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Abstract

We consider a consistent pricing model of government bonds, interest-rate swaps and basis swaps in one currency within the no-arbitrage framework. To this end, we propose a three yield-curve model; one for discounting cashflows, one for calculating LIBOR deposit rates and one for calculating coupon rates of government bonds. The derivation of the yield curves from observed data is presented, and the option prices on a swap or a government bond are studied. A one-factor quadratic Gaussian model is proposed as a specific model, and shown to provide a very good fit to the current Japanese low interest-rate environment.

1 Introduction

Risk management of interest rates is becoming more important than ever due to growing economic activities by individuals, corporate firms, institutional investors and governments, since interest rates reflect demand for money to support their business lives. Many transactions of interest-rate products in financial markets determine the consensus of interest rates. Interest rates are not "local" but "global" in the sense that interest rates of one currency are affected by those of other currencies and other financial products with the globalization of business activities.

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Interest-rate products frequently traded in the market include bonds, swaps, basis swaps, and other interest-rate derivatives. An interest-rate (plain vanilla) swap is an exchange of a fixed-coupon bond and a floating-coupon bond, whereas a basis swap is a cross-currency swap to exchange a floating-coupon bond in the US dollar (USD) and a floating-coupon bond in another currency, e.g. Japanese yen (JPY).[1] If the market were frictionless, the swap rate would coincide with the bond yield with the same maturity and the basis swap spread would diminish. However, several yield spreads are observed in the actual market due to some frictions.

It is understood that swap spreads between swap rates and bond yields are caused by credit risk, a preference for government bonds that can be used as a collateral,[2] and other reasons. On the other hand, basis swap spreads emerge due to uneven demands in one currency against another currency as a result of economic activities or credit issues of the governments. For example, suppose that a Japanese bank needs a USD fund. It will raise a fund in JPY in the home market and enters into a basis swap on which it pays USD LIBOR and receives JPY LIBOR plus the basis swap spread so that it borrows USD with a collateral in JPY. If it is very aggressive in the USD funding, the basis swap spread will become negative and further.[3] The aim of this paper is to construct a consistent pricing model for the interest-rate products that are traded with frictions.

There exists a vast literature for the term-structure models of interest rates. A seminal paper by Vasicek (1977) used the Ornstein–Uhlenbeck process to model the short-rate dynamics. Since then, many short-rate models have been proposed, including the square-root model by Cox, Ingersoll and Ross (1985), the quadratic-Gaussian model by Pelsser (1997), and the affine term structure model by Duffie and Kan (1996).[4] On the other hand, Heath, Jarrow and Morton (1992) developed the forward-rate model to preclude arbitrage opportunities among bonds of different maturities. Flesaker and Hughston (1996) proposed the potential approach to model directly the bond prices, and showed that they actually consider a class of positive HJM models.

[1] More precisely, a basis swap exchanges interests of the USD 3-month LIBOR and interests in a different currency, say JPY, of the 3-month LIBOR with some spread, in addition to the principal exchanges at the starting date and the maturity.

[2] Government bonds are excessively demanded compared to swaps, because the bonds are the objective of investment of many funds, while swaps are not.

[3] If the funds in any currencies are equally available to all market participants, the basis swap spreads will diminish.

[4] The Vasicek model as well as the CIR model belongs to the class of one-factor affine models, while the quadratic Gaussian model does not.

While the literature of swap spreads is also huge,[5] academic researchers have paid little attention to the basis swap spread, although these two spreads are carefully watched by practitioners in the actual market, with the exception of Fruchard, Zammouri and Willems (1995) and Boenkost and Schmidt (2005). In particular, Boenkost and Schmidt (2005) proposed two models for the basis swap spread by using two different curves, the one for discounting cashflows and the other for calculating forward LIBOR rates. The first model being widely used in practice is inconsistent to the traditional bootstrapping model,[6] whence admitting an arbitrage opportunity. The second model is consistent, but mark-to-market valuation differs from the results of the standard method.

This paper adopts and extends the idea of Boenkost and Schmidt (2005) to model three different curves (D, L, G-curves) in one currency under a stochastic interest-rate economy within the no-arbitrage paradigm.[7] More specifically, we assume that market participants trade government bonds, interest-rate swaps, and basis swaps, and use the D-curve to discount all the cashflows. The LIBOR as a reference rate of the floating rate on a swap contract is a deposit rate whose yield curve is called the L-curve. The difference between the L-curve and the D-curve reflects the unequal demands to currencies. The G-curve is constructed similar to the L-curve. The key idea here is to regard the purchase of a government bond as a swap contract on the floating rates determined by the G-curve. Hence, excessive demands on government bonds to swaps are reflected in the difference between the L-curve and the G-curve. By making use of these ideas, it is shown that a bootstrapping method calculates zero-coupon bond prices from observed swap rates and bond prices.

This paper is organized as follows. After describing the model setup in the next section, we obtain valuation formulas of the interest-rate products in Section 3. Section 4 is devoted to constructing specific models for the dynamics of the D-curve and the two spreads, $L - D$ and $G - D$. Namely, while the spreads are modelled by one-factor Vasicek models, we consider a one-factor quadratic Gaussian model and a one-factor Vasicek model for the risk-free short rate process. In either case, the three curves (D, L, G-curves) are derived in closed form. Section 5 considers the pricing of options such as swaptions and bond options. The option prices are approximated by the Gram–Charlier expansion that decomposes the density function of the value of the underlying asset at the expiry. In Section 6, we show

[5] Among them, we refer to Collin-Dufresne and Solnik (2001), Feldhütter and Lando (2006) and references therein.

[6] The model proposed by Fruchard, Zammouri and Willems (1995) is a special case of this model.

[7] We call our model *the DLG model*, where "D" stands for "Discount", "L" for "LIBOR", and "G" for "Government", respectively, that is a direct extension of the LG model previously studied by Kijima and Tanaka (2007). Note that any interest-rate model can be applied to construct each curve.

some empirical results. After showing the historical rates, we present some calibration results and approximation accuracy of the Gram–Charlier expansion for option prices. Section 7 concludes this paper.

2 Model Setup

Throughout the paper, we assume that the denominated currency of both interest-rate swaps and government bonds is in JPY, and the basis swap is an exchange of JPY LIBOR plus the *basis swap spread* with USD LIBOR.[8] A government bond is priced with a premium due to several risk resources and/or the convenience yield, i.e. a benefit of holding the government bond. An interest-rate swap is priced also with a premium due to several risk resources including credit risk.[9]

Since interest-rate swaps and government bonds in the actual market are traded with different yields and since a non-zero spread in the basis swap exists, it is justified to use a different discounting curve for each trade. In this paper, we consider the three curves, "D-curve", "L-curve" and "G-curve" for this purpose. The short rate associated with the k-curve, $k = D, L, G$, is denoted by $r_k(t)$.

Market participants observe the interest rates determined by these curves. For example, LIBOR deposit rates are implied by the L-curve, and we assume that the participants evaluate any claim by discounting all cashflows along the D-curve. Similarly, government bonds are priced by using the G-curve for coupons and the D-curve for discounting cashflows. Hence, all transactions of swaps and bonds are priced based on the short rates $r_k(t)$, $k = D, L, G$, of JPY under the risk-neutral probability measure. We shall derive three fundamental equations for the three products in the next section. Note that our discussion is similar to, but different from those in Boenkost and Schmidt (2005), since our framework is based on the no-arbitrage paradigm. Also, we consider a stochastic interest-rate economy.

The uncertainty is represented by a probability space $(\Omega, \mathcal{F}, Q_D)$ on which the short rates $r_k(t)$, $k = D, L, G$, are defined using a three-dimensional standard Brownian motion $(W_D(t), W_L(t), W_G(t))$.[10] The filtration generated by the Brownian motion is denoted by $\{\mathcal{F}_t\}$. The probability measure Q_D is the risk-neutral measure, since we are interested in

[8]As in Boenkost and Schmidt (2005), we assume that there exist no frictions in the USD currency.

[9]According to Collin-Dufresne and Solnik (2001), whereas corporate bonds carry default risk, swap contracts are free of *default risk*.

[10]In this paper, we take the short rate approach rather than the market model (or forward rate) approach simply because calibration becomes simpler.

the pricing of financial products.[11] Hence, the short rate $r_D(t)$ is regarded as the risk-free interest rate. The expectation of random variable X with respect to probability measure P is denoted by $E^P[X]$.

Associated with the short rate $r_k(t)$ is the *money market account* defined by

$$B_k(t) = e^{\int_0^t r_k(s)ds}, \qquad k = D,\ L,\ G.$$

Recall that any price process discounted by $B_D(t)$ is a martingale under Q_D.

Before proceeding, we introduce a useful pricing kernel $Z_k(t)$ associated with the short rate $r_k(t)$ by

$$\frac{dZ_k(t)}{Z_k(t)} = -r_k(t)dt - \sum_{j=D,L,G} \lambda_k^j(t)dW_j(t), \quad Z_k(0) = 1, \qquad k = D, L, G, \tag{2.1}$$

where $\lambda_D^D = \lambda_D^L = \lambda_D^G = 0$,[12] which implies $Z_D(t) = 1/B_D(t)$. Here, we call $(\lambda_k^D, \lambda_k^L, \lambda_k^G)$ the *market price of risk* associated with the k-curve. The zero-coupon bond price $P_k(t,T)$, $t \leq T$, on the k-curve can then be evaluated by

$$P_k(t,T) = E^{Q_D}\left[\frac{Z_k(T)}{Z_k(t)}\Big|\mathcal{F}_t\right], \qquad k = D, L, G. \tag{2.2}$$

Of course, we have $P_k(T,T) = 1$ for all $k = D, L, G$.

For the valuation purpose, it is also useful to introduce measures Q_ℓ equivalent to the risk-neutral measure Q_D by

$$\left.\frac{dQ_\ell}{dQ_D}\right|_{\mathcal{F}_t} = \zeta_\ell(t) \equiv B_\ell(t)Z_\ell(t), \qquad \ell = L, G. \tag{2.3}$$

The probability measure Q_ℓ is the equivalent martingale measure with respect to the numéraire of the money market account B_ℓ. Similarly, we define the T-forward measure Q_D^T with the numéraire of $P_D(t,T)$.

3 Valuation of Interest-Rate Products

In the ordinary interest-rate swaps, a swap rate is the fixed rate to be exchanged with the floating rate LIBOR for a certain period. Similarly, as we shall show, both basis swaps and

[11]Other measures such as Q_L defined later can be chosen as a reference pricing measure. Our choice of Q_D as a reference measure is based on the intuition that all cashflows should be discounted along the D-curve, and L-curve is used for the derivation of the forward LIBOR rates.

[12]This form of pricing kernel is consistent with a result in Cuoco (1997). He proves that a pricing kernel $Z_\nu(t)$ in an incomplete market due to a portfolio constraint ν satisfies a stochastic differential equation similar to (2.1).

government bonds can be regarded as swap contracts. Using this idea, we formulate a swap rate, a basis swap spread and a government bond price in this section. To this end, we define the deposit rate under the k-curve for period $[T_{i-1}, T_i]$ observed at time t as

$$c_k(t, T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{P_k(t, T_{i-1})}{P_k(t, T_i)} - 1 \right), \qquad k = L, G. \tag{3.1}$$

Whereas c_L is called the LIBOR, we call c_G the Govt rate.

In the following, for the sake of simplicity, we assume that the same day-count convention is applied to all the products, and the relevant dates $T_0 < T_1 < \cdots < T = T_N$ are set at regularly spaced time intervals with $\delta = T_i - T_{i-1}$ for all i, and the time t is any time on or prior to T_0.

3.1 Interest-Rate Swaps

Let us denote the time-t swap rate for period $[T_0, T_N]$ by $S(t, T_0, T_N)$. An interest-rate swap is equivalent to an exchange at the same price of a fixed-coupon bond with rate $S(t, T_0, T_N)$ and a floating-coupon bond with the LIBOR rates $c_L(T_{i-1}, T_{i-1}, T_i)$, $i = 1, 2, \ldots, N$.

By our assumption, market participants evaluate at time t the value of the floating coupon $c_L(T_{i-1}, T_{i-1}, T_i)$ as

$$E^{Q_D}\left[\frac{Z_D(T_i)}{Z_D(t)} c_L(T_{i-1}, T_{i-1}, T_i) \Big| \mathcal{F}_t \right] = \frac{1}{\delta} E^{Q_D}\left[\frac{Z_D(T_i)}{Z_D(t)} \left(\frac{1}{P_L(T_{i-1}, T_i)} - 1 \right) \Big| \mathcal{F}_t \right]. \tag{3.2}$$

Using the T_i-forward measure $Q_D^{T_i}$, it follows that

$$E^{Q_D}\left[\frac{Z_D(T_i)}{Z_D(t)} c_L(T_{i-1}, T_{i-1}, T_i) \Big| \mathcal{F}_t \right] = \frac{1}{\delta} P_D(t, T_i) E^{Q_D^{T_i}}\left[\frac{1}{P_L(T_{i-1}, T_i)} - 1 \Big| \mathcal{F}_t \right]. \tag{3.3}$$

In the following, we denote

$$L(t, T_{i-1}, T_i) = \frac{1}{\delta} \left(E^{Q_D^{T_i}}\left[\frac{1}{P_L(T_{i-1}, T_i)} \Big| \mathcal{F}_t \right] - 1 \right). \tag{3.4}$$

Let us regard a swap contract as an exchange of a fixed-coupon bond and a floating-coupon bond. While the value of the fixed leg at time t is given by

$$-P_D(t, T_0) + \delta S(t, T_0, T_N) \sum_{i=1}^{N} P_D(t, T_i) + P_D(t, T_N),$$

the value of the floating leg at time t is given by

$$-P_D(t, T_0) + \delta \sum_{i=1}^{N} L(t, T_{i-1}, T_i) P_D(t, T_i) + P_D(t, T_N), \tag{3.5}$$

where we have used (3.3) and (3.4). Since the values of the both legs must be equal, we obtain the *first fundamental equation*

$$S(t, T_0, T_N) = \frac{\sum_{i=1}^{N} L(t, T_{i-1}, T_i) P_D(t, T_i)}{\sum_{i=1}^{N} P_D(t, T_i)}. \tag{3.6}$$

Note, however, that the value of each leg may not be zero. This implies that the valuation of an off-the-market swap under this method is different from a traditional bootstrapping result (see Remark 3.1 below), which is a well-known fact for international financial institutions, as pointed out by Boenkost and Schmidt (2005).

3.2 Basis Swaps

Let $bs(t, T_0, T_N)$ denote the basis swap spread for period $[T_0, T_N]$, originated at time t. Namely, coupons of USD LIBOR are exchanged with those of JPY LIBOR plus $bs(t, T_0, T_N)$ in addition to the principal exchanges at the starting date and the maturity.

Since we have assumed that there exist no frictions in the USD currency, the USD-denominated floating bond is priced at par[13] . Then, on the JPY leg to be exchanged with the USD leg on the basis swap, we must have

$$0 = -P_D(t, T_0) + \delta \sum_{i=1}^{N} \left(L(t, T_{i-1}, T_i) + bs(t, T_0, T_N)\right) P_D(t, T_i) + P_D(t, T_N), \tag{3.7}$$

which implies the *second fundamental equation*

$$bs(t, T_0, T_N) = \frac{P_D(t, T_0) - P_D(t, T_N)}{\delta \sum_{i=1}^{N} P_D(t, T_i)} - S(t, T_0, T_N). \tag{3.8}$$

Remark 3.1. When the L-curve and the D-curve are identical as in Kijima and Tanaka (2007), we have from (3.2) that

$$E^{Q_D}\left[\frac{Z_D(T_i)}{Z_D(t)} c_L(T_{i-1}, T_{i-1}, T_i) \middle| \mathcal{F}_t\right] = \frac{1}{\delta}[P_D(t, T_{i-1}) - P_D(t, T_i)].$$

It follows from (3.5) that the value of the floating leg is given by

$$-P_D(t, T_0) + \sum_{i=1}^{N}[P_D(t, T_{i-1}) - P_D(t, T_i)] + P_D(t, T_N) = 0,$$

[13] In the case of the presence of some frictions in the USD market, the USD floating bond is not a par. Hence, the right-hand side of equation (3.7) is not equal to zero. The following discussion can be modified by considering the JPY-value of the USD leg with the process of the foreign exchange rate, D- and L-curves of USD.

and the swap rate is determined as

$$S(t,T_0,T_N) = \frac{P_D(t,T_0) - P_D(t,T_N)}{\delta \sum_{i=1}^{N} P_D(t,T_i)}.$$

Therefore, we have $bs(t,T_0,T_N) = 0$ in this case. The converse statement is also true; i.e., zero basis swap spreads along all the maturities implies that the L-curve and the D-curve are identical in the DLG model.

3.3 Government Bonds

The swap spread will be zero if all the cashflows of a government bond are evaluated along the L-curve. The swap spread is usually positive, meaning that the market price of the government bond is more expensive than evaluated along the L-curve. The premium can be interpreted as a benefit of holding the government bond (i.e. a convenience yield) or a reflection of credit issues surrounding banks and the government. In order to describe such a benefit formally, we regard the purchase of a government bond as a swap transaction to exchange the fixed-coupon bond against a floating-coupon bond with relatively low rates. Namely, we derive the deposit rate calculated along the G-curve and consider a swap transaction on the floating deposit rates. The notional fixed-coupon bond in the swap transaction is understood to be the government bond. We formulate this idea as follows.

Let $V(t,T_N)$ be the time-t price of the government bond ("JGB") with maturity T_N and fixed-coupon rate $C(T_N)$. Since the purchase of a government bond is equivalent to a swap transaction to exchange the fixed-coupon bond against a floating-coupon bond, we obtain

$$\begin{aligned} &-V(t,T_N) + \delta C(T_N) \sum_{i=1}^{N} P_D(t,T_i) + P_D(t,T_N) \\ &\quad = -P_D(t,T_0) + \delta \sum_{i=1}^{N} G(t,T_{i-1},T_i) P_D(t,T_i) + P_D(t,T_N), \end{aligned} \tag{3.9}$$

where

$$G(t,T_{i-1},T_i) = \frac{1}{\delta}\left(E^{Q_D^{T_i}}\left[\frac{1}{P_G(T_{i-1},T_i)} \middle| \mathcal{F}_t \right] - 1 \right) \tag{3.10}$$

as for (3.4). The derivation of (3.9) is the same as (3.5) for the ordinary interest-rate swaps.

By solving (3.9) with respect to the government bond price $V(t,T_N)$, we obtain the *third fundamental equation*

$$V(t,T_N) = P_D(t,T_0) + \delta \sum_{i=1}^{N} [C(T_N) - G(t,T_{i-1},T_i)] P_D(t,T_i) \tag{3.11}$$

for the G-curve. Note that the par yield (the bond price is at par) $C_p(t, T_0, T_N)$ is given by

$$C_p(t, T_0, T_N) = \frac{\sum_{i=1}^{N} G(t, T_{i-1}, T_i) P_D(t, T_i)}{\sum_{i=1}^{N} P_D(t, T_i)}, \tag{3.12}$$

which is completely parallel to the swap rate (3.6).

Remark 3.2. Equation (3.9) represents that the value of cashflows arising from the purchase of a government bond is equal to the value of the floating-rate bond whose floating rates are the Govt rates $c_G(T_{i-1}, T_{i-1}, T_i) = G(T_{i-1}, T_{i-1}, T_i)$. However, note that the value of the both sides in (3.9) is not equal to zero, because the floating-rate bond is not at par. Moreover, note that the coupon cashflows of the fixed bond are evaluated along the D-curve, while the redemption value of the bond is evaluated with the D- and G-curves under the DLG model. Namely, from (3.11), one can see that by setting $C(T_N) = 0$ in (3.11), the price of the zero-coupon government bond is given by

$$P_D(t, T_0) - \delta \sum_{i=1}^{N} G(t, T_{i-1}, T_i) P_D(t, T_i),$$

which implies that the coupon value of the coupon-bearing bond is equal to

$$\delta C(T_N) \sum_{i=1}^{N} P_D(t, T_i).$$

Hence, the convenience yield appears only in the original principal.

If all the forward rates of the G-curve are lower than the ones of the L-curve (forward LIBOR) plus the basis swap spread, i.e.

$$G(t, T_{i-1}, T_i) < L(t, T_{i-1}, T_i) + bs(t, T_0, T_N), \qquad i = 1, 2, \ldots, N,$$

then, from (3.7) and (3.11), we obtain

$$V(t, T_N) > \delta C(T_N) \sum_{i=1}^{N} P_D(t, T_i) + P_D(t, T_N).$$

That is, the value of each leg in (3.9) is negative and the JGB price becomes more expensive than evaluated along the L-curve. This is the situation that a government bond yield is lower than the corresponding swap rate as usually observed in the actual market.

3.4 Bootstrapping

By the three fundamental equations (3.6), (3.8) and (3.11), we can construct a bootstrapping to calculate the prices (P_D, L, G) from the observed data (S, bs, V) by starting from the nearest maturity T_0 and going through further maturities recursively. Namely, from (3.8), we first obtain the D-curve

$$P_D(t,T_N) = \frac{P_D(t,T_0) - \delta[S(t,T_0,T_N) + bs(t,T_0,T_N)] \sum_{i=1}^{N-1} P_D(t,T_i)}{1 + \delta[S(t,T_0,T_N) + bs(t,T_0,T_N)]}. \tag{3.13}$$

Next, from (3.6), we have with $P_D(t,T_i)$ at hand that

$$L(t,T_{N-1},T_N) = \frac{1}{P_D(t,T_N)} \left[S(t,T_0,T_N) \sum_{i=1}^{N} P_D(t,T_i) - \sum_{i=1}^{N-1} L(t,T_{i-1},T_i) P_D(t,T_i) \right]. \tag{3.14}$$

Finally, from (3.11), we obtain

$$\begin{aligned} G(t,T_{N-1},T_N) &= C(T_N) - \frac{1}{\delta P_D(t,T_N)} \Big[V(t,T_N) - P_D(t,T_0) \\ &\quad - \delta \sum_{i=1}^{N-1} [C(T_N) - G(t,T_{i-1},T_i)] P_D(t,T_i) \Big]. \end{aligned} \tag{3.15}$$

4 Short-Rate Models

In this section, we study two simple short-rate models to derive formulas in closed form for the three curves $P_k(t,T)$, $k = D, L, G$. The D-curve on the first model is built by a quadratic Gaussian model and the second one is by the Vasicek model[14]. For this purpose, we introduce the short-rate spreads defined by

$$h_L(t) = r_L(t) - r_D(t), \qquad h_G(t) = r_G(t) - r_D(t),$$

and model them together with $r_D(t)$, rather than the three short rates $r_k(t)$ directly.

The spreads $h_L(t)$ and $h_G(t)$ can be interpreted in our framework as follows. First, $h_L(t)$ can be viewed as an averaged cost of JPY funding over $r_D(t)$ among reference banks contributing to the LIBOR panel. The cost may be incurred due to heterogeneous demands among currencies. Next, $-h_G(t)$ is a convenience yield, or a benefit of holding the government bond for market participants. These costs can be retrieved from the bootstrapped forward rates $L(t,T_{i-1},T_i)$ and $G(t,T_{i-1},T_i)$. Note that

$$r_L(t) - r_G(t) = h_L(t) - h_G(t)$$

represents the swap spread. Hence, we have

$$r_G(t) = r_L(t) - [h_L(t) - h_G(t)],$$

meaning that the JGB yield is the swap rate with the same maturity less the swap spread.

[14]The Hull-White model (1990,1994) is another candidate for our purpose, provided that the initial curve is available. The formulation can be carried out in a completely parallel fashion to the Vasicek case.

4.1 Quadratic Gaussian Model

As a specific model, we first apply the quadratic Gaussian model proposed by Pelsser (1997). That is, the short rate $r_D(t)$ is assumed to follow the process

$$\begin{aligned} r_D(t) &= (y(t)+\alpha+\beta t)^2, \\ dy(t) &= -a_D y(t)dt + \sigma_D dW_D(t), \end{aligned} \tag{4.1}$$

where α, β, a_D, σ_D are some constants.

On the other hand, the spreads are modelled by the Vasicek model (i.e., the Ornstein–Uhlenbeck process) as

$$\begin{aligned} dh_L(t) &= a_L(b_L - h_L(t))dt + \sigma_L dW_L(t), \\ dh_G(t) &= a_G(b_G - h_G(t))dt + \sigma_G dW_G(t), \end{aligned} \tag{4.2}$$

respectively, where a_k, b_k, σ_k $(k = L, G)$ are some constants and the Brownian motions $W_D(t)$, $W_L(t)$, $W_G(t)$ are independent of each other under Q_D. The market prices of risk are assumed to be given by

$$\begin{pmatrix} \lambda_D^D(t) & \lambda_D^L(t) & \lambda_D^G(t) \\ \lambda_L^D(t) & \lambda_L^L(t) & \lambda_L^G(t) \\ \lambda_G^D(t) & \lambda_G^L(t) & \lambda_G^G(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_L & 0 \\ 0 & 0 & \lambda_G \end{pmatrix} \tag{4.3}$$

with some constants λ_L, λ_G.

According to Pelsser (1997), the zero-coupon bond price derived from (4.1) is given by

$$P_D(t,T) = \exp\left\{A_D(t,T) - B_D(t,T)y(t) - C_D(t,T)y(t)^2\right\}, \tag{4.4}$$

where

$$\begin{aligned} \gamma &= \sqrt{a_D^2 + 2\sigma_D^2}, \\ F_D(t,T) &= 2\gamma \mathrm{e}^{\gamma(T-t)}\left((\gamma+a_D)\mathrm{e}^{2\gamma(T-t)} + \gamma - a_D\right)^{-1}, \\ C_D(t,T) &= \left(\mathrm{e}^{2\gamma(T-t)} - 1\right)\left((\gamma+a_D)\mathrm{e}^{2\gamma(T-t)} + \gamma - a_D\right)^{-1}, \\ B_D(t,T) &= 2F_D(t,T)\int_t^T \frac{\alpha+\beta s}{F_D(s,T)}ds, \\ A_D(t,T) &= \int_t^T \left(\frac{1}{2}\sigma_D^2 B_D(s,t)^2 - \sigma_D^2 C_D(s,T) - (\alpha+\beta s)^2\right)ds. \end{aligned}$$

Explicit formulas for B_D and A_D are given in Appendix A.

We next derive an explict formula for the L-curve. Because $W_D(t)$ and $W_L(t)$ are independent under Q_D, one can show that

$$P_L(t,T) = E^{Q_D}\left[\frac{Z_L(T)}{Z_L(t)}\middle|\mathcal{F}_t\right] = P_D(t,T)H_L(t,T), \tag{4.5}$$

where

$$H_L(t,T) = E^{Q_L}\left[e^{-\int_t^T h_L(s)ds} \middle| \mathcal{F}_t \right]$$

after an appropriate change of measures.

For the explicit calculation of $H_L(t,T)$, using Girsanov's theorem, it is easy to show that

$$dh_L(s) = a_L \left[\left(b_L - \frac{\sigma_L \lambda_L}{a_L} \right) - h_L(s) \right] ds + \sigma_L dW_L^L(s)$$

where $W_L^L(t) = W_L(t) + \lambda_L t$ is a standard Brownian motion under Q_L. Note that the process $h_L(t)$ is also the Vasicek process (the Ornstein–Uhlenbeck process) under Q_L. Recalling the usual discussions for the Vasicek model, we then obtain

$$H_L(t,T) = \exp\left\{ A_L(t,T) + B_L(t,T) h_L(t) \right\}, \tag{4.6}$$

where the functions A_L and B_L are given by

$$\begin{aligned} B_L(t,T) &= -\frac{1}{a_L}\left(1 - e^{-a_L(T-t)}\right), \\ A_L(t,T) &= -\left(B_L(t,T) + (T-t)\right)\left(b_L - \frac{\sigma_L\lambda_L}{a_L} - \frac{\sigma_L^2}{2a_L^2}\right) - \frac{\sigma_L^2 B_L(t,T)^2}{4a_L}, \end{aligned}$$

respectively. We note from (4.5) that

$$-\log\frac{P_L(t,T)}{T-t} + \log\frac{P_D(t,T)}{T-t} = -\log\frac{H_L(t,T)}{T-t}$$

represents the spread in the zero rates between the "discounting curve" and the "LIBOR curve".

In order to calculate the forward LIBOR, we invoke the fact that the dynamics of $h_L(t)$ is not affected by the change of measures from Q_D to $Q_D^{T_1}$. It follows from (4.6) that

$$\begin{aligned} \delta L(t,T_1,T_2) &= E^{Q_D^{T_2}}\left[\frac{1}{P_L(T_1,T_2)} \middle| \mathcal{F}_t \right] - 1 \\ &= \frac{P_D(t,T_1)}{P_D(t,T_2)} \exp\Big\{ -A_L(T_1,T_2) - B_L(T_1,T_2) E^{Q_D}[h_L(T_1) \mid \mathcal{F}_t] \\ &\qquad + \frac{1}{2} B_L(T_1,T_2)^2 Var^{Q_D}[h_L(T_1) \mid \mathcal{F}_t] \Big\} - 1. \end{aligned}$$

Thanks to the conditional normality of $h_L(T_1)$, we then obtain the forward LIBOR rates as

$$L(t,T_1,T_2) = \frac{1}{\delta}\left(\frac{P_D(t,T_1)}{P_D(t,T_2)} K_L(t,T_1,T_2) - 1 \right), \tag{4.7}$$

where

$$\begin{aligned} K_L(t,T_1,T_2) &= \exp\Big\{ -A_L(T_1,T_2) - B_L(T_1,T_2)\left(h_L(t) e^{-a_L(T_1-t)} + b_L\left(1 - e^{-a_L(T_1-t)}\right)\right) \\ &\qquad + \frac{\sigma_L^2 B_L(T_1,T_2)^2}{4a_L}\left(1 - e^{-2a_L(T_1-t)}\right) \Big\}, \end{aligned}$$

which represents the spread in the forward rates between the discounting curve and the forward LIBOR curve.

Finally, using the same arguments, we can derive an explicit formula for the G-curve. Namely,

$$P_G(t,T) = P_D(t,T)H_G(t,T), \tag{4.8}$$

and

$$H_G(t,T) = \exp\left\{A_G(t,T) + B_G(t,T)h_G(t)\right\}, \tag{4.9}$$

where

$$\begin{aligned} B_G(t,T) &= -\frac{1}{a_G}\left(1 - e^{-a_G(T-t)}\right), \\ A_G(t,T) &= -\left(B_G(t,T) + (T-t)\right)\left(b_G - \frac{\sigma_G\lambda_G}{a_G} - \frac{\sigma_G^2}{2a_G^2}\right) - \frac{\sigma_G^2 B_G(t,T)^2}{4a_G}. \end{aligned}$$

Furthermore, we have

$$G(t,T_1,T_2) = \frac{1}{\delta}\left(\frac{P_D(t,T_1)}{P_D(t,T_2)}K_G(t,T_1,T_2) - 1\right), \tag{4.10}$$

where

$$\begin{aligned} K_G(t,T_1,T_2) &= \exp\Big\{-A_G(T_1,T_2) - B_G(T_1,T_2)\left(h_G(t)e^{-a_G(T_1-t)} + b_G\left(1 - e^{-a_G(T_1-t)}\right)\right) \\ &\qquad + \frac{\sigma_G^2 B_G(T_1,T_2)^2}{4a_G}\left(1 - e^{-2a_G(T_1-t)}\right)\Big\}. \end{aligned}$$

4.2 The Vasicek Model

The quadratic Gaussian model (4.1) has an apparent merit that the short rate $r_D(t)$ stays non-negative. However, in order to derive formulas in closed form, we assumed that $r_D(t)$ and the short-rate spreads $h_L(t)$, $h_G(t)$ are mutually independent. In reality, however, these processes are correlated significantly, so the independent assumption may cause a limited flexibility of our model when fitted to the actual data.

In this subsection, at the sacrifice of non-negativity in the short rate, we develop a correlated Gaussian model. Namely, the short rate $r_D(t)$ is assumed to follow the Vasicek model

$$dr_D(t) = a_D(b_D - r_D(t))dt + \sigma_D dW_D(t), \tag{4.11}$$

where a_D, b_D and σ_D are some constants. The spreads $h_L(t)$ and $h_G(t)$ follow the same Vasicek processes as in (4.2). In order to introduce correlations between them, we assume that the Brownian motions $W_D(t)$, $W_L(t)$, $W_G(t)$ are correlated as

$$dW_D(t)dW_L(t) = \rho_{DL}dt, \quad dW_D(t)dW_G(t) = \rho_{DG}dt, \quad dW_L(t)dW_G(t) = \rho_{LG}dt.$$

The market prices of risk are the same as (4.3).

By making use of the conditional normality of the above processes, one can show that the relevant prices and the forward rates are given by[15]

$$\begin{aligned}
P_D(t,T) &= \exp\left\{A_D(t,T) + B_D(t,T)r_D(t)\right\}, \\
P_L(t,T) &= E^{Q_D}\left[\frac{Z_L(T)}{Z_L(t)}\Big|\mathcal{F}_t\right] = P_D(t,T)H_L(t,T), \\
P_G(t,T) &= E^{Q_D}\left[\frac{Z_G(T)}{Z_G(t)}\Big|\mathcal{F}_t\right] = P_D(t,T)H_G(t,T), \\
L(t,T_1,T_2) &= \frac{1}{\delta}\left(E^{Q_D^{T_2}}\left[\frac{1}{P_L(T_1,T_2)}\Big|\mathcal{F}_t\right] - 1\right) = \frac{1}{\delta}\left(\frac{P_D(t,T_1)}{P_D(t,T_2)}K_L(t,T_1,T_2) - 1\right), \\
G(t,T_1,T_2) &= \frac{1}{\delta}\left(E^{Q_D^{T_2}}\left[\frac{1}{P_G(T_1,T_2)}\Big|\mathcal{F}_t\right] - 1\right) = \frac{1}{\delta}\left(\frac{P_D(t,T_1)}{P_D(t,T_2)}K_G(t,T_1,T_2) - 1\right),
\end{aligned}$$

where

$$\begin{aligned}
B_k(t,T) &= -\frac{1}{a_k}\left(1 - e^{-a_k(T-t)}\right), \\
A_k(t,T) &= -\left(B_k(t,T) + (T-t)\right)\left(b_k - \frac{\sigma_k^2}{2a_k^2}\right) - \frac{\sigma_k^2 B_k(t,T)^2}{4a_k}, \\
B_{Dk}(t,T) &= -\frac{1 - e^{-(a_D+a_k)(T-t)}}{a_D + a_k}, \\
A_{D\ell}(t,T) &= \lambda_\ell\frac{\sigma_\ell}{a_\ell}(B_\ell(t,T) + T - t) + \lambda_\ell\frac{\rho_{D\ell}\sigma_D}{a_D}(B_D(t,T) + T - t) \\
&\quad + \frac{\rho_{D\ell}\sigma_D\sigma_\ell}{a_D a_\ell}\left(B_D(t,T) + B_\ell(t,T) - B_{D\ell}(t,T) + T - t\right), \\
H_\ell(t,T) &= \exp\left\{A_\ell(t,T) + A_{D\ell}(t,T) + B_\ell(t,T)h_\ell(t)\right\}, \\
K_\ell(t,T_1,T_2) &= \exp\Big\{-A_\ell(T_1,T_2) - B_\ell(T_1,T_2)\left(h_\ell(t)e^{-a_\ell(T_1-t)} + b_\ell\left(1 - e^{-a_\ell(T_1-t)}\right)\right) \\
&\quad + \frac{\sigma_\ell^2 B_\ell(T_1,T_2)^2}{4a_\ell}\left(1 - e^{-2a_\ell(T_1-t)}\right) \\
&\quad - A_{D\ell}(T_1,T_2) - \frac{\rho_{D\ell}\sigma_D\sigma_\ell}{a_D}B_\ell(T_1,T_2)\left(B_D(t,T_1) - B_{DD}(t,T_1)\right)\Big\},
\end{aligned}$$

for $k = D, L, G$ and $\ell = L, G$.

[15] A detailed derivation is provided from the authors upon request.

5 The Pricing of Options

The three curves obtained in the previous section look too complicated to be used for the evaluation of swaptions and bond options in the DLG model. However, regardless of the complexity, we can obtain approximated prices of such options by using the Gram–Charlier expansion and bond moments as discussed in Tanaka, Yamada and Watanabe (2007). In this section we outline the basic procedure of the approximation.

As the first step, we need to calculate the moments of the bond prices, called the *bond moments*, involved in the valuation of the cashflow upon the exercise of the option in question under the T-forward measure. Suppose that the zero-coupon bond price $P(t, T)$ is given as a function of Markov state variable $X(t)$. For a given set of dates T, T_0, $U_1, \ldots, U_m$ ($T \le T_0 \le U_i$ for all $i = 1, \ldots, m$), the bond moment is defined as

$$\mu^T(t, T_0, \{U_1, \ldots, U_m\}) \equiv E^T\left[\prod_{i=1}^{m} P(T_0, U_i) \mid X(t)\right].$$

What is useful in practice is that the bond moments can be obtained as a function of X_t either analytically or numerically.

Next, by regarding the cashflow $\sum_{i=0}^{N} a_i P(T_0, T_i)$ of a fixed-coupon bond with cashflow a_i on date T_i as the value of a swap, we obtain the mth swap moment, using the bond moments and the cashflows, as

$$\begin{aligned} M_m(t) &= E^{T_0}\left[\left(\sum_{i=0}^{N} a_i P(T_0, T_i)\right)^m \middle| X_t\right] \\ &= \sum_{0 \le i_1, \ldots, i_m \le N} a_{i_1} \cdots a_{i_m} \mu^{T_0}(t, T_0, \{T_{i_1}, \ldots, T_{i_m}\}). \end{aligned}$$

The kth cumulant $c_k(t)$ of the cashflow $\sum_{i=0}^{N} a_i P(T_0, T_i)$ is calculated from the set of moments $M_m(t)$.

Finally, define the coefficients $q_k(t)$, $k \ge 1$, as $q_0 = 1$, $q_1 = q_2 = 0$, and

$$q_k = \sum_{m=1}^{[k/3]} \sum_{k_1 + \cdots + k_m = k, k_i \ge 3} \frac{c_{k_1} \cdots c_{k_m}}{m! k_1! \cdots k_m!} \left(\frac{1}{\sqrt{c_2}}\right)^k, \qquad k = 3, 4, \ldots,$$

where we omit the argument of time t to save the space. In particular, we have

$$q_3 = \frac{c_3}{3! c_2^{3/2}}, \quad q_4 = \frac{c_4}{4! c_2^2}, \quad q_5 = \frac{c_5}{5! c_2^{5/2}}, \quad q_6 = \frac{c_6 + 10 c_3^2}{6! c_2^3}, \quad q_7 = \frac{c_7 + 35 c_3 c_4}{7! c_2^{7/2}},$$

and so on. Recall that the Gram–Charlier expansion of the density function $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n\left(\frac{x - c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x - c_1}{\sqrt{c_2}}\right),$$

where $\phi(x)$ is the density function of the standard normal distribution and $H_n(x)$ denotes the nth Hermite polynomial defined by

$$H_n(x) = \frac{(-1)^n}{\phi(x)} \frac{d^n}{dx^n}\phi(x).$$

It follows that, by working under the T_0-forward measure, the option price $V(t)$ at time t can be approximated as

$$V(t) \approx P(t, T_0) \left[c_1 N\left(\frac{c_1}{\sqrt{c_2}}\right) + \sqrt{c_2}\phi\left(\frac{c_1}{\sqrt{c_2}}\right) + \sqrt{c_2}\phi\left(\frac{c_1}{\sqrt{c_2}}\right) \sum_{k=3}^{L} (-1)^k q_k H_{k-2}\left(\frac{c_1}{\sqrt{c_2}}\right)\right] \tag{5.1}$$

for some positive integer L.[16] Here, $N(x)$ denotes the distribution function of the standard normal distribution.

Now, let us consider the pricing of a receiver's swaption with strike rate C under the setting of Section 4.1. The value of the underlying swap at the expiry is given by

$$\begin{aligned} V(T_0) &= \delta \sum_{i=1}^{N} (C - L(T_0, T_{i-1}, T_i)) P_D(T_0, T_i) \\ &= \sum_{i=1}^{N} \left((1 + \delta C) P_D(T_0, T_i) - P_D(T_0, T_{i-1}) K_L(T_0, T_{i-1}, T_i)\right), \end{aligned}$$

which is a polynomial of the exponentials of $y(T_0)$, $y(T_0)^2$ and $h_L(T_0)$. Hence, it is possible to obtain the moments of $V(T_0)$ by making use of the fact that, when Y is normally distributed with mean μ and variance σ^2, we have

$$E\left[e^{aY + bY^2}\right] = \frac{1}{\sqrt{1 - 2b\sigma^2}} \exp\left\{\frac{\sigma^2(a + 2b\mu)^2}{2(1 - 2b\sigma^2)} + a\mu + b\mu^2\right\},$$

where a and b are some constants with $b < (2\sigma^2)^{-1}$. The Gram–Charlier expansion can then be applied to this payoff.

Next, for the pricing of a call option written on the bond with coupon C and strike price K, we note that the value of the underlying bond at the expiry is given by

$$V(T_0) = 100 - K + \sum_{i=1}^{N} \left((1 + \delta C) P_D(T_0, T_i) - P_D(T_0, T_{i-1}) K_G(T_0, T_{i-1}, T_i)\right),$$

which is again a polynomial of the exponentials of $y(T_0)$ and $h_G(T_0)$. The same procedure as the swaptions can apply.

[16] Tanaka, Yamada and Watanabe (2007) suggested to use either $L = 3$ or $L = 6$ for a practical application.

6 Empirical Results

In this section, we provide some numerical examples. Before proceeding, a brief discussion on the historical movement of the USD/JPY basis swap spreads is given.

6.1 Overview of the Historical JPY Rates

Until recently, the USD/JPY basis swap spreads showed the relative strength of the demand for JPY against USD. Since Japanese banks needed USD funding to support their Japanese customers' businesses in abroad, the basis swap spread had fluctuated with negative values as shown in Figure 1.

After the crush of the bubble economy in early 1990's, Japanese banks faced the so-called "Japan premium" problem[17] in 1998 due to several financial problems surrounding themselves. The affect of the problem is clearly observed in the historical movement of the USD/JPY basis swap spreads in Figure 1. To support Japanese firms and banks through a monetary policy, Bank of Japan conducted the zero interest-rate policy between 1998 and 2006. After 2006, the basis swap spreads turned to be positive. As the result of the positive basis swap spreads, JGB's become more expensive for USD-based investors than JPY-based investors.

The volatilities of the basis swap spreads are very low recently, while they were relatively high in 1997 and 1998 as shown in Table 1.

6.2 Calibration

As shown in Kijima and Tanaka (2007), the one-factor quadratic Gaussian model is well-fitted to the low interest-rate environment such as JPY,[18] while the performance of the Vasicek model (as well as the CIR model) is not so good. Therefore, in the following, we focus on the calibration of the quadratic Gaussian model described in Section 4.1, where for the sake of simplicity, we assume that $\lambda_L = \lambda_G = 0$ throughout the calibration. Also, we verify the fitness of the DLG model by calibrating the parameters to the market data

[17]Foreign banks were reluctant to lend money, especially in USD, to Japanese banks.

[18]As observed in Kijima (2002), the short-rate process implied by the quadratic Gaussian model has a square-root volatility with a non-linear drift, which is consistent with the empirical findings by Ait-Sahalia (1996). Also, although some recent papers such as Gorovoi and Linetsky (2004) and Kabanov, Kijima and Rinaz (2007) proposed one-factor short-rate models to fit the low interest-rate economy, their models are not easy to calibrate.

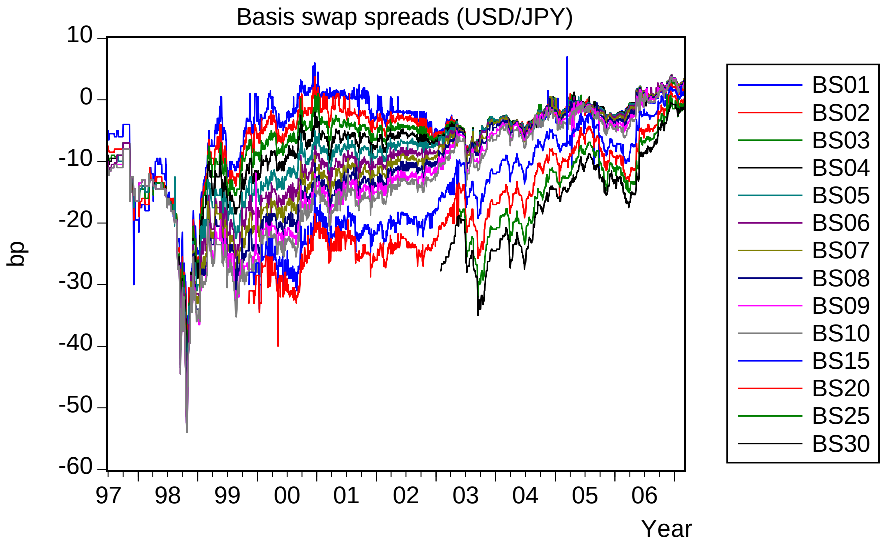

Figure 1: USD/JPY basis swap spreads

Table 1: Historical absolute volatilities (bp/yr, 1997-2007)

	[Basis swap spreads]					[Swap rates]				
	1yr	5yr	10yr	20yr	30yr	1yr	5yr	10yr	20yr	30yr
1997	27.5	12.6	10.7			34.1	43.2	48.0		
1998	18.7	23.5	22.8			30.7	57.0	69.2		
1999	16.2	12.5	13.9	10.4		25.7	64.5	74.6	57.6	
2000	11.6	10.0	10.6	21.7		32.6	36.9	46.1	53.9	
2001	11.0	7.5	10.2	15.9		14.8	33.8	46.9	50.1	
2002	11.7	6.5	6.6	9.4		11.9	21.9	33.9	39.3	
2003	6.2	7.4	5.7	15.7	8.6	14.5	46.9	61.4	70.9	72.7
2004	7.3	6.2	5.4	5.2	5.9	12.9	40.9	50.1	50.3	50.6
2005	8.9	5.0	6.2	9.7	5.3	7.9	33.5	35.7	34.9	36.3
2006	5.4	5.1	6.8	6.4	6.4	23.9	48.3	45.2	41.0	41.6
2007	3.1	3.8	4.3	4.2	4.3	21.7	32.4	31.2	36.7	30.4
average	12.8	10.8	11.0	13.0	6.5	22.1	44.1	52.6	49.7	51.0

Table 2: Observed rates and yields (pct) on November 6, 2007

Maturity (yrs)	Swap	Basis swap	Bond yield	(swap spread)
1	0.9670	-0.0015	0.6605	(0.3065)
2	1.0210	0.0075	0.7720	(0.2490)
3	1.1060	0.0200	0.8415	(0.2645)
4	1.2030	0.0280	0.9969	(0.2061)
5	1.3010	0.0325	1.1055	(0.1955)
7	1.4960	0.0325	1.2548	(0.2412)
10	1.7735	0.0313	1.5871	(0.1864)
20	2.3125	-0.0250	2.1332	(0.1793)
30	2.5075	-0.0155	2.3772	(0.1303)
40	2.5915	-0.0160	2.4375	(0.1540)

observed on November 6, 2007 [19], using the Nelder–Mead algorithm[20] with 1,000 times of iterated calculation within the ranges specified in the second and third columns of Table 3. Table 2 is a snapshot of the market data of swap rates, basis swap spreads and JGB yields on the same day.[21]

Our calibration procedure consists of three stages. As the first stage, we conduct the calibration for the parameters of $r_L(t)$ by observing LIBOR rates, swap rates and basis swap spreads in the following manner. Since the sum of the swap rate and the basis swap spread is expressed by P_D according to the second fundamental equation (3.8), the parameters related to r_D can be calibrated by minimizing the sum of squared errors in the "synthetic swap rates"

$$\min_{\theta_1} \sum_{i=1}^{n} \left[S_{model}(T_i) + bs_{model}(T_i) - (S_{obs}(T_i) + bs_{obs}(T_i))\right]^2,$$

where θ_1 denotes the set of parameters to be calibrated in the first stage, and where

$$S_{model}(T_i) + bs_{model}(T_i) = \frac{P_D(0,T_0) - P_D(0,T_i)}{\delta \sum_{k=1}^{i} P_D(0,T_k)},$$

$$S_{obs}(T_i) = \text{Observed swap rate with maturity } T_i,$$

$$bs_{obs}(T_i) = \text{Observed basis swap spread with maturity } T_i.$$

[19]This is the day of first issuance of 40-year JGB auctioned by Ministry of Finance, Japan.
[20]See Nelder and Mead (1965).
[21]Data Source: Bloomberg and Japan Securities Dealers Association

Here, the maturities T_i run over 6 month, every 1-10 years, and 12, 15, 20, 25, 30 years.[22]

Next, for the second stage, given the calibrated parameters of $r_D(t)$, calibration for the parameters of $h_L(t)$ is conducted by observing swap rates in the following manner. By the first fundamental equation (3.6) and (4.7), we see that the swap rate is calculated from P_D and K_L as

$$S(0,T_i) = \frac{P_D(0,T_N) + \sum_{i=1}^{N}(1 - K_L(0,T_{i-1},T_i))P_D(0,T_{i-1})}{\delta \sum_{i=1}^{N} P_D(0,T_i)}.$$

Since all the parameters related to P_D are already fixed, we can calibrate the parameters related to h_L by minimizing the errors in the swap rates as

$$\min_{\theta_2} \sum_{i=1}^{n} \left[S_{model}(T_i) - S_{obs}(T_i)\right]^2,$$

where θ_2 denotes the set of parameters to be calibrated in the second stage, and where

$$S_{model}(T_i) = \frac{P_D(0,T_N) + \sum_{i=1}^{N}(1 - K_L(0,T_{i-1},T_i))P_D(0,T_{i-1})}{\delta \sum_{i=1}^{N} P_D(0,T_i)}.$$

Finally, as the third stage, given the calibrated parameters of r_D and h_L, calibration for the parameters of h_G is conducted by observing government bond prices. According to the third fundamental equation (3.11), we set the objective function as the sum of the absolute errors[23] in the yields of synthetic zero-coupon bonds implied by the observed JGB's and the calibrated parameters of r_D as

$$\min_{\theta_3} \sum_{i=1}^{n} |Z_{model}(T_i) - Z_{obs}(T_i)|,$$

where θ_3 denotes the set of parameters to be calibrated in the third stage, and where

$$\begin{aligned}
Z_{model}(T_i) &= -\ln\left(P_D(0,T_i) + \sum_{k=1}^{i}[1 - K_G(0,T_{k-1},T_k)]P_D(0,T_{k-1})\right) \Big/ (T_i - T_0), \\
Z_{obs}(T_i) &= -\ln\left(V_{obs}(T_i) - \delta C(T_i)\sum_{k=1}^{i} P_D(0,T_k)\right) \Big/ (T_i - T_0), \\
V_{obs}(T_i) &= \text{Closing price of JGB with maturity } T_i \text{ and coupon rate } C(T_i).
\end{aligned}$$

Here, the bonds are taken from 2, 5, 10, 20, 30 year JGB's (one issue for a calendar year).[24] The calibrated parameters obtained by the above procedure are listed in Table 3.

[22]Data Source: British Bankers Association and Bloomberg

[23]We evaluate the error in the absolute deviation rather than the squared deviation for the purpose of robust estimation.

[24]Data Source: Japan Securities Dealers Association

Table 4 presents the calibration results of swap rates, basis swap spreads and bond par yields calculated from the calibrated numbers listed in Table 3, while Table 5 shows the difference between the calibration results and the observed rates[25]. The absolute differences are basically less than 5 basis points. In particular, the prediction error for the newly issued 40-year bond is less than 2 basis points.

6.3 Numerical Examples for Options

In this subsection, we calculate receiver's swaption prices using the Gram–Charlier expansion (5.1) with calibrated parameters listed in Table 3, and compare them with the Monte Carlo simulation results. Here, we employ the third order Gram–Charlier expansion (GC3), i.e. $L = 3$ in (5.1), and Monte Carlo simulation with 100,000 runs. Option prices by GC3 are shown in Table 6 for at-the-money-forward (ATMF) receiver's swaptions, while Figures 2 and 3 illustrate the pricing errors produced by GC3 when Monte Carlo prices are used as the benchmark.

In Figure 2, we examine the pricing errors of the GC3 approximation for a 1-into-10 receiver's swaption across various strike rates, i.e. a receiver's swaption expiring at 1 year with the underlying being a 10 year swap. The pricing errors are at most 3 basis points. On the other hand, Figure 3 compares the two approaches for the ATMF receiver's swaptions across a typical range of expiries and maturities. For all the cases, the pricing errors are less than 5 basis points. The Gram–Charlier expansion approach seems to provide a good approximation for swaption prices even in the DLG model.

As to the computational time, the GC3 takes 7 seconds to calculate the price of the 1-into-10 receiver's swaption, while it takes about 35 seconds to produce the price using Monte Carlo simulation with 100,000 runs. The computational time depends strongly on the maturity of the underlying swap. For example, for a 1-into-5 receiver's swaption, the computational time is reduced to 17 seconds for Monte Carlo simulation and 1 second for GC3. Hence, the Gram–Charlier approach is not only accurate but also very efficient in terms of the computational time.

[25] Calibration errors for basis swaps are of the same magnitude as the market quotes of basis swaps, because the the current level is very low. Hence it might make no sense to calibrate basis swaps for the current market. However, basis swaps play an important role in our model, especially under the situations in late 1990's, since they produce the spread between D-curve and L-curve, which implies a friction in banks' funding.

Table 3: Calibrated parameters and searched ranges in the calibration

Parameter	Lower bound	Upper bound	Calibrated number
a_D	0.0500	0.8000	0.0909
σ_D	0.0000	1.0000	0.0282
$y(0)$	-1.0000	1.0000	-0.1340
α	-1.0000	1.0000	0.2204
β	-0.5000	0.5000	-0.0012
a_L	0.0001	1.0000	0.9892
b_L	-0.2000	0.2000	-0.0003
σ_L	0.0001	0.5000	0.0002
$h_L(0)$	-0.2000	0.2000	0.0007
a_G	0.0001	1.0000	0.3383
b_G	-0.2000	0.2000	-0.0012
σ_G	0.0001	0.5000	0.0001
$h_G(0)$	-0.2000	0.2000	-0.0035

Table 4: Calibrated rates and yields (pct)

Maturity (yrs)	Swap	Basis swap	Bond yield	(swap spread)
1	0.9152	-0.0337	0.5618	(0.3534)
2	1.0226	-0.0134	0.7171	(0.3055)
3	1.1325	-0.0017	0.8611	(0.2714)
4	1.2406	0.0056	0.9944	(0.2462)
5	1.3446	0.0103	1.1178	(0.2268)
7	1.5362	0.0160	1.3370	(0.1992)
10	1.7795	0.0203	1.6057	(0.1738)
20	2.2741	0.0251	2.1338	(0.1403)
30	2.4763	0.0267	2.3471	(0.1292)
40	2.5467	0.0274	2.4227	(0.1240)

Table 5: Difference between calibrated rates and observed rates (bp)

Maturity (yrs)	Swap	Basis swap	Bond yield	(swap spread)
1	-5.18	-3.22	-9.87	(4.69)
2	0.16	-2.09	-5.49	(5.65)
3	2.65	-2.17	1.96	(0.69)
4	3.76	-2.24	-0.25	(4.01)
5	4.36	-2.22	1.23	(3.13)
7	4.02	-1.65	8.22	(-4.20)
10	0.60	-1.10	1.86	(-1.26)
20	-3.84	5.01	0.06	(-3.90)
30	-3.12	4.22	-3.01	(-0.11)
40	-4.48	4.34	-1.48	(-3.00)

Table 6: ATMF receiver's swaption premium by GC3 with calibrated parameters (bp). Numbers in the first column depict the years to the expiry of the option, and those in the first row depict the years to the maturity of the underlying swap.

Exp / Mat	1	3	5	7	10
1	20.9	60.6	95.7	125.4	160.1
3	38.1	106.6	164.4	211.7	265.6
5	48.6	133.3	202.6	258.2	320.3
7	54.9	148.6	223.6	282.8	348.2
10	58.9	157.3	234.3	294.2	360.0

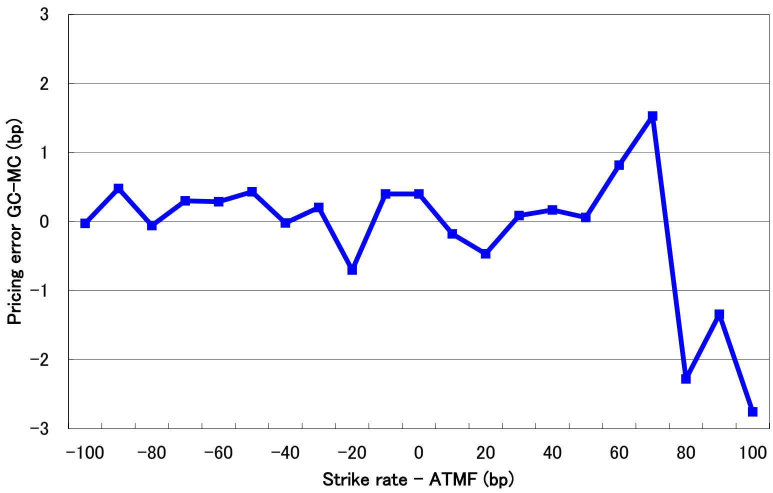

Figure 2: Errors of GC3 prices relative to Monte Carlo prices for a 1-into-10 receiver's swaption across various strikes.

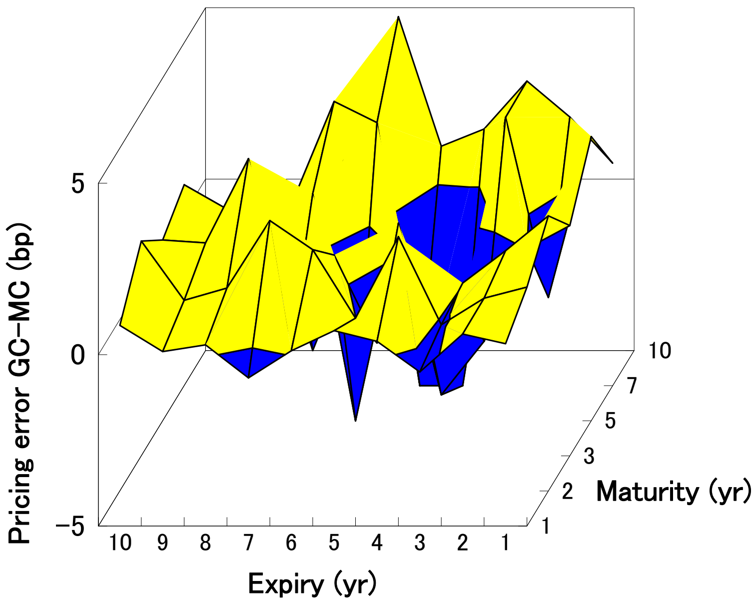

Figure 3: Errors of GC3 prices relative to Monte Carlo prices for receiver's swaptions with various expiries and maturities.

7 Conclusion

We study a three yield-curve model, called the DLG model, for the consistent pricing of government bonds, interest-rate swaps and basis swaps. The derivation is based on the two ideas; the first one is to distinguish the discounting rates from the deposit rates, and the other one is to regard a bond purchase as a swap contract to exchange a fixed-coupon bond against a floating-coupon bond. As a result, on-the-market swap values are always zero while off-the-market swap values are different from the classical NPV due to the existence of basis swap spreads, which is a well-known fact for international financial institutions.

However, the pricing of derivatives based on our model seems rather weak, given the presence in the market of reliable quotes. The model calibration to market quotes including option prices is left to the future research.

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A Formula

$$\begin{aligned}
B_D(t,T) &= 2F_D(t,T)\int_t^T \frac{\alpha+\beta s}{F_D(s,T)}ds = \frac{2B_1(t,T)}{\gamma^2 A_5(t,T)},\\
A_D(t,T) &= \int_t^T \left(\frac{1}{2}\sigma_D^2 B_D(s,t)^2 - \sigma_D^2 C_D(s,T) - (\alpha+\beta s)^2\right) ds\\
&= -\sigma_D^2\left(\frac{A_4(t,T)}{\gamma^5 A_5(t,T)} + A_6(t,T)\right) - \alpha^2(T-t) - \alpha\beta(T^2-t^2) - \frac{1}{3}\beta^2(T^3-t^3),
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_a &= \gamma - a_D,\\
\Gamma_b &= \gamma + a_D,\\
A_{1a}(t,T) &= -e^{\gamma(T-t)} + 4 - e^{-\gamma(T-t)}(3+2\gamma(T-t)),\\
A_{1b}(t,T) &= e^{-\gamma(T-t)} - 4 + e^{\gamma(T-t)}(3-2\gamma(T-t)),\\
A_{2a}(t,T) &= e^{\gamma(T-t)}(1-\gamma T) - 2(1-\gamma(t+T)) + e^{-\gamma(T-t)}(1-\gamma(2t+T)+\gamma^2(t^2-T^2)),\\
A_{2b}(t,T) &= e^{-\gamma(T-t)}(1+\gamma T) - 2(1+\gamma(t+T)) + e^{\gamma(T-t)}(1+\gamma(2t+T)+\gamma^2(t^2-T^2)),\\
A_{3a}(t,T) &= -4\gamma t(1-\gamma T) - e^{\gamma(T-t)}(1-\gamma T)^2\\
&\quad + e^{-\gamma(T-t)}\left(1+2\gamma t-\gamma^2(2t^2+T^2)+\frac{2}{3}\gamma^3(t^3-T^3)\right),\\
A_{3b}(t,T) &= -4\gamma t(1+\gamma T) + e^{-\gamma(T-t)}(1+\gamma T)^2\\
&\quad + e^{\gamma(T-t)}\left(-1+2\gamma t+\gamma^2(2t^2+T^2)+\frac{2}{3}\gamma^3(t^3-T^3)\right),\\
A_4(t,T) &= \Gamma_a\left(\alpha^2\gamma^2 A_{1a}(t,T) + 2\alpha\beta\gamma A_{2a}(t,T) + \beta^2 A_{3a}(t,T)\right)\\
&\quad + \Gamma_b\left(\alpha^2\gamma^2 A_{1b}(t,T) + 2\alpha\beta\gamma A_{2b}(t,T) + \beta^2 A_{3b}(t,T)\right),\\
A_5(t,T) &= \Gamma_a e^{-\gamma(T-t)} + \Gamma_b e^{\gamma(T-t)},\\
A_6(t,T) &= -\frac{1}{2}(T-t)\left(\Gamma_a^{-1} - \Gamma_b^{-1}\right) + \frac{1}{2\gamma}\left(\Gamma_a^{-1} + \Gamma_b^{-1}\right)\ln\frac{A_5(t,T)}{2\gamma},\\
B_1(t,T) &= -\alpha\gamma\left(e^{-\gamma T} - e^{-\gamma t}\right)\left(\Gamma_a e^{\gamma t} + \Gamma_b e^{\gamma T}\right)\\
&\quad + \beta\left(\Gamma_a e^{-\gamma(T-t)}(1-\gamma t) + \Gamma_b e^{\gamma(T-t)}(1+\gamma t) - \Gamma_a(1-\gamma T) - \Gamma_b(1+\gamma T)\right).
\end{aligned}$$