i^2}}\right)\right] \phi(u)\mathrm{d}u. \tag{3.13}$$

Hence, we recover the one-factor Gaussian copula model, the standard market model for the pricing of CDO's, first developed by Li (2000). Note that, in the joint CDF $F^Q(\boldsymbol{t})$, the risk aversion index λ is embedded in the marginal default CDF $F_i^Q(t)$ given by (3.9), and the joint CDF $F^Q(\boldsymbol{t})$ under Q is similar to that under P; cf. (3.6). The CDF $F_i^Q(t)$ can be calibrated from the market quotes for, e.g., credit default swaps (CDS's).

The distribution of the cumulative loss $L(t)$ under Q (and also under P) can be obtained relatively easily. Let $\varphi(s|t)$ be the characteristic function of $L(t)$. Since τ_i are conditionally independent given the common factor U, we obtain

$$\varphi(s|t) \equiv E^Q\left[\mathrm{e}^{isL(t)}\right] = E^Q\left[E^Q\left[\mathrm{e}^{isL(t)}\middle|U\right]\right] = E^Q\left[\prod_{i=1}^{n}\varphi_i(s|t,U)\right],$$

where E^Q denotes the expectation operator under Q and the conditional characteristic function $\varphi_i(s|t,U)$ is given by

$$\varphi_i(s|t,U) \equiv E^Q\left[\mathrm{e}^{isM_iN_i(t)}\middle|U\right] = q_i^Q(t|U)\mathrm{e}^{isM_i} + (1-q_i^Q(t|U)).$$

Here, $q_i^Q(t|U)$ is the conditional default probability given by (3.12). The distribution of $L(t)$ can then be numerically inverted back from $\varphi(s|t)$ using, e.g., the fast Fourier transform.[12]

4 A Risk-Adjusted Gaussian Copula Model

We have seen that the one-factor Gaussian copula model (3.13) has a sound economic interpretation within the Bühlmann's equilibrium pricing framework. However, it is also very well known that the model cannot explain the market prices of CDO tranches. In this section, we modify the standard model (3.13) to include risk aversion indices for tranches of CDO's. The risk aversion indices are calibrated from market quotes for tranches of standard CDO's.

[12] Alternatively, we can apply the bucketing method developed by Hull and White (2004).

4.1 Risk adjustment

It is often said that the CDO market is segmented into tranches according to investor's preference against risks. Hence, it is natural to assume that the risk aversion index differs over tranches. More precisely, for a tranche with detachment point D, which we call tranche D for short, we assume that the risk aversion index is given by $\lambda + \lambda_D$ and the transformed risks X_i^* in (3.8) are rewritten as

$$(X_1^*, X_2^*, \ldots, X_n^*) \stackrel{\mathrm{d}}{=} (X_1 - (\lambda + \lambda_D)C_1, X_2 - (\lambda + \lambda_D)C_2, \ldots, X_n - (\lambda + \lambda_D)C_n), \quad (4.1)$$

when evaluating the tranche D. The parameter λ_D can be negative, because it represents the risk adjustment for tranche D.

Recall that the CDF's $F_i^Q(t)$ are calibrated from market quotes for CDS's. In other words, the default boundary x remains the same as in (3.10). It follows from (3.11) that the joint CDF $F^Q(\boldsymbol{t})$ of default times τ_i under Q is given by

$$F^Q(\boldsymbol{t}) = \Phi_{n:\Sigma_\rho}\left(\Phi^{-1}(F_1^Q(t_1)) + \lambda_D C_1, \ldots, \Phi^{-1}(F_n^Q(t_n)) + \lambda_D C_n\right), \quad (4.2)$$

where λ_D is the risk adjustment parameter for tranche D.

4.2 Parameter estimation and calibration

Our model (4.2) involves parameters λ_D, ρ_i and the marginal CDF's $F_i^Q(t)$, $i = 1, 2, \ldots, n$ to be estimated or calibrated from the market data.

Calibration is necessary for parameters that reflect the risk attitude of investors. In our model, this role is taken by the risk adjustment parameters λ_D and the CDF's $F_i^Q(t)$ of default times τ_i under Q. Recall that the CDF's $F_i^Q(t)$ can be calibrated from market quotes for CDS's.

On the other hand, since we start from Merton's structural model with the assumption that

$$X_i = \log V_i = \rho_i U + \sqrt{1 - \rho_i^2}\, U_i,$$

it is natural to estimate the parameters ρ_i using the conventional regression with the (standardized) stock return X_i, if name i issues a stock, to the (standardized) return of the market index.

Given the CDF's $F_i^Q(t)$ of default times and the correlation parameters ρ_i, the risk adjustment parameters λ_D are calibrated from the market quotes for tranches of standard CDO's. Non-standard credit derivatives such as bespoke CDO's are then priced using, e.g., appropriate interpolation of the risk adjustment parameters.

We note that, for the standard market models including the one-factor Gaussian copula model, default correlation among the underlying names is the only unobservable element and, as a result, the correlation plays a role of risk preference of investors. The industry convention for this purpose is to assume that $\rho_i = \rho$ for all i, and the parameter ρ implied by the market prices is used as in much the same way as implied volatilities for the Black–Scholes model. In contrast, our model involves risk adjustment parameters λ_D of the representative agent in the market and the correlation structure remains the same after the change of measures.

4.3 A numerical example

In this numerical example, we consider 5 data sets consisting of market quotes on synthetic CDO's and underlying CDS's for the most liquid 5 year maturity. The CDO quotes are available on the five benchmark tranches trading on the Dow Jones iTraxx index, consisting of 125 European investment grade companies, with 0-3%, 3-6%, 6-9%, 9-12% and 12-22% tranches, and iTraxx Japan index, consisting of 80 Japanese investment grade companies, with the same tranches. All quotes are obtained from Bloomberg. The market quotes for the tranches are listed in Table 1.

The calibrated risk adjustment parameters λ_D are also listed in Table 1. It is interesting to see that, while λ_D are all positive on July 30, 2007 in the iTraxx Japan index, they are all negative on December 14, 2007. This is so, because the CDO spreads were relatively wide compared to the CDS spreads on July 30, 2007, while the CDS spreads became wider on December 14, 2007. Recall that the parameter λ represents the risk aversion index for CDS in our Gaussian copula model, and $\lambda + \lambda_D$ does for CDO tranche D.

5 A Risk-Adjusted Model with t Copula

The risk-adjusted Gaussian copula model (4.2) has an apparent advantage to the existing models, because it can perfectly fit market quotes for all tranches of standard CDO's by calibrating the risk adjustment parameters λ_D. A non-standard basket credit derivative such as a bespoke CDO can then be priced by using an appropriate risk adjustment parameter. However, the model may be difficult to interpret economically, because we have considered $\lambda + \lambda_D$ as the risk aversion index for tranche D of the representative agent in the market. Hence, it makes sense economically to assume that the parameter λ_D is increasing in D, because more risk averse investors will invest higher tranches. For example, a plausible

Table 1: Market and model prices for CDO tranches

Index	Tranches	0-3%	3-6%	6-9%	9-12%	12-22%
DJ iTraxx 8/23/2004	Mid Price	25.5%	146.0	60.3	36.3	19.3
	Bid/Ask Spread	1.3%	10.0	5.5	5.5	3.5
	Calibrated λ_D	0.0012	-0.0009	-0.0013	-0.0014	-0.0016
DJ iTraxx 12/05/2005	Mid Price	26.3%	80.6	23.1	10.3	5.8
	Bid/Ask Spread	0.6%	3.3	2.6	2.0	1.3
	Calibrated λ_D	0.0031	-0.0002	-0.0012	-0.0016	-0.0020
iTraxx Japan 7/05/2007	Mid Price	12.2%	40.0	11.0	6.5	3.5
	Bid/Ask Spread	1.5%	10.0	4.0	5.0	3.0
	Calibrated λ_D	0.0023	-0.0003	-0.0011	-0.0014	-0.0014
iTraxx Japan 7/30/2007	Mid Price	32.0%	164.0	51.0	31.0	19.8
	Bid/Ask Spread	2.0%	20.0	10.0	10.0	2.0
	Calibrated λ_D	0.0066	0.0032	0.0019	0.0015	0.0016
iTraxx Japan 12/14/2007	Mid Price	24.5%	192.5	62.5	30.5	16.5
	Bid/Ask Spread	3.5%	45.0	29.0	15.0	9.0
	Calibrated λ_D	-0.0042	-0.0043	-0.0046	-0.0047	-0.0042

The market prices were obtained from Bloomberg. The risk adjustment parameters λ_D are calibrated from the market quotes based on the joint CDF given by (4.2). Interest rates are constant at 3%, and the recovery rate is 40%.

choice for λ_D will be

$$\lambda_D = a + b \log D \tag{5.1}$$

and the parameters a and b are to be calibrated. In this section, we accommodate the Student t copula to the risk-adjusted model (4.2) to overcome this deficiency.

5.1 Fat-tail distribution

It is often said that a drawback of the Wang transform (and its extension to the multivariate setting) is the normal CDF $\Phi(x)$ appeared in (2.1) ($\Phi_{n:\Sigma}(\boldsymbol{x})$ in (2.4), respectively), that never matches the fat-tailness observed in the actual markets. In fact, some empirical studies suggest to use t distributions, whose CDF is denoted by $t_\nu(x)$, with $\nu = 3$ to 7 degrees of freedom for return distributions of financial and insurance assets (see, e.g., Platen and Stahl (2003) and Wang (2004)). Hence, in order to overcome this deficiency, Wang (2002) proposed the following two-parameter transformation:

$$G^Q(x) = t_\nu[\Phi^{-1}(G(x)) + \theta], \qquad \theta > 0, \tag{5.2}$$

and reported that (5.2) is much better to fit, although the two-parameter transform is not consistent with the economic premium principle (2.2).

In our multivariate setting, we also adopt this idea to the joint CDF (4.2) and propose the following multivariate extension of the two-parameter Wang transform:[13]

$$F^Q(\boldsymbol{t}) = t_{n:\nu,\Sigma_\rho}\left(\Phi^{-1}(F_1^Q(t_1)) + \lambda_D C_1,\, \ldots,\, \Phi^{-1}(F_n^Q(t_n)) + \lambda_D C_n\right), \tag{5.3}$$

where $t_{n:\nu,\Sigma_\rho}(\boldsymbol{x})$ denotes the CDF of the n-variate standard t distribution with ν degrees of freedom and correlation matrix $\boldsymbol{\Sigma}_\rho$ for the underlying standard normal random variables.

Remark 5.1 Hull and White (2004) suggested to use the Student t distribution for U and U_i in (3.3) under the risk-neutral measure Q, and reported that the model fits to the market data very well. The good performance of the model is reported in other papers as well. See, e.g., Burtschell, Gregory and Laurent (2005) for details. In this model, however, explicit form for the CDF of X_i is not known.

[13] Alternatively, we can apply this idea to the joint CDF (3.7). However, the resulting joint CDF

$$F^Q(\boldsymbol{t}) = t_{n:\nu,\Sigma_\rho}\left(t_\nu^{-1}(F_1^Q(t_1)) + \lambda_D C_1,\, \ldots,\, t_\nu^{-1}(F_n^Q(t_n)) + \lambda_D C_n\right)$$

does not have a fatter tail than the Gaussian counterpart. In fact, Kijima and Muromachi (2008) found from numerical experiments that the risk-adjusted distribution $F^Q(t)$ has a fatter tail when the inside distribution is less fat-tailed. That is, the two-parameter transformation (5.3) is justified for practical use when the risk-adjusted distribution is fat-tailed.

The joint CDF (5.3) can be expressed as follows. Let the default boundary x be given by (3.10), and define the transformed risks X_i^* as

$$X_i^* = \frac{X_i}{Y(\nu)} - (\lambda + \lambda_D)C_i, \qquad Y(\nu) = \sqrt{\chi_\nu^2/\nu}, \tag{5.4}$$

where X_i are given by (3.3) and χ_ν^2 denotes a random variable that follows the chi-square distribution with ν degrees of freedom, independent of other random variables. It is readily checked that the joint CDF (5.3) of default times τ_i is given by

$$F^Q(\boldsymbol{t}) = Q\{X_1^* \le x_1, X_2^* \le x_2, \ldots, X_n^* \le x_n\}.$$

Now, given the common factor U in (3.3) and $Y(\nu)$ in (5.4), the conditional default probability of name i under Q is obtained as

$$\begin{aligned} q_i^Q(t|U,Y(\nu)) &\equiv Q\{\tau_i \le t|U,Y(\nu)\} = Q\{X_i^* \le x|U,Y(\nu)\} \\ &= Q\left\{\rho_i U + \sqrt{1-\rho_i^2}U_i \le \left(\Phi^{-1}(F_i^Q(t)) + \lambda_D C_i\right) Y(\nu) \Big| U, Y(\nu)\right\} \\ &= \Phi\left(\frac{(\Phi^{-1}(F_i^Q(t)) + \lambda_D C_i)Y(\nu) - \rho_i U}{\sqrt{1-\rho_i^2}}\right). \end{aligned} \tag{5.5}$$

The joint CDF $F^Q(\boldsymbol{t})$ under the risk-neutral measure Q is then obtained as

$$F^Q(\boldsymbol{t}) = \int_{-\infty}^{\infty}\int_0^{\infty}\left[\prod_{i=1}^n \Phi\left(\frac{(\Phi^{-1}(F_i^Q(t_i)) + \lambda_D C_i)y - \rho_i u}{\sqrt{1-\rho_i^2}}\right)\right]\chi_\nu(y)\mathrm{d}y\phi(u)\mathrm{d}u, \tag{5.6}$$

where $\chi_\nu(y)$ is the PDF of $Y(\nu)$.

5.2 Single integral formula

In order to avoid the double integral in (5.6),[14] we employ the following approximation instead of (5.4):

$$X_i^* = \frac{X_i}{Y_i(\nu)} - (\lambda + \lambda_D)C_i, \qquad Y_i(\nu) = \sqrt{\chi_{i:\nu}^2/\nu}, \tag{5.7}$$

where $\chi_{i:\nu}^2$ are independent, identically distributed chi-square random variables with ν degrees of freedom. Then, we can consider the conditional default probability (5.5) given the common factor U only.

More specifically, we define random variables

$$\xi_i(u) = \frac{U_i + \delta_i(u)}{Y_i(\nu)}, \qquad \delta_i(u) = \frac{\rho_i u}{\sqrt{1-\rho_i^2}}, \qquad i = 1, 2, \ldots, n. \tag{5.8}$$

[14] The double integral is computationally time consuming when the parameters are calibrated to the market data. We aim to derive a simple model for practical use.

It is well known that $\xi_i(u)$ follows a non-central t distribution with ν degrees of freedom and non-centrality parameter $\delta_i(u)$. In the following, we denote the CDF of $\xi_i(u)$ by $P_{\nu:\delta(u)}(x)$.

Now, given the common factor U in (3.3), we have from (5.7) and (5.8) that

$$\{X_i^* < x\} = \left\{ \xi_i(U) < \frac{x + (\lambda + \lambda_D)C_i}{\sqrt{1-\rho_i^2}} \right\},$$

where x is defined by (3.10). It follows that, given the common factor U, the conditional default probability of name i under Q is obtained as

$$\begin{aligned} q_i^Q(t|U) &\equiv Q\{\tau_i \le t|U\} = Q\{X_i^* < x|U\} \\ &= Q\left\{ \xi_i(U) < \frac{\Phi^{-1}(F_i^Q(t)) + \lambda_D C_i}{\sqrt{1-\rho_i^2}} \Big| U \right\} \\ &= P_{\nu:\delta_i(U)}\left(\frac{\Phi^{-1}(F_i^Q(t)) + \lambda_D C_i}{\sqrt{1-\rho_i^2}} \right). \end{aligned} \tag{5.9}$$

The joint CDF $F^Q(\boldsymbol{t})$ of default times τ_i under the risk-neutral measure Q is then obtained as

$$\begin{aligned} F^Q(\boldsymbol{t}) &= \int_{-\infty}^{\infty} \left[\prod_{i=1}^n P_{\nu:\delta_i(u)}\left(\frac{\Phi^{-1}(F_i^Q(t_i)) + \lambda_D C_i}{\sqrt{1-\rho_i^2}} \right) \right] \phi(u) \mathrm{d}u, \\ \delta_i(u) &= \frac{\rho_i u}{\sqrt{1-\rho_i^2}}. \end{aligned} \tag{5.10}$$

The distribution of the cumulative loss $L(t)$ under Q can be obtained by using, e.g., the bucketing algorithm of Hull and White (2004).[15]

5.3 Empirical application

The model (5.10) involves new parameters a and b in (5.1) and ν, the degree of freedom in the Student t distribution. Given the marginal CDF's $F_i^Q(t)$ and the correlation parameters ρ_i, these parameters are calibrated from market quotes for CDO tranches by minimizing the following root mean square price errors (RMSE) relative to bid/ask spreads:

$$\text{RMSE} = \sqrt{\frac{1}{5}\sum_{j=1}^{5}\left(\frac{S_{j:\text{mid}} - S_j}{S_{j:\text{ask}} - S_{j:\text{bid}}} \right)^2},$$

where S_j is the spread of tranche j calculated by the model, $S_{j:\text{mid}}$ the market mid-spread, $S_{j:\text{ask}}$ the market ask spread, and $S_{j:\text{bid}}$ the market bid spread; see Mortensen (2006) for details.

[15] The CDF $P_{\nu:\delta_i}(x)$ can be evaluated easily with enough accuracy using the algorithm developed by Lenth (1989). See Kijima and Muromachi (2008) for details.

The calibrated results are shown in Tables 2 and 3, where the calibration results for other models are taken from Mortensen (2006). Our risk-adjusted t copula model fits the market prices very well, or at least comparable with the existing models. It is interesting to note that the mid price in equity tranche on December 5, 2005 is higher than than on August 23, 2004, while the mid prices in higher tranches on December 5, 2005 are smaller than those on August 23, 2004. In our model, this tranche structure of mid prices is captured by the degree of freedom ν of t distribution as well as the slpe b of the risk adjustment function λ_D in (5.1). More specifically, the slope b on December 5, 2005 is steeper than that on August 23, 2004, whereas the degree of freedom ν on December 5 is bigger than that on August 23, 2004.

6 Conclusion

This paper showed that, contrary to the criticism, the one-factor Gaussian copula model is consistent with Bühlmann's equilibrium pricing model, whence it has a *sound* economic interpretation. Based on this finding, we then develop an alternative within the Bühlmann's framework to the Gaussian copula to fit market quotes for tranches of standard CDO's better by taking the well-recognized facts in the credit derivatives market into consideration.

Namely, in our model, we introduce the risk aversion index for each tranche to be calibrated from the market quotes for CDO tranches, while keeping the correlation structure as given under the actual probability measure, since the CDO market is segmented into tranches according to investor's preference against risks. We also apply the Student t copula for the multivariate Wang transform, because some empirical studies suggest to use t distributions with $\nu = 3$ to 7 degrees of freedom for return distributions of financial and insurance assets. Numerical experiments reveal that our model provide a better fit than the existing models in the literature.

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Table 2: Comparison of calibration results for DJ iTraxx index on August 23, 2004

Tranches	0-3%	3-6%	6-9%	9-12%	12-22%	RMSE
Mid Price	25.5%	146.0	60.3	36.3	19.3	
Bid/Ask Spread	1.3%	10.0	5.5	5.5	3.5	
Jump-diffusion intensities	25.0%	145.1	58.6	38.1	17.7	0.34
Pure diffusion intensities	30.0%	187.1	27.4	3.5	0.1	5.11
Gaussian copula	27.4%	222.3	52.5	13.8	1.6	4.58
EFL Gaussian copula	25.3%	148.9	52.4	43.4	17.9	0.90
Double t copula	24.0%	153.4	56.5	32.4	16.4	0.84
Risk adjusted t copula	25.8%	145.2	49.1	28.4	17.1	1.15

Calibrated parameters: $\nu = 3$ $\lambda_D = 0.0045 \times \log D - 0.0372$

Table 3: Comparison of calibration results for DJ iTraxx index on December 5, 2005

Tranches	0-3%	3-6%	6-9%	9-12%	12-22%	RMSE
Mid Price	26.3%	80.6	23.1	10.3	5.8	
Bid/Ask Spread	0.6%	3.3	2.6	2.0	1.3	
Jump-diffusion intensities	28.7%	86.3	18.7	14.4	10.4	2.88
Pure diffusion intensities	32.5%	104.3	8.9	0.8	0.0	6.99
Gaussian copula	34.6%	99.9	2.9	0.1	0.0	8.44
EFL Gaussian copula	27.0%	83.2	9.4	7.4	7.3	2.54
Double t copula	29.8%	101.1	24.4	13.2	6.6	3.99
Risk adjusted t copula	26.5%	77.2	18.5	12.6	8.3	1.37

Calibrated parameters: $\nu = 1$ $\lambda_D = 0.024 \times \log D - 0.3675$

The market prices were obtained from Bloomberg. The results in the row 'Risk adjusted t copula' are calibrated from the market quotes based on the joint CDF given by (5.10). The results in the other rows are taken from Mortensen (2006). Interest rates are constant at 3%, and the recovery rate is 40%.

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