Research Paper Series

No. 42

Applications of Gram-Charlier Expansion and Bond Moments
For Pricing of Interest Rates and Credit Risk

Keiichi Tanaka, Takeshi Yamada and Toshiaki Watanabe

(December 2007)
Rev. December 2008

Graduate School of Social Sciences, Tokyo Metropolitan University

Applications of Gram–Charlier Expansion and Bond Moments for Pricing of Interest Rates and Credit Risk

Keiichi Tanaka
Graduate School of Social Sciences, Tokyo Metropolitan University
E-mail: tanaka-keiichi@tmu.ac.jp

Takeshi Yamada
Financial Systems and Bank Examination Department, Bank of Japan
E-mail: takeshi.yamada@boj.or.jp

Toshiaki Watanabe
Institute of Economic Research, Hitotsubashi University
E-mail: watanabe@ier.hit-u.ac.jp

December 31, 2008

Abstract

The purpose of this paper is to demonstrate the powerful and flexible applicability of Gram–Charlier expansion to pricing of a wide variety of interest rate related products involving interest rate risk and credit risk. In this paper, we develop easily implemented approximations of the prices of several derivatives; swaptions, CMS, CMS options, and vulnerable options. Associated with the default risk, a survival contingent forward measure is constructed.

Key words: swaption, CMS, affine term structure model, convexity adjustment, credit derivative, survival contingent measure

1 Introduction

Many financial institutions hold large portfolios of interest rate derivatives transactions. Therefore, an efficient calculation method is needed, not only for pricing specific transaction, but also for evaluation and risk management of the portfolio, with two features. (1) it is not computer-intensive for the valuation of a portfolio. (2) it can handle many types of products within the same model to avoid inconsistent valuations among products. In this paper, we develop easily implemented approximations of the prices of several interest rate related derivatives; swaptions, CMS, CMS options, and credit derivatives. The available models include general affine term structure models (Duffie and Kan, 1996), Gaussian Heath-Jarrow-Morton model (Heath et al., 1992) and quadratic interest rate models (Pelsser, 1997 and Ahn et al., 2002). The method is based on the Gram–Charlier expansion and the bond moments introduced by Collin-Dufresne and Goldstein (2002) (hereafter, "CDG"). There are the literature on an approximation technique for one specific product but, to our best knowledge, this paper is the first one to develop one technique applicable to many products.

A closed formula for swaptions has not been obtained in multi-factor models due to the difficulty of identifying the exercise boundary. By focusing on the exercise boundary carefully, several approximation methods have been proposed by Wei (1997), Brace (1997), Munk (1999), Singleton and Umantsev (2002). Schrager and Pelsser (2006) obtain good results of swaption pricing by approximating the stochastic volatility of a swap rate with deterministic one under the swap measure. To be sure, the swap measure approach of Jamshidian (1997) is useful for swaption pricing, but the relevant processes become difficult to use in association with the change of measure between a forward measure and a swap measure. Our approach differs from theirs in that we focus on zero-coupon bond prices as building blocks of the evaluation of all derivatives so that the approach is suitable to not only swaptions but also other products. A closely related literature to ours is CDG. They introduce the idea of bond moments and develop an approximation method of swaption prices under many forward measures associated with the swaption expiry and the cash-flow timing of the underlying swap. We simplify the methods of CDG by using the Gram–Charlier expansion under one forward measure rather than many forward measures. The coefficients are expressed by the cumulants of the underlying swap value, or equivalently by the bond moments. Our numerical studies confirm that our methods improve the results of CDG.

Approximation of the convexity adjustment of a CMS rate is discussed by Benhamou (2000), Pelsser (2003) and Hunt and Kennedy (2004). Benhamou (2000) derives the convexity adjustment for multi-factor Gaussian models by using a Wiener Chaos expansion. The linear swap model (LSM) of Hunt and Kennedy (2004) provides a convenient way to calculate the convexity adjustment of a CMS rate. The result is naturally related to the volatility of the swap rate. Pelsser (2003) unify the theories of similar adjustments as the side-effect of the change of numéraire. For general models under which the bond moments are available, we show convexity adjustments of CMS and CMS option prices with the bond moments by approximating the reciprocal of a duration (or PVBP) with a polynomial of zero-coupon bonds. Our results of CMS convexity adjustments are obtained in similar forms as Benhamou (2000). The point of our observation to express everything with zero-coupon bonds under a forward measure is in contrast with one of the LSM to represent a relative price of zero-coupon bond as a linear function of a swap rate under a swap measure. Our approach is closer to Benhamou (2000) in that point. Nevertheless, our results are consistent with LSM.

As the applications to credit derivatives, we present pricing of vulnerable options. It may be natural to think of a defaultable security as a numéraire in a defaultable environment. The corresponding martingale measure is called a survival measure which is investigated by Collin-Dufresne et al. (2004) and Schönbucher (2000a). However, it is not equivalent to the original risk-neutral measure since the numéraire gets worthless when default. On the other hand, a survival contingent measure is an equivalent martingale measure, and it is first proposed by Schönbucher (2000b). Our idea is that the pre-default price can be a numéraire after modifying it so that the relative price be a martingale under the original risk-neutral measure. We construct a survival contingent forward measure with a pre-default price of a fictitious defaultable asset. The choice of numéraire relies on not only the non-defaultability but also the recovery rule.

The rest of this paper is organized as follows. After setting up the model in Section 2, we develop approximation methods of swaptions by using the Gram–Charlier expansion and bond moments in Section 3. Then we provide CMS and CMS options based on these methods in Section 4. In Section 5, we discuss the survival contingent measure and the pricing of several vulnerable credit derivatives. Section 6 concludes the paper.

2 Setup

2.1 Gram–Charlier expansion

The Gram–Charlier expansion gives the density function of a random variable from the cumulants. In a similar way as Stuart and Ord (1987) who show an expansion for a random variable with the mean of zero, we derive the Gram–Charlier expansion around an arbitrary mean value in order to make the successive formulation easy-to-use.

First, we define the Hermite polynomial as $H_n(x) = (-1)^n \phi(x)^{-1} D^n \phi(x)$ with $H_0(x) = 1$, where $D = \frac{d}{dx}$ and $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.[1] The Hermite polynomials have the orthogonal property $\int_{-\infty}^{\infty} H_m(x) H_n(x) \phi(x) dx = \delta_{mn} n!$ with respect to the Gaussian measure which has a standard normal distribution. The Gram–Charlier expansion is an orthogonal decomposition with $\{H_n \phi\}_n$ of a density function. The proof of the following proposition is shown in the appendix.

Proposition 1. *Assume that a random variable Y has the continuous density function f and has cumulants c_k $(k \geq 1)$, all of which are finite and known. Then the followings hold.*

[1] By definition,

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3,$$
$$H_5(x) = x^5 - 10x^3 + 15x, \quad H_6(x) = x^6 - 15x^4 + 45x^2 - 15, \quad H_7(x) = x^7 - 21x^5 + 105x^3 - 105x.$$

(*i*) f *can be expanded as follows*

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right)\phi\left(\frac{x-c_1}{\sqrt{c_2}}\right), \quad \textit{where} \quad q_0 = 1, \quad q_1 = q_2 = 0,$$

$$q_n = \frac{1}{n!}E\left[H_n\left(\frac{Y-c_1}{\sqrt{c_2}}\right)\right] = \sum_{m=1}^{[n/3]}\sum\nolimits_{k_1+\cdots+k_m=n, k_i\geq 3} \frac{c_{k_1}\cdots c_{k_m}}{m!k_1!\cdots k_m!}\left(\frac{1}{\sqrt{c_2}}\right)^n, \quad (n \geq 3). \tag{1}$$

(*ii*) *for any* $a \in \mathbb{R}$,

$$E[1_{\{Y>a\}}] = N\left(\frac{c_1-a}{\sqrt{c_2}}\right) + \sum_{k=3}^{\infty}(-1)^{k-1}q_k H_{k-1}\left(\frac{c_1-a}{\sqrt{c_2}}\right)\phi\left(\frac{c_1-a}{\sqrt{c_2}}\right).$$

The advantages of the Gram–Charlier expansion are that it is written in additive form and the coefficients q_n are easily expressed by the given cumulants as follows

$$q_3 = \frac{c_3}{3!c_2^{3/2}}, \quad q_4 = \frac{c_4}{4!c_2^2}, \quad q_5 = \frac{c_5}{5!c_2^{5/2}}, \quad q_6 = \frac{c_6+10c_3^2}{6!c_2^3}, \quad q_7 = \frac{c_7+35c_3c_4}{7!c_2^{7/2}},$$

among which $3!q_3$ represents skewness and $4!q_4$ represents the excess kurtosis. The cumulants c_j can be calculated from the moments μ_j around zero[2].

The Gram–Charlier expansion is related to the Edgeworth expansion. Actually, in the proof of Proposition 1, the inverse Fourier transforms of both (22) and (23) yield the Edgeworth expansion

$$f(x) = \frac{1}{\sqrt{c_2}}\exp\left(\sum_{k=3}^{\infty}\frac{(-1)^k c_k}{k!}D^k\right)\phi\left(\frac{x-c_1}{\sqrt{c_2}}\right).$$

Thus, both the Gram-Charlier and the Edgeworth expansion give the same value when the summation is taken over infinite terms, but there may be a difference between the truncated sums which will be discussed later.

2.2 Affine Term Sructure Models

Our methodology in this paper can be applied to many models. For the tractability we apply the affine term structure models ("ATSMs", see Duffie and Kan, 1996) and assume the followings through the paper.

[2]See Stuart and Ord (1987). For example,

$$\begin{aligned}
c_1 &= \mu_1, \quad c_2 = \mu_2 - \mu_1^2, \quad c_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3, \quad c_4 = \mu_4 - 4\mu_1\mu_3 - 3\mu_2^2 + 12\mu_1^2\mu_2 - 6\mu_1^4,\\
c_5 &= \mu_5 - 5\mu_1\mu_4 - 10\mu_2\mu_3 + 20\mu_1^2\mu_3 + 30\mu_1\mu_2^2 - 60\mu_1^3\mu_2 + 24\mu_1^5,\\
c_6 &= \mu_6 - 6\mu_1\mu_5 - 15\mu_2\mu_4 + 30\mu_1^2\mu_4 - 10\mu_3^2 + 120\mu_1\mu_2\mu_3 - 120\mu_1^3\mu_3\\
&\quad + 30\mu_2^3 - 270\mu_1^2\mu_2^2 + 360\mu_1^4\mu_2 - 120\mu_1^6,\\
c_7 &= \mu_7 - 7\mu_1\mu_6 - 21\mu_2\mu_5 - 35\mu_3\mu_4 + 140\mu_1\mu_3^2 - 630\mu_1\mu_2^3 + 210\mu_1\mu_2\mu_4\\
&\quad - 1260\mu_1^2\mu_2\mu_3 + 42\mu_1^2\mu_5 + 2520\mu_1^3\mu_2^2 - 210\mu_1^3\mu_4 + 210\mu_2^2\mu_3 + 840\mu_1^4\mu_3 - 2520\mu_1^5\mu_2 + 720\mu_1^7.
\end{aligned}$$

The time horizon in our study is finite $[0, T^*]$ with some $T^* < \infty$. The relevant dates of financial contracts in question are $T_0 < T_1 < \cdots < T_N < \cdots \leq T^*$, which are set at regularly spaced time intervals, with $\delta = T_i - T_{i-1}$ for all i. We assume that tradable assets comprise zero coupon bonds and a money-market account, and that there is an equivalent spot martingale measure (or risk-neutral measure) Q. $(\Omega, \mathcal{F}, Q)$ is a complete probability space with an n-dimensional standard Brownian motion W, and $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T^*]\}$ is the augmented filtration generated by the Brownian motion W.

An n-dimensional Markov-state vector X satisfies a stochastic differential equation (SDE) ,

$$dX_t = K(\theta - X_t)dt + \Sigma D(X_t)dW_t,$$

where

$$\begin{aligned} &\alpha_i \in \mathbb{R} \quad (i = 1, 2, \ldots, n), \quad \theta \in \mathbb{R}^n, \quad \beta_i \in \mathbb{R}^n \quad (i = 1, 2, \ldots, n), \quad K \in \mathbb{R}^{n \times n} \\ &D(x) = \operatorname{diag}\left[\sqrt{\alpha_1 + \beta_1{}^\top x}, \ldots, \sqrt{\alpha_n + \beta_n{}^\top x}\right], \quad x \in \mathbb{R}^n, \end{aligned}$$

and $\Sigma \in \mathbb{R}^{n \times n}$ is a matrix such that $\Sigma\Sigma^\top$ is a positive definite matrix. The risk-free short rate is given as $r_t = \delta_0 + \delta_X^\top X_t$, where $\delta_0 \in \mathbb{R}$, $\delta_X \in \mathbb{R}^n$. The time-$t$ price of a zero coupon bond with a maturity date of T is denoted by $P(t, T)$. We assume that the T-forward measure Q^T exists for any $T \in [T_0, T^*]$.

It is well-known that under the ATSM the bond price is expressed in the form of an exponentially affine function,

$$P(t, T) = \exp(A(t, T) + B(t, T)^\top X(t)).$$

The Feynman–Kac formula yields the following system of ordinary differential equations of A and B

$$\begin{aligned} \frac{\partial}{\partial t} A(t, T) &= -(K\theta)^\top B(t, T) - \frac{1}{2} \sum_{j=1}^{n} (\Sigma^\top B(t, T))_j{}^2 \alpha_j + \delta_0, \quad A(T, T) = 0, \\ \frac{\partial}{\partial t} B(t, T) &= K^\top B(t, T) - \frac{1}{2} \sum_{j=1}^{n} (\Sigma^\top B(t, T))_j{}^2 \beta_j + \delta_X, \quad B(T, T) = 0. \end{aligned} \tag{2}$$

This system can be solved in a closed form for special cases and can be solved numerically in many other cases.

For given dates, the bond moment is defined under the T-forward measure as

$$\mu^T(t, T_0, \{T_1, \cdots, T_m\}) \equiv E^{T_0}\left[\prod_{i=1}^{m} P(T_0, T_i) \mid X_t\right].$$

As is the bond price, the bond moment is also an exponentially affine function of $X(t)$

$$\mu^T(t, T_0, \{T_1, \cdots, T_m\}) = \frac{\exp(M(t) + N(t)^\top X(t))}{P(t, T)},$$

where $M(t) = M(t, T, T_0, \{T_1, \ldots, T_m\})$ and $N(t) = N(t, T, T_0, \{T_1, \ldots, T_m\})$ satisfy the same system of ordinary differential equations as (2)

$$\begin{aligned}
\frac{\partial}{\partial t} M(t) &= -(K\theta)^\top N(t) - \frac{1}{2} \sum_{j=1}^{n} \left(\Sigma^\top N(t)\right)_j^2 \alpha_j + \delta_0, \\
\frac{\partial}{\partial t} N(t) &= K^\top N(t) - \frac{1}{2} \sum_{j=1}^{n} \left(\Sigma^\top N(t)\right)_j^2 \beta_j + \delta_X,
\end{aligned}$$

with the terminal conditions

$$M(T_0) = \sum_{i=1}^{m} A(T_0, T_i) + A(T_0, T), \quad N(T_0) = \sum_{i=1}^{m} B(T_0, T_i) + B(T_0, T).$$

For Gaussian ATSMs and uncorrelated CIR models, there exist explicit solutions of A, B, M, and N, which are given in the Appendix.

3 Swaptions

In this section we present a way to approximate a swaption price with the technique of the Gram–Charlier expansion and the bond moments.

3.1 Approximation

Consider a receiver's swaption with the expiry T_0 and the fixed rate K during a period $[T_0, T_N]$. By the linearity of the valuation, the value $SV(t)$ of the underlying swap at time t is written as a linear combination of the zero coupon bond prices

$$SV(t) = -P(t, T_0) + \delta K \sum_{i=1}^{N} P(t, T_i) + P(t, T_N) \equiv \sum_{i=0}^{N} a_i P(t, T_i),$$

where a_i is the amount of cash flow at time T_i. Then the swaption value $SOV(t)$ at time t is the discounted value of the expectation of the gain from exercising under the T_0-forward measure Q^{T_0}

$$SOV(t) = P(t, T_0) E^{T_0} \left[1_{\{SV(T_0) > 0\}} SV(T_0) \mid \mathcal{F}_t \right] = P(t, T_0) \int_0^\infty x f(x) dx, \tag{3}$$

where f is the density function of the swap value $SV(T_0)$ at the expiry date T_0 under the T_0-forward measure conditioned on $\mathcal{F}_t$. Therefore, it is enough to obtain the density function of the value of the underlying swap under the T_0-forward measure for the calculation of the swaption price.

For the application of the Gram–Charlier expansion to (3), we have only to calculate the coefficients q_n in (1) from the bond moments under a underlying model. Firstly, the m-th swap moment can be obtained with bond moments and cash flow amounts as

$$M_m(t) = E^{T_0} \left[SV(T_0)^m \mid X_t \right] = \sum_{0 \le i_1, \ldots, i_m \le N} a_{i_1} \cdots a_{i_m} \mu^{T_0} \left(t, T_0, \{T_{i_1}, \cdots, T_{i_m}\} \right).$$

Secondly, we can calculate the n-th swap cumulant $c_n(t)$ from the set of the swap moments $\{M_m(t)\}_m$. Define the weighted cumulant $C_n = c_n(t)P(t,T_0)^n$ for $n \geq 1$, and rewrite the coefficients q_n in (1) for $n \geq 3$ as

$$q_n = \sum_{m=1}^{[n/3]} \sum_{k_1+\cdots+k_m=n, k_i\geq 3} \frac{C_{k_1}\cdots C_{k_m}}{m!k_1!\cdots k_m!}\left(\frac{1}{\sqrt{C_2}}\right)^n.$$

Finally, by applying the Gram–Charlier expansion to $Y = SV(T_0)$ we have

$$SOV(t) = C_1 N\left(\frac{C_1}{\sqrt{C_2}}\right) + \sqrt{C_2}\phi\left(\frac{C_1}{\sqrt{C_2}}\right) + \sqrt{C_2}\phi\left(\frac{C_1}{\sqrt{C_2}}\right)\sum_{n=3}^{\infty}(-1)^n q_n H_{n-2}\left(\frac{C_1}{\sqrt{C_2}}\right).$$

Therefore, the truncated sum of the above formula yields an approximation of the swaption value.

Proposition 2. *The swaption value is approximated as*

$$SOV(t) \approx C_1 N\left(\frac{C_1}{\sqrt{C_2}}\right) + \sqrt{C_2}\phi\left(\frac{C_1}{\sqrt{C_2}}\right) + \sqrt{C_2}\phi\left(\frac{C_1}{\sqrt{C_2}}\right)\sum_{n=3}^{L}(-1)^n q_n H_{n-2}\left(\frac{C_1}{\sqrt{C_2}}\right),$$

to which we refer as the L-th order approximated price GCL.

Hence, the calculation of the swaption is reduced to the calculation of the bond moments $\mu^{T_0}(t, T_0, \{T_1, \ldots, T_m\})$. It is straightforward to extend this method to a short rate model where the initial yield curve is fitted to the observed curve in the market by adding a deterministic shift in the risk-free rate process (see Brigo and Mercurio, 2001), and/or where the volatility is a deterministic function of time. Similarly, this method can be applied when the value of the underlying asset of an option in question is represented as a polynomial of zero coupon bond prices, and the bond moments can be calculated numerically. The available models include general ATSMs, Gaussian Heath-Jarrow-Morton models (Heath et al., 1992) and quadratic interest rate models (Pelsser, 1997 and Ahn et al., 2002).

3.2 Comparison with Edgeworth expansion

It is worth mentioning other approaches based on the Edgeworth expansion which is closely related to the Gram–Charlier expansion.

In existing studies including CDG, the swaption value is often decomposed into cash-flow values weighted by the exercise probability under a forward measure associated with the swaption expiry and the cash-flow timing of the underlying swap,

$$SOV(t) = \sum_{i=0}^{N} a_i P(t,T_i) E^{T_i}\Big[1_{\{SV(T_0)>0\}} \mid \mathcal{F}_t\Big]. \quad (4)$$

CDG use a seventh-order Edgeworth expansion when calculating the probability of ending up in-the-money under each forward measure. They ignore higher terms than D^8 in the Taylor expansion of the exponential,

$$\begin{aligned} f(x) &\approx \frac{1}{\sqrt{c_2}} \exp\Big(\sum_{k=3}^{7} \frac{(-1)^k c_k}{k!} D^k\Big) \phi\Big(\frac{x-c_1}{\sqrt{c_2}}\Big) \\ &\approx \frac{1}{\sqrt{c_2}} \Big(1 + \sum_{k=3}^{7} \frac{(-1)^k c_k}{k!} D^k + \frac{1}{2}\big((\frac{c_3}{3!} D^3)^2 - 2\frac{c_3}{3!}\frac{c_4}{4!} D^3 D^4\big)\Big) \phi\Big(\frac{x-c_1}{\sqrt{c_2}}\Big) \\ &= \frac{1}{\sqrt{c_2}} \Big(1 + q_3 H_3 + q_4 H_4 + q_5 H_5 + q_6 H_6 + q_7 H_7\Big) \phi\Big(\frac{x-c_1}{\sqrt{c_2}}\Big), \end{aligned}$$

where $H_k = H_k\Big(\frac{x-c_1}{\sqrt{c_2}}\Big)$. This approximation itself is exactly same as the truncated sum of the Gram–Chariler expansion for the term $E^{T_i}\Big[1_{\{SV(T_0)>0\}} \mid \mathcal{F}_t\Big]$. In general, the approximation based on the Gram–Charlier expansion ignores the higher "standardized moments" q_n in (1), whereas the approximation based on the Edgeworth expansion ignores the higher cumulants c_n. In many practical applications, however, the finite sums become same after the Edgeworth expansion is futher approximated and reordered.

Although (3) and (4) bring the same values, they differ in the number of measures. (3) can be calculated under the T_0-forward measure Q^{T_0} only. On the other hand, CDG carry out their calculations in accordance with (4) under several forward measures. It follows that the accumulated approximation error in each expectation in (4) may be bigger than the error of (3), and using (3) will be faster in the computation of the relevant bond moments than (4). In order to reduce the computational time, CDG ignore c_6 and c_7 which are negligible in (1) relative to c_3^2 and $c_3 c_4$ in the calculation of q_6 and q_7. We call the method GC7d. Since the Gram–Charlier expansion is an orthnormal expansion, the error term cannot be evaluated analytically. In the next subsection on numerical examples, we compare the results of the two approaches.

3.3 Numerical examples

We consider the three ATSMs whose parameter values are specified in Table 1 for the numerical studies of swaptions and CMS in Sections 3 and 4. Model 1 (three factor Gaussian model) is constructed to build a lower yield and higher volatility environment than Model 2 (three factor Gaussian model) and Model 3 (two factor CIR model) whose parameter values are taken from CDG and Schrager and Pelsser (2006) for comparison purpose. Throughout the numerical examples in the paper, swaps and bonds are assumed to have semi-annual coupon payments[3]. Table 2 shows the option data implied by the three models, which include at-the-money-forward swap rates (ATMF), the ATMF receivers' swaption prices in basis points calculated by the Monte Carlo method, yield volatilities and absolute volatilities[4].

[Table 1 and Table 2 around here]

[3]It seems that CDG and Schrager and Pelsser (2006) assume annual coupon payments. Thus, the true prices are slightly different by a few basis points due to the different frequency.

[4]An absolute volatility is obtained as the yield volatility times the ATMF and it roughly represents the annual standard deviation of a particular swap rate to be observed.

The numerical performance of our methods is shown in Figures 1 - 4, Table 3 and 4. The price difference is calculated as the approximated price (4) based on the Gram–Charlier expansion ("GC price") with a specified approximation order minus the price from the Monte Carlo simulation ("MC price"). GC7d is obtained from GC7 with the sixth and seventh cumulants being set equal to 0 in (1) to save calculaion time as CDG apply. The MC price is obtained by simulating 400 million times (20 million runs multiplied by 20 to calculate MC error) with the negative correlation technique, and using Gaussian distribution of state variables at the expiry to avoid the discretizing error. The standard error is of the order of 10^{-6} for a one-year into 10-year swaption.

Figures 1, 2 and 3 illustrate absolute price differences for a one-year into a 10-year swaption across the stike rates under Model 1, Model 2 and Model 3, respectively. The numbers of the results for GC3 and GC6 are presented in Table 3. The order of the difference is basically determined by the underlying model and the volatility level. All price differences are within 0.3 bp under Model 1 (Figure 1) while they are within 0.01 bp under Model 2 (Figure 2) and within 0.10 bp under Model 3 (Figure 3). Figure 4 shows a result for a five-year into a 10-year swaption where one can observe a "zoomed" wave pattern of the result of a one-year into a 10-year swaption with higher difference by 1-digit. All of them give satisfactory numerical results for practical purposes.

Among several approximation orders, GC6 and GC7d show good results. Note that the higher-order approximations (GC4 and GC5) do not necessarily produce more accurate prices than lower order approximations (GC3) since the Gram–Charlier expansion is an orthogonal expansion. Table 4 shows price differences for ATMF swaptions for combinations of option expiries and swap maturities, based on GC6. It turns out that these performance is very competitive with other methods if one compares the results for Model 2 and 3 with ones of CDG and Schrager and Pelsser (2006). Figure 5 compares the performance of the CDG approach to our approach for coupon-bond option prices (a two-year option on 12-year bond) by using the same parameters used by CDG (Model 3). This figure confirms that each of our methods improves the results of CDG. Especially, a difference between GC7d and CDG implies an accumulated difference due to the number of forward measures. In addition, our results with GC6 compare favorly with Schrager and Pelsser (2006), except for for swaptions with short-dated maturities under Model 3.

The calculation time increases with the number of bond moments in use, or equivalently, the approximation order L and the time to maturity of underlying swap though it does not depend on the time to the option exipry so much. In our calculation for a one-year into a 5-year swaption under a three-factor Gaussian model, it took less than 10^{-3} seconds with up to GC7d, 0.063 seconds with GC7 and 78 seconds with MC, using Visual C in a 2.4 GHz Pentium 4 CPU. For a one-year into a 10-year swaption, the time was less than 10^{-3} seconds with GC3, 0.156 seconds with GC6 and GC7d, 3.453 seconds with GC7, and 109 seconds with MC. These speedy calculation of GC prices can be carried out with analytical solutions of the bond moments. For more general ATSMs than Models 1-3 such as a correlated multi-factor CIR model, one needs to solve a set of Ricatti equations numerically to obtain bond moments so that it will take much longer time. On the other hand, for quadratic Gaussian models, we expect to have a similar efficient approximation as Gaussian ATSMs. It is another feature of the bond moments that a value of a bond moment for a set of dates can be shared among derivative transactions which involve cash flows in the same dates. This fact allows us to evaluate a portfolio of derivative transactions efficiently with our methods and it differs from other valuation methods which need to evaluate each transaction in a portfolio independently.

We suggest either GC3, GC6 or GC7d for a practical application. By considering the accuracy and the computational time, we conclude that a higher order approximation GC6 or GC7d yields very accurate prices enough to price a specific transaction, and that a lower order one GC3 attains good approximation in a very short time so that it is suitable for the portfolio evaluation and the risk management.

[Figure 1 - 5, Table 3 - 4 around here]

4 CMS

Our approximation method can be applied to the convexity adjustment of a CMS rate and an option on CMS.

4.1 CMS

A CMS is a swap contract between two parties to exchange a fixed rate and a floating rate whose reference rate is a swap rate with a specified time to maturity. The fixed rate to be exchanged on a CMS is called the CMS rate.

We consider a CMS to be traded at time $t < T_0$ for the exchange of a fixed rate $CMSR(t)$ with the prevailing swap rates for a maturity of $M = m\delta$ in arrears during the period $[T_0, T_N]$. The fixing (observation) dates are T_{i-1} $(i = 1, \dots, N)$ and the payment dates are T_i. By the usual discussion, the fixed rate on the CMS is given by

$$CMSR(t) \quad = \quad \frac{\sum_{i=1}^{N} P(t,T_i) E^{T_i}[SR(T_{i-1}, T_{i-1}, M) \mid \mathcal{F}_t]}{\sum_{i=1}^{N} P(t,T_i)}, \tag{5}$$

where $SR(u, T_{i-1}, M)$ is the forward swap rate for the period $[T_{i-1}, T_{i-1} + M]$ at time u,

$$SR(u, T_{i-1}, M) = \frac{P(u, T_{i-1}) - P(u, T_{i-1} + M)}{\delta \sum_{j=i}^{i+m-1} P(u, T_j)}.$$

A swap rate is a martingale under the swap measure, but it may not be so under other measures since a swap rate is not a price of some traded security. The expectation of the swap rate $E^{T_i}[SR(T_{i-1}, T_{i-1}, M) \mid \mathcal{F}_t]$ may be close to the forward swap rate $SR(t, T_{i-1}, M)$ but they don't coincide in general. Thus, we call a difference between them the convexity adjustment in a broad sense (bCA)[5].

For a pricing of CMS, it is sufficient to consider the expectation of a swap rate starting on a particular future date in the numerator of (5). It is equivalent to the bCA of the single-period CMS rate

$$\text{bCA} = E^{T_1}[SR(T_0, T_0, M) \mid \mathcal{F}_t] - SR(t, T_0, M).$$

For that purpose let us consider a receiving swap with a coupon rate of $SR(t, T_0, M)$ initiated at time t for a period of $[T_0, T_0 + M]$. For $t \le u \le T_0$, recall that the time-u swap value is given by the rate difference times the duration

$$SV(u, S(t, T_0, M), T_0, M) = (SR(t, T_0, M) - SR(u, T_0, M)) Dur(u, T_0, M), \tag{6}$$

[5]bCA for the LIBOR is zero because a forward LIBOR of maturity date T_i is a martingale under the T_i-forward measure $SR(t, T_{i-1}, \delta) = E^{T_i}[S(T_{i-1}, T_{i-1}, \delta) \mid \mathcal{F}_t]$.

where $Dur(u, T_0, M) = \delta \sum_{j=1}^{m} P(u, T_j)$ is the duration (or PVBP) of the underlying swap. Then by dividing the both sides of the above equality evaluated at time $u = T_0$ by the duration and taking the expectations, we see the bCA is given by

$$E^{T_1}[SR(T_0, T_0, M) \mid \mathcal{F}_t] - SR(t, T_0, M) = -E^{T_1}\left[\frac{SV(T_0, S(t, T_0, M), T_0, M)}{Dur(T_0, T_0, M)}\middle|\mathcal{F}_t\right]. \quad (7)$$

A major problem in evaluating the CMS is that no general analytical expression exists for the right hand side of (7).

To overcome the problem, we propose an approximation with a polynomial of zero-coupon bond prices so as to calculate the expectation (7) with the bond moments. Let us denote the forward duration of the swap by

$$D(t, T_0, M) = \frac{\delta \sum_{j=1}^{m} P(t, T_j)}{P(t, T_0)}.$$

Thus, a random variable $Dur(T_0, T_0, M)$ will be distributed around $D(t, T_0, M)$[6]. By making use of a first-order approximation[7] $(1 + x)^{-1} \approx 1 - x$, one can approximate the reciprocal of the stochastic duration as a linear function of the zero-coupon bonds as

$$\frac{1}{Dur(T_0, T_0, M)} = \frac{D(t, T_0, M)^{-1}}{1 + \frac{Dur(T_0, T_0, M) - D(t, T_0, M)}{D(t, T_0, M)}} \approx \frac{2}{D(t, T_0, M)} - \frac{Dur(T_0, T_0, M)}{D(t, T_0, M)^2}. \quad (8)$$

Let us denote by a_j $(j = 0, \ldots, m)$ the equivalent bond cash flow at time T_j in the swap with the fixed rate of $S(t, T_0, M)$ under consideation,

$$a_j = \begin{cases} -1 & \text{if } j = 0, \\ \delta SR(t, T_0, M) & \text{if } 1 \leq j \leq m - 1, \\ 1 + \delta SR(t, T_0, M) & \text{if } j = m. \end{cases}$$

Then SV is also represented a linear function of the zero-coupon bonds

$$SV(T_0, S(t, T_0, M), T_0, M) = \sum_{j=0}^{m} a_j P(T_0, T_j). \quad (9)$$

By plugging (8) and (9) into (7), we obtain the expected value of the swap rate as

$$\begin{aligned} & E^{T_1}[SR(T_0, T_0, M) \mid \mathcal{F}_t] \\ \approx\ & SR(t, T_0, M) - \sum_{j=0}^{m} a_j \left(\frac{2\mu^{T_1}(t, T_0, \{T_j\})}{D(t, T_0, M)} - \delta \sum_{k=1}^{m} \frac{\mu^{T_1}(t, T_0, \{T_j, T_k\})}{D(t, T_0, M)^2} \right). \end{aligned}$$

Therefore, the bond moments allow us to calculate the convexity adjustment easily.

[6]This idea is similar to the low-variance martingales argument of Schrager and Pelsser (2006) who approximate $P(T_0, T_0 + M)/Dur(T_0, T_0, M) \approx P(t, T_0 + M)/Dur(t, T_0, M)$ under a swap measure.

[7]The second-order approximation $(1 + x)^{-1} \approx 1 - x + x^2$ is also applicable and the results can be easily modified accordingly

Proposition 3. *The convexity adjustment in a broad sense can be approximated as*

$$\begin{aligned} bCA \quad \approx \quad & \frac{2\left(1-\mu^{T_1}(t,T_0,\{T_m\})\right)}{D(t,T_0,M)} - \delta \sum_{k=1}^{m} \frac{\mu^{T_1}(t,T_0,\{T_k\}) - \mu^{T_1}(t,T_0,\{T_k,T_m\})}{D(t,T_0,M)^2} \\ & - \delta SR(t,T_0,M) \sum_{j=1}^{m} \left(\frac{2\mu^{T_1}(t,T_0,\{T_j\})}{D(t,T_0,M)} - \delta \sum_{k=1}^{m} \frac{\mu^{T_1}(t,T_0,\{T_j,T_k\})}{D(t,T_0,M)^2} \right). \quad (10) \end{aligned}$$

There are several related derivations of bCA. By (8) one may have

$$\begin{aligned} & E^{T_1}[SR(T_0,T_0,M) \mid \mathcal{F}_t] \approx E^{T_1} \left[\frac{1-P(T_0,T_m)}{D(t,T_0,M)} \left(2 - \frac{Dur(T_0,T_0,M)}{D(t,T_0,M)} \right) \Big| \mathcal{F}_t \right] \\ = \quad & \frac{2\left(1-\mu^{T_1}(t,T_0,\{T_m\})\right)}{D(t,T_0,M)} - \delta \sum_{k=1}^{m} \frac{\mu^{T_1}(t,T_0,\{T_k\}) - \mu^{T_1}(t,T_0,\{T_k,T_m\})}{D(t,T_0,M)^2}, \end{aligned}$$

which is the first two terms of (10). However, this formulation does not make a full use of the covariance among the zero-coupon bonds (or the second order bond moments) so that the equality of (7) may not be guranteed. On the other hand, the linear swap model (LSM)[8] of Hunt and Kennedy (2004) is consistent with (7). Under LSM, the relative price of a zero-coupon bond with respect to the duration is assumed to be a linear function of the swap rate

$$\frac{P(u,T_j)}{Dur(u,T_0,M)} \approx A + B_j SR(u,T_0,M), \quad t \le u \le T_0 \qquad (11)$$

with $A = \dfrac{1}{m\delta}$, $B_j = \dfrac{P(t,T_j) - ADur(t,T_0,M)}{SR(t,T_0,M)Dur(t,T_0,M)}$. The constants A and B_j are determined so that the identity $\sum_{j=1}^{m} P(u,T_j) = Dur(u,T_0,M)$ holds and the left hand side of (11) is a martingale under the swap measure. Then we see $\sum_{j=0}^{m} a_j A = SR(t,T_0,M)$, $\sum_{j=0}^{m} a_j B_j = -1$, and (7) holds. By applying (8) to the left hand side of (11), multiplying the cash flow a_j and taking the summation, we attain (10) again.

The bCA represents convexity adjustment with different timings for the observation, T_0, and the payment, T_1. We can consider a convexity adjustment with the same timing of the observation and the payment, and call it the convexity adjustment in a narrow sense (nCA). By noting that $E^{T_0}[SV(T_0,S(t,T_0,M),T_0,M) \mid \mathcal{F}_t] = 0$, equation (7) can be decomposed into two terms as

$$\begin{aligned} \text{bCA} \quad = \quad & -Cov^{T_0}[SV(T_0,S(t,T_0,M),T_0,M), Dur(T_0,T_0,M)^{-1} \mid \mathcal{F}_t] \\ & + \Big(E^{T_0}[SV(T_0,S(t,T_0,M),T_0,M)Dur(T_0,T_0,M)^{-1} \mid \mathcal{F}_t] \\ & \quad - E^{T_1}[SV(T_0,S(t,T_0,M),T_0,M)Dur(T_0,T_0,M)^{-1} \mid \mathcal{F}_t] \Big) \\ = \quad & \text{nCA} + \text{TA}. \end{aligned}$$

[8] Pelsser (2003) derives the convexity adjustment in terms of the volatility of the swap rates under LSM.

We call the first term $-Cov_t^{T_0}[\cdots,\cdots]$ the convexity adjustment in a narrow sense (nCA) that represents adjustment based on the same timing for the observation and the payment[9]

$$\begin{aligned}\text{nCA} &= -\delta \sum_{k=1}^{m} \frac{\mu^{T_0}(t,T_0,\{T_k\}) - \mu^{T_0}(t,T_0,\{T_k,T_m\})}{D(t,T_0,M)^2} \\ &\quad + \delta^2 SR(t,T_0,M) \sum_{j,k=1}^{m} \frac{\mu^{T_0}(t,T_0,\{T_j,T_k\})}{D(t,T_0,M)^2},\end{aligned}$$

which is obtained by replacing μ^{T_1} with μ^{T_0} in (10). The remaining term in the bracket, $E^{T_0}[\cdots] - E^{T_1}[\cdots]$, represents the timing adjustment (TA) due to the different timing of the payments.

Table 5 reports the convexity adjustments (bCA, nCA, and TA) of one-period CMS rates under the three models calculated by the first-order approximation. The longer the time to the observation or the longer the maturity of the observed swaps, the bigger the adjustments are. By comparing the results with the Monte Carlo method, we find that the pricing deviations reported in Tables 6 are at most 0.29 bp for Model 1, and less than 0.04 bp for Model 2 and 3. These results are very good though obviously the level of volatility affects the performance in a similar way as swaptions. When necessary such as extremely high volatility environment, the second-order approximation can be applied and will show relatively better performance.

[Table 3 and 4 around here]

4.2 CMS options

The approximated price of an option contract on a CMS can be obtained by combining the two methods to approximate a swaption price by the Gram–Charlier expansion in Section 3 and a convexity adjustments of a CMS rate with bond moments in the previous subsection. We discuss the approximated price of a CMS floor and a CMS swaption.

Let's consider a CMS floorlet first. A swap rate is observed at T_0 for a period of $M = m\delta$. The strike rate of the floor is K and the payment of the floor is made at T_1. The value is then given by

$$CMSF(t) = \delta P(t,T_1) E^{T_1}\left[\max(K - SR(T_0,T_0,M),0) \mid \mathcal{F}_t\right].$$

From (6) and (8), we can approximate the observed swap rate as an affine function of bond prices,

$$\begin{aligned}SR(T_0,T_0,M) &\approx SR(t,T_0,M) \\ &\quad - \frac{SV(T_0,S(t,T_0,M),T_0,M)}{D(t,T_0,M)}\left(2 - \frac{Dur(T_0,T_0,M)}{D(t,T_0,M)}\right). \quad (12)\end{aligned}$$

[9] Benhamou (2000) obtains nCA and bCA in a similar form as ours with Wiener Chaos expansion under multi-factor Gaussian models.

Then we can obtain an approximated price of the CMS floor as

$$\begin{aligned} & CMSF(t) \\ \approx\ & \delta P(t,T_1)E^{T_1}\Big[\max\Big(K - SR(t,T_0,M) + \frac{2}{D(t,T_0,M)}SV(T_0,SR(t,T_0,M),T_0,M) \\ & \quad - \frac{1}{D(t,T_0,M)^2}SV(T_0,SR(t,T_0,M),T_0,M)Dur(T_0,T_0,M),0\Big)\Big|\mathcal{F}_t\Big]. \end{aligned}$$

Since the underlying value is expressed as a polynomial of zero-coupon bond prices, the CMS floor price can be approximated further by using the Gram–Charlier expansion and the bond moments so as to obtain an approximated price. While it is straightforward to calculate the moments of the underlying rate, this requires higher order bond moments because of the quadratic term $SV(T_0)Dur(T_0)$.

We test an approximated price of 10-year CMS floors on five-year swap rates. The strike rate is 2 percent under Model 1 and 6 percent under Model 2. The true prices by the Monte-Carlo method are 528.3 bp, and 106.33 bp, respectively, while our method results in 525.8 bp with with GC3, and 106.31 bp with GC3, respectively.

Lastly, we consider a CMS swaption which is a right to enter into a CMS swap on M-maturity swap for a period of $[T_0, T_N]$ with the expiry T_0 and the strike rate of K. The arbitrage-free price can be given by

$$\begin{aligned} & CMSOV(t) \\ =\ & P(t,T_0)E^{T_0}\left[\max\left(\sum_{i=1}^{N}\delta P(T_0,T_i)(K - E^{T_i}[SR(T_{i-1},T_{i-1},M) \mid \mathcal{F}_{T_0}]),0\right) \mid \mathcal{F}_t\right]. \end{aligned}$$

Each coupon in the underlying swap can be approximated by taking the expectation of (12) to obtain the expression with bond moments. Therefore, one can apply the Gram-Charlier expansion to obtain the approximated price in a similar way as swaptions and CMS floorlets.

5 Vulnerable options

We apply the techniques developed so far to the valuation of the credit derivatives. The forward measure in a default-free setting is replaced by the survival contingent forward measure in the defaultable setting.

5.1 Default risk

We make more standing assumptions. There are three firms A, B, C subject to default risk[10]. The default time of firm i ($i = A, B, C$) is denoted by τ^i and the default indicator is denoted by $H_t^i = \mathbf{1}_{\{\tau^i \leq t\}}$. At time $t = 0$ all of these three firms are solvent almost surely. A filtration $\mathbb{G} = \{\mathcal{G}_t : t \in [0, T^*]\}$ is generated by the default indicators and $\mathbb{F}$; $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(H_s^A, H_s^B, H_s^C : 0 \leq s \leq t)$. Any $(Q, \mathbb{F})$-martingale is also a $(Q, \mathbb{G})$-martingale. H_t^i has the intensity $h^i(X_t)$ with an affine function $h^i(x) = l_0^i + {l_1^i}^\top x$, $(l_0^i, l_1^i) \in \mathbb{R} \times \mathbb{R}^n$, of the state vector X_t.

[10] The discussion here can be easily extended to a case of a finite number of defaultable firms.

The default time of firm i ($i = A, B, C$) τ^i is constructed by

$$\tau^i = \inf\left\{t \geq 0 : \int_0^t h^i(X_s)ds \geq \eta^i\right\},$$

where η^A, η^B, and η^C are independent random variables with a unit exponential law under Q. We assume that simultaneous defaults by any two firms do not occur almost surely; $Q(\tau^i = \tau^j) = 0$ ($i \neq j$). It will be convenient to introduce an auxiliary notation,

$$\tau^{ABC} = \min\{\tau^A, \tau^B, \tau^C\}.$$

An adjusted short rate R is a $\mathbb{F}$-predictable process defined by

$$R_t = r_t + \delta^A h_t^A + \delta^B h_t^B + \delta^C h_t^C$$

such that a process $\exp(-\int_0^t R_s ds)$ is square-integrable.

We study a defaultable security which pays Y at the maturity T. In case of default of firm i ($i = A, B, C$) prior to T, the security holder receives $1 - \delta^i$ times the pre-default price and the contract is terminated without further payments. Such a recovery rule is called a fractional recovery of market value (RMV).

Then one can show that the time-t price S_t of a defaultable security with the payment Y at the maturity T in case of no default subject to a fractional recovery of market value is given by

$$S_t = \mathbf{1}_{\{\tau^{ABC} > t\}} E\left[\exp\left(-\int_t^T R(X_u)du\right) Y \mid \mathcal{F}_t\right], \tag{13}$$

which is a natural extension of the well known result by Duffie and Singleton (1999) by recalling that $R_t = r_t + \delta^A h_t^A + \delta^B h_t^B + \delta^C h_t^C$. The proof is omitted since it can be accomplished in a similar fashion as Duffie and Huang (1996). When the defaultable security price is written as $S_t = \mathbf{1}_{\{\tau^{ABC} > t\}} V_t$ with some $\mathbb{F}$-adapted process V_t, we call V_t the pre-default price of the security. When considering the applications, a typical form of Y is a payoff from an option $Y = \max(G, 0)$ where $G = \sum_i a_i \widetilde{P}(T, T_i)$ with a pre-default price of some defaultable bonds $\widetilde{P}(T, T_i)$.

5.2 Change of measure

When the payoff is complicated, the valuation gets difficult in general and a change of measure to a kind of a forward measure turns out useful. For the construction of an equivalent measure, it is natural to consider a fictitious defaultable bond which pays 1 at maturity T if all of firms A, B and C are solvent. If either firm i defaults prior to the maturity, the bond pays $1 - \delta^i$ times the pre-default price as the recovery.

By (13), the pre-default price $F(t, T)$ at time t is given by

$$F(t,T) = E\left[\exp\left(-\int_t^T R(X_s)ds\right) \mid \mathcal{F}_t\right], \tag{14}$$

which satisfies a SDE

$$dF(t,T) = F(t,T)R(X_t)dt + F(t,T)\sigma^F(t,T)^\top dW_t,$$

for some $\mathbb{F}$-progressively measurable process $\sigma^F(\cdot,T):[0,T^*]\times\Omega\to\mathbb{R}^n$. For an arbitrary fixed T let us define a process $\widetilde{L}^T$ by

$$\widetilde{L}_t^T = \frac{F(t\wedge T,T)\widetilde{\Lambda}_{t\wedge T}}{\exp\left(\int_0^{t\wedge T} r(X_s)ds\right)F(0,T)},$$
$$\widetilde{\Lambda}_t = \exp\left(-\int_0^t \left(\delta^A h^A(X_s)+\delta^B h^B(X_s)+\delta^C h^C(X_s)\right)ds\right).$$

$\widetilde{\Lambda}$ is constructed so that the relative price of a modified fictitious bond price $F(t,T)\widetilde{\Lambda}_t$ is a martingale under Q. The modification is made due to the RMV feature with non-zero recovery and the difference between the price and the pre-default price (or the jump on the default). Moreover, by using Ito's lemma, we see that $d\widetilde{L}_t^T=\widetilde{L}_t^T\sigma^F(t,T)^\top dW_t$. By the above observation and the assumptions, $\widetilde{L}^T$ can be a density process to define an equivalent measure $\widetilde{Q}^T$ with respect to Q by

$$\frac{d\widetilde{Q}^T}{dQ}\Big|_{\mathcal{G}_t}=\widetilde{L}_t^T. \tag{15}$$

We call $\widetilde{Q}^T$ the T-survival contingent forward measure for the RMV and we will use the symbol $\widetilde{E}^T$ for the expectation operator with respect to $\widetilde{Q}^T$. It should be emphasized that the measure $\widetilde{Q}^T$ depends on the recovery rule upon default of firms A,B and C although we omit the dependence in the notation. By construction, $\widetilde{Q}^T$ is the corresponding martingale measure with a numéraire of a modified pre-default bond price $F(t,T)\widetilde{\Lambda}_t$.

To obtain a better understanding of the change of the measure, it is useful to see how the processes are transformed. One can see that the change of measure transforms the Brownian motion only and keeps the jump process unaffected. Namely,

$$W_t^T=W_t-\int_0^t\sigma^F(s,T)ds$$

is a Brownian motion under $\widetilde{Q}^T$, and the process h^i remains the intensity of H^i under $\widetilde{Q}^T$.

By the change of measure (15), the price S of a contingent claim which pays Y at time T with the recovery payoff based on fractional recovery of market value for firms A,B, and C is given by

$$S_t=\mathbf{1}_{\{\tau^{ABC}>t\}}F(t,T)\widetilde{E}^T\left[Y\mid\mathcal{F}_t\right],\quad t<T. \tag{16}$$

The pre-default price of a contingent claim with the RMV is expressed under the survival contingent forward measure $\widetilde{Q}^T$ as if $F(t,T)$ were a bond price.

The importance of change of numéraire cannot be overstressed. By construction, for any asset price S, the relative price $S/(F(\cdot,T)\Lambda)$ is a $(\widetilde{Q}^T,\mathbb{G})$-martingale. On the other hand, the pre-default price V of a contingent claim paying Y with maturity T with the same RMV is written as $V_t=F(t,T)\widetilde{E}^T[Y\mid\mathcal{F}_t]$. Hence, the relative price $V/F(\cdot,T)$ is a $(\widetilde{Q}^T,\mathbb{F})$-martingale thus it is also a $(\widetilde{Q}^T,\mathbb{G})$-martingale. The fictitious bond price $F(\cdot,T)$ plays the same role as a numéraire for pre-default prices of T-maturity contingent claims with RMV. However, it is not a numéraire for other claims.

The idea of a survival contingent measure is first proposed by Schönbucher (2000b) for the case of zero-recovery. The contribution of this paper is to extend to more general cases and give two meanings in the change of measure to the survival contingent forward measure $\widetilde{Q}^T$. One meaning is that it is associated with the change of numéraire to the pre-default price of a fictitious bond. Another one is that it reflects the possibility of a change of cash flow schedule before the maturity due to a default and the recovery rule.

The survival contingent forward measure $\widetilde{Q}^T$ should not be confused with a survival forward measure $\overline{Q}^T$ which is introduced by Schönbucher (2000a). A survival contingent measure is equivalent while a survival measure is only absolutely continuous and is not equivalent. The survival forward measure $\overline{Q}^T$ is defined by the density process

$$\overline{L}_t^T = \frac{\overline{F}(t,T)\overline{\Lambda}_t}{\exp\left(\int_0^t r(X_s)ds\right)\overline{F}(0,T)},$$

where $\overline{F}(t,T) = \mathbf{1}_{\{\tau^{ABC}>t\wedge T\}}F(t,T)$ is the price of the fictitious zero coupon bond with the RMV feature and

$$\overline{\Lambda}_t = \exp\left(\int_0^t \left((1-\delta^A)h^A(X_s) + (1-\delta^B)h^B(X_s) + (1-\delta^C)h^C(X_s)\right)ds\right)$$

is a modifying term reflecting the RMV feature.

In a similar way as Schönbucher (2004), we can give a justification of the name of a survival contingent measure. It is clear that the survival contingent forward measure $\widetilde{Q}^T$ and the survival forward measure $\overline{Q}^T$ are constructed in a different way. However, these two measures are identical if the domain is restricted to the sub σ-field $\mathcal{F}_t$ $(t \le T)$ because one can show that they assign the same probability $\overline{Q}^T(G) = \widetilde{Q}^T(G)$ for any event $G \in \mathcal{F}_t$ by making use of a well-known formula

$$E\left[\mathbf{1}_{\{\tau^{ABC}>T\}}Y \mid \mathcal{G}_t\right] = \mathbf{1}_{\{\tau^{ABC}>t\}}E\left[\exp\left(-\sum_{i=A,B,C}\int_t^T h_s^i ds\right)Y \mid \mathcal{F}_t\right], \qquad (17)$$

which holds for any $\mathcal{F}_T$-measurable, integrable random variable Y and any $t \le T$. It implies that given no default, these two measures assign the same probability. But for other events involving a default they assign different probabilities. That is a justification of the name of a survival contingent measure. The relationship (17) is an instructive expression that leads us to a survival contingent forward measure rather than a survival measure.

5.3 Vulnerable option on a defaultable coupon bond

In this subsection and the subsequent one we will see that regardless of the complicated valuation form of vulnerable options, the pricing formulae become tractable in the computation owing to the affine structure and the change of measure to the survival contingent forward measure.

First, we consider the following vulnerable option on a defaultable bond:

- The firm A buys from the firm B a defaultable option (either call or put) with strike price K and the expiry T_0 on a defaultable coupon bond issued by the firm C.

- The bond pays a_j at T_j $(j = 1, \cdots, N)$ as long as C is solvent. The firm C also issues zero coupon bonds with maturity T_j $(j = 1, \cdots, N)$. In case of the default of C the recovery payoff of these bonds is subject to RT, that is, the bond of unit face amount is assumed to be replaced by the corresponding default-free bonds with face amount of $1 - \delta^C_{bond}$ upon the default of C. [11]

- If a firm i $(i = A, B, C)$ defaults before the expiry T_0, the seller B pays $1 - \delta^i$ times the market value prior to the default to the buyer A and the option contract is terminated. We set $\delta^A = 0$.

Time t-price of the zero coupon bond with the maturity date T issued by C is denoted by $P^C(t, T)$. Since the bond is subject to RT, it is well known that $P^C(t, T)$ is represented as $P^C(t, T) = \mathbf{1}_{\{\tau^C > t\}} \widetilde{P}^C(t, T)$ where $\widetilde{P}^C(t, T)$ is the pre-default price of the bond that is a linear combination of default-free bond price and defaultable bond price with zero recovery

$$\widetilde{P}^C(t, T) = (1 - \delta^C_{bond}) E\left[e^{-\int_t^T r(X_s)ds} \mid X_t\right] + \delta^C_{bond} E\left[e^{-\int_t^T \left(r(X_s) + h^C(X_s)\right)ds} \mid X_t\right]. \quad (18)$$

This bond price is expressed as a linear combination of exponentially affine functions of the state variable X_t.

A defaultable coupon bond has cash flows of a_j at T_j $(j = 1, \cdots, N)$ as long as C is solvent. Then we obtain the price S_t of the vulnerable put option on a defaultable coupon bond as $S_t = \mathbf{1}_{\{\tau^{ABC} > t\}} V_t$ with

$$V_t = F(t, T_0) \widetilde{E}^{T_0}\left[\max\left(K - \sum_{j=1}^N a_j \widetilde{P}^C(T_0, T_j), 0\right) \mid X_t\right]. \quad (19)$$

This is the same form as a swaption valuation. Since X is an affine diffusion, the price $F(t, T)$ can be written as $F(t, T) = \exp\left(\alpha(t, T) + \beta(t, T)^\top X_t\right)$ with some deterministic functions α, β. Hence the state vector X is also an affine diffusion under $\widetilde{Q}^T$ and we can calculate the approximated price of the option (19) with the Gram–Charier expansion and bond moments as discussed earlier.

5.4 Vulnerable option on CDS

Let's consider a vulnerable option contract to enter into the following credit default swap (CDS):

- A buys from B an option to enter into a CDS which starts at T_0.

- The reference bond on the CDS is a zero coupon bond with maturity U issued by C $(U \geq T_N)$ subject to fractional recovery of Treasury.

- As the premium the CDS buyer A pays κ to the CDS seller B at T_j $(j = 1, \cdots N)$ if C survives at T_j ; $\tau^C > T_j$.

- As the protection B pays $a + bP(T_j, U)$ to A at T_j if $T_{j-1} < \tau^C \leq T_j$.

[11] Other recovery rules such as RMV can be applied to these bonds issued by C if appropriate modifications are made on the successive discussion.

- If a firm i ($i = A, B, C$) defaults before the expiry T_0, the seller B pays $1 - \delta^i$ times the market value prior to the default to the buyer A and the option contract is terminated whereas we assume $\delta^A = 0$.

In order to make our discussion simple, the vulnerability is only up to the exercise time T_0 of the option and we don't assume the default of A and B after the CDS starts. Although the above option is a call-type option in the sense that the option buyer becomes the CDS buyer upon the exercise, a put-type option contract can be discussed in a similar fashion.

The payoff at T_0 upon the exercise of the call option is given by

$$E\left[\sum_{j=1}^{N} e^{-\int_{T_0}^{T_j} r(X_u)du}\left(-\kappa \mathbf{1}_{\{\tau^C > T_j\}} + \mathbf{1}_{\{T_{j-1} < \tau^C \le T_j\}}\left(a + bP(T_j, U)\right)\right) \mid \mathcal{G}_{T_0}\right].$$

To simplify this expression it is convenient to introduce a function

$$G(t, S, T) = E\left[\exp\left(-\int_t^S r(X_u)du - \int_t^T h^C(X_u)du\right) \mid X_t\right], \tag{20}$$

which is an exponentially affine function of the state vector X_t. Then by (17) the payoff upon the exercise can be written as $\mathbf{1}_{\{\tau^{ABC} > T_0\}} H$ where

$$\begin{aligned} H \quad = \quad & \sum_{j=1}^{N}\Big(-\kappa G(T_0, T_j, T_j) + a\Big(G(T_0, T_j, T_{j-1}) - G(T_0, T_j, T_j)\Big) \\ & + b\left(G(T_0, U, T_{j-1}) - G(T_0, U, T_j)\right)\Big). \end{aligned}$$

Hence the price of the option on the CDS is given by

$$S_t \quad = \quad \mathbf{1}_{\{\tau^{ABC} > t\}} F(t, T_0) \widetilde{E}^{T_0}\left[\max\left(H, 0\right) \mid X_t\right]. \tag{21}$$

Thanks to the form of G, this option price can also be approximated by the Gram–Charlier expansion and bond moments.

5.5 Numerical examples

In order to see numerical examples of these vunerable options discussed in Sections 5.3 and 5.4, let us assume that the risk-free rate and the intensities[12] are independent CIR processes satisfying

$$\begin{aligned} dr(t) \quad &= \quad K_r(\theta_r - r(t))dt + \sigma_r\sqrt{r(t)}dW_r(t), \\ dh^i(t) \quad &= \quad K_i(\theta_i - h^i(t))dt + \sigma_i\sqrt{h^i(t)}dW_i(t), \quad (i = B, C), \end{aligned}$$

where (W_r, W_B, W_C) is a three-dimensional standard Brownian motion, and

$$\begin{array}{rllll} r(0) = 0.03, & K_r = 0.5, & \theta_r = 0.03, & \sigma_r = 0.15, & \\ h^B(0) = 0.03, & K_B = 0.4, & \theta_B = 0.03, & \sigma_B = 0.10, & \delta^B = 0.6, \\ h^C(0) = 0.04, & K_C = 0.3, & \theta_C = 0.04, & \sigma_C = 0.10, & \delta^C = \delta^C_{bond} = 0.6. \end{array}$$

[12] Since the parameter δ^A is assumed zero in the previous subsections, the process h^A does not matter.

For a Markov process y, let us define

$$P(t,T;y) = E\left[\exp\left(-\int_t^T y(s)ds\right) \mid y(t)\right],$$

which is the zero-coupon bond price with maturity T by regarding y as the risk-free rate. Then we can write (14), (20) and (18) as,

$$\begin{aligned} F(t,T) &= P(t,T;r)P(t,T;\delta^B h^B)P(t,T;\delta^C h^C), \\ G(t,S,T) &= P(t,S;r)P(t,T;h^C), \\ \widetilde{P}^C(t,T) &= (1-\delta^C_{bond})P(t,T;r) + \delta^C_{bond}G(t,T,T). \end{aligned}$$

For the vulnerable option on the defaultable coupon-bearing bond in Section 5.3, the pre-default price $\widetilde{P}^C(t,T)$ is a polynomial of exponentially affine functions of r and h^C. It follows that the bond moments can be calculated analytically and we can work for (19). As the example we consider a 6-year 6% coupon bearing bond of which the current price is 103.792. Table 7 and Figure 6 show the price differences of 1-yr vulnerable put options on the bond with several strike prices. Similar features as swaptions are observed. As a reference, we obtain the at-the-money-forward price K_{ATMF} for the vulnerable option on the coupon-bearing bond as

$$K_{ATMF} \equiv \widetilde{E}^{T_0}\left[\widetilde{P}^C(T_0,T)\right] = \textstyle\sum_{j=1}^N a_j\left((1-\delta^C_{bond})\frac{P(0,T_j;r)}{P(0,T_0;r)} + \delta^C_{bond}J(0,T_0,T_j,T_j)\right),$$

where a_j is the cash flow on T_j $(j = 1, 2, \cdots, N)$ from the bond, and

$$\begin{aligned} J(t,T_0,S,T) &\equiv \widetilde{E}^{T_0}\left[G(T_0,S,T) \mid X_t\right] = \frac{P(t,S;r)Q(t,T_0,T;\delta^C,h^C)}{P(t,T_0;r)P(t,T_0;\delta^C h^C)}, \\ Q(t,T,U;\delta^C,h^C) &= E\left[\exp\left(-\int_t^T \delta^C h^C(s)ds\right)P(T,U;h^C) \mid h^C(t)\right]. \end{aligned}$$

$Q(t,T;\delta^C,h^C)$ for a CIR process h^C can be calculated via Laplace transform and the formula is found in Theorem 4.8 of Cairns (2004).

For a vulnerable CDS option with $a = 0, b = 1$ in Section 5.4, the ATMF rate of the underlying CDS is

$$\kappa_{ATMF} = \frac{J(0,T_0,U,T_0) - J(0,T_0,U,T_N)}{\sum_{j=1}^N J(0,T_0,T_j,T_j)}.$$

We take 5-yr into 5-yr CDS options to buy the protection as the example. Table 8 shows the prices of vulnerable CDS options. The quality is at a satisfactory level for practical purposes.

[Table 7 and Figure 6 around here]

[Table 8 around here]

6 Conclusion

We have demonstrated that the Gram–Charlier expansion and bond moments are so powerful that we can develop easy-to-use approximation methods for pricing several interest rate derivatives and credit derivatives. The accuracy and the computational speed is very competitive. Therefore, our methods can be applied to not only a pricing of a transaction but also an evaluation of a portfolio with keeping model consistency among products.

Acknowledgements

The authors are grateful to two anonymous referees whose detailed comments enabled us to make significants improvements to the paper.

References

[1] Ahn, D.H., Dittmar, R.F. and Gallant, A.R., Quadratic Term Structure Models: Theory and Evidence. *Review of Financial Studies*, 2002, **15**, 243–288.

[2] Benhamou, E., Pricing Convexity Adjustment with Wiener Chaos. Working paper, London School of Economics, 2000.

[3] Brace, A., Rank-2 Swaption Formulae. Working paper, University of New South Wales, 1997.

[4] Brigo, D. and Mercurio, F., A deterministic-shift extension of analytically-tractable and time-homogeneous short-rate models. *Finance and Stochastics*, 2001, **5**, 369–387.

[5] Cairns, A., *Interest Rate Models: An Introduction*, 2004, Princeton University Press.

[6] Collin-Dufresne, P. and Goldstein, R., Pricing Swaptions Within an Affine Framework. *Journal of Derivatives*, 2002, **10**, 1–18.

[7] Collin-Dufresne, P., Goldstein, R. and Hugonnier, J., A general formula for pricing defaultable securities. *Econometrica*, 2004, **72**, 1377–1407.

[8] Duffie, D. and Huang, M., Swap rates and credit quality. *Journal of Finance*, 1996, **51**, 921–949.

[9] Duffie, D. and Kan, R., A Yield Factor Model of Interest Rates. *Mathematical Finance*, 1996, **6**, 379–406.

[10] Duffie, D. and Singleton, K., Modeling term structures of defaultable bonds. *Review of Financial Studies*, 1999, **12**, 687–720.

[11] Heath, D., Jarrow R. and A. Morton A., Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, 1992, **60**, 77–105.

[12] Jamshidian, F., LIBOR and Swap Market Models and Measures. *Finance and Stochastics*, 1997, **1**, 293–330.

[13] Hunt, P. and Kennedy, J., *Financial Derivatives in Theory and Practice*, 2004 John Wiley & Sons Ltd.

[14] Munk, C., Stochastic Duration and Fast Coupon Bond Option Pricing in Multi-Factor Models. *Review of Derivatives Research*, 1999, **3**, 157–181.

[15] Pelsser A., A tractable yield-curve model that guarantees positive interest rates. *Review of Derivatives Research*, 1997, **1**, 269–284.

[16] Pelsser A., Mathematical foundation of convexity correction. *Quantitative Finance*, 2003, **3**, 59–65.

[17] Schönbucher, P., A libor market model with default risk. Working Paper, ETH Zurich, 2000a.

[18] Schönbucher, P., Credit risk modelling and credit derivatives. PhD thesis, Bonn University, 2000b.

[19] Schönbucher, P., A measure of survival. *RISK*, 2004 August, 79–85.

[20] Schrager, D. and Pelsser, A., Pricing Swaptions and Coupon Bond Options in Affine Term Structure Models. *Mathematical Finance*, 2006, **16**, 673–694.

[21] Singleton, K. and Umantsev, L., Pricing Coupon-Bond Options and Swaptions in Affine Term Structure Models. *Mathematical Finance*, 2002, **12**, 427–446.

[22] Stuart, A. and Ord. J.K., *Kendall's Advanced Theory of Statistics, Volume 1 Distribution Theory*, 1987, Oxford University Press.

[23] Wei, J., A Simple Approach to Bond Option Pricing. *Journal of Futures Markets*, 1997, **17**, 131–160.

A Proof of Proposition 1

The characteristic function G_Y of a random variable Y is defined by the Fourier transform of f as

$$G_Y(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx = e^{itc_1} \int_{-\infty}^{\infty} e^{i\sqrt{c_2}tx} \sqrt{c_2} f(c_1 + \sqrt{c_2}x)dx. \tag{22}$$

On the other hand, by the definitions of the cumulants, this can be expressed as

$$G_Y(t) = \exp\left(\sum_{k=1}^{\infty} \frac{c_k}{k!}(it)^k\right) = e^{itc_1} \int_{-\infty}^{\infty} e^{i\sqrt{c_2}tx} \exp\left(\sum_{k=3}^{\infty} \frac{(-1)^k c_k}{k!} \left(\frac{D}{\sqrt{c_2}}\right)^k\right) \phi(x)dx. \tag{23}$$

This is because, for any sequence $\{a_n\}$, it holds that

$$\exp\left(-\frac{c_2}{2}t^2 + \sum_{n=1}^{\infty} a_n(-i\sqrt{c_2}t)^n\right) = \int_{-\infty}^{\infty} e^{i\sqrt{c_2}tx} \exp\left(\sum_{n=1}^{\infty} a_n D^n\right) \phi(x)dx.$$

We further expand the integrand of (23) by using the Taylor expansion. We then reorder the terms as follows

$$\begin{aligned}
&\exp\left(\sum_{k=3}^{\infty} \frac{(-1)^k c_k}{k!} \left(\frac{D}{\sqrt{c_2}}\right)^k\right) \phi(x) \\
= &\left(1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{k=3}^{\infty} \frac{(-1)^k c_k}{k!} \left(\frac{D}{\sqrt{c_2}}\right)^k\right)^m\right) \phi(x) \\
= &\left(1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{k_1,\cdots,k_m \geq 3} \frac{(-1)^{k_1+\cdots+k_m} c_{k_1} \cdots c_{k_m}}{k_1! \cdots k_m!} \left(\frac{D}{\sqrt{c_2}}\right)^{k_1+\cdots+k_m}\right) \phi(x) \\
= &\left(1 + \sum{}^{*} \frac{c_{k_1} \cdots c_{k_m}}{m! k_1! \cdots k_m!} \left(\frac{1}{\sqrt{c_2}}\right)^n H_n(x)\right) \phi(x),
\end{aligned}$$

where $\sum^* = \sum_{n=3}^{\infty} \sum_{m=1}^{[n/3]} \sum_{k_1+\cdots+k_m=n, k_i \geq 3}$. We use the relationship $H_n(x)\phi(x) = (-1)^n D^n \phi(x)$ in the last equality. Then, (23) can be written as

$$e^{itc_1} \int_{-\infty}^{\infty} e^{i\sqrt{c_2}tx} \phi(x)dx + e^{itc_1} \int_{-\infty}^{\infty} e^{i\sqrt{c_2}tx} \sum{}^{*} \frac{c_{k_1} \cdots c_{k_m}}{m! k_1! \cdots k_m!} \left(\frac{1}{\sqrt{c_2}}\right)^n H_n(x)\phi(x)dx. \tag{24}$$

By using the inverse Fourier transforms of both (22) and (24) and by changing the relevant variable, we obtain the following Gram–Charlier expansion around the mean [13]

$$f(x) = \frac{1}{\sqrt{c_2}} \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right) + \frac{1}{\sqrt{c_2}} \sum{}^{*} \frac{c_{k_1} \cdots c_{k_m}}{m! k_1! \cdots k_m!} \left(\frac{1}{\sqrt{c_2}}\right)^n H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right).$$

The proof of (*ii*) is straightforward by using (*i*) and the properties of Hermite polynomials.

[13] For the density function of a standardized random variable, an expansion around zero $f(x) = \sum_{k=0}^{\infty} q_k H_k(x)\phi(x)$, where $q_k = \frac{1}{k!}E[H_k(Y)] = \sum_{l=0}^{[k/2]} \frac{(-1)^l}{l!(k-2l)!2^l} E[Y^{k-2l}]$, is known as a Gram–Charlier series of type A (Stuart and Ord, 1987).

B Affine term structure models

B.1 $\mathbb{A}_0(n)$ Gaussian Model

The coefficients of an n-factor Gaussian model, $\mathbb{A}_0(n)$, are given by

$$\begin{aligned}
&\delta_X = \mathbf{1}_n, \quad K = \mathrm{diag}[K_1, \ldots, K_n], \\
&\Sigma = \mathrm{diag}[\sigma_1, \ldots, \sigma_n]V, \quad \text{where } VV^\top = (\rho_{ij})_{ij}, \quad D(X(t)) = I_n.
\end{aligned}$$

Bond prices and bond moments can be obtained from

$$\begin{aligned}
A(t,T) &= -(T-t)\Big(\delta_0 + \sum_{i=1}^{n}(1 - D(K_i(T-t)))\theta_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\rho_{ij}\sigma_i\sigma_j}{K_iK_j} \\
&\quad \times \big(1 - D(K_i(T-t)) - D(K_j(T-t)) + D((K_i+K_j)(T-t))\big)\Big), \\
B_j(t,T) &= -\tau D(K_j(T-t)), \\
M(t) &= A(t,T_0) + F_0 + \tau\sum_{j=1}^{n} K_j\theta_j F_j D(K_j\tau) + \frac{\tau}{2}\sum_{i,j=1}^{J}\rho_{ij}\sigma_i\sigma_j\Big(F_iF_jD((K_i+K_j)\tau) \\
&\quad + F_i\frac{D((K_i+K_j)\tau) - D(K_i\tau)}{K_j} + F_j\frac{D((K_i+K_j)\tau) - D(K_j\tau)}{K_i}\Big), \\
N_j(t) &= B_j(t,T_0) + F_j\exp(-K_j(T_0-t)),
\end{aligned}$$

where $D(x) = \frac{1-e^{-x}}{x}$, $\tau = T_0 - t$, $F_0 = \sum_{i=1}^m A(T_0,T_i) + A(T_0,T)$ and $F_j = \sum_{i=1}^m B_j(T_0,T_i) + B_j(T_0,T)$.

B.2 $\mathbb{A}_n(n)$ CIR Model

The coefficients of an n-factor CIR model, $\mathbb{A}_n(n)$, are given by

$$\begin{aligned}
&\delta_X = 1_n, \quad K = \mathrm{diag}[K_1, \ldots, K_n], \quad \theta = (\theta_1, \ldots, \theta_n)^\top, \quad (\theta_j > 0), \\
&\Sigma = \mathrm{diag}[\sigma_1, \ldots, \sigma_n], \quad D(X(t)) = \mathrm{diag}[\sqrt{X_1(t)}, \ldots, \sqrt{X_n(t)}].
\end{aligned}$$

Bond prices and bond moments can be obtained from

$$\begin{aligned}
A(t,T) &= -\delta_0(T-t) - \sum_{j=1}^{n} K_j\theta_j\Big[\frac{2}{\sigma_j^2}\ln\frac{(K_j+\gamma_j)(e^{\gamma_j(T-t)}-1) + 2\gamma_j}{2\gamma_j} + \frac{2}{K_j - \gamma_j}(T-t)\Big], \\
B_j(t,T) &= \frac{-2(e^{\gamma_j(T-t)}-1)}{(K_j+\gamma_j)(e^{\gamma_j(T-t)}-1) + 2\gamma_j}, \\
M(t) &= F_0 - \delta_0\tau - \sum_{j=1}^{n} K_j\theta_j\Big[\frac{2}{\sigma_j^2}\ln\frac{(K_j+\gamma_j-\sigma_j^2F_j)(e^{\gamma_j\tau}-1) + 2\gamma_j}{2\gamma_j} \\
&\quad + \frac{(K_j+\gamma_j)F_j + 2}{K_j - \gamma_j - \sigma_j^2F_j}\tau\Big], \\
N_j(t) &= \frac{-((K_j-\gamma_j)F_j + 2)(e^{\gamma_j\tau}-1) + 2\gamma_jF_j}{(K_j+\gamma_j-\sigma_j^2F_j)(e^{\gamma_j\tau}-1) + 2\gamma_j},
\end{aligned}$$

where $\gamma_j = \sqrt{K_j^2 + 2\sigma_j^2}$, $\tau = T_0 - t$, $F_0 = \sum_{i=1}^m A(T_0,T_i) + A(T_0,T)$ and $F_j = \sum_{i=1}^m B_j(T_0,T_i) + B_j(T_0,T)$.

Table 1: Parameter values

	Model 1 (Gaussian)	Model 2 (Gaussian)	Model 3 (CIR)
δ_0	-0.0065	0.06	0.02
δ	$[1 \quad 1 \quad 1]^\top$	$[1 \quad 1 \quad 1]^\top$	$[1 \quad 1]^\top$
K	$[0.05 \quad 0.1 \quad 1]^\top$	$[1.0 \quad 0.2 \quad 0.5]^\top$	$[0.2 \quad 0.2]^\top$
θ	$[0.015 \quad 0.02 \quad 0.02]^\top$	$[0 \quad 0 \quad 0]^\top$	$[0.03 \quad 0.01]^\top$
Σ	$\text{diag}[0.01 \quad 0.02 \quad 0.03]V$	$\text{diag}[0.01 \quad 0.005 \quad 0.002]V$	$\text{diag}[0.04 \quad 0.02]V$
V	$\begin{pmatrix} 1 & -0.8 & 0.7 \\ -0.8 & 1 & -0.9 \\ 0.7 & -0.9 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -0.2 & -0.1 \\ -0.2 & 1 & 0.3 \\ -0.1 & 0.3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$D(x)$	$\text{diag}[1 \quad 1 \quad 1]$	$\text{diag}[1 \quad 1 \quad 1]$	$\text{diag}[\sqrt{x_1} \quad \sqrt{x_2}]$
X_0	$[0.005 \quad -0.02 \quad 0.02]^\top$	$[0.01 \quad 0.005 \quad -0.02]^\top$	$[0.04 \quad 0.02]^\top$

Table 2: Yields and volatilities

Option	Model 1 (Gaussian)				Model 2 (Gaussian)				Model 3 (CIR)			
	Swap Maturity				Swap Maturity				Swap Maturity			
Expiry	1	3	5	10	1	3	5	10	1	3	5	10
ATMF rate (pct)												
1	0.47	0.82	1.12	1.70	5.72	5.87	5.95	6.03	7.33	7.12	6.96	6.71
3	1.17	1.45	1.69	2.14	6.01	6.06	6.09	6.10	6.90	6.76	6.65	6.48
5	1.72	1.95	2.14	2.51	6.12	6.13	6.13	6.12	6.61	6.52	6.44	6.33
10	2.68	2.81	2.93	3.15	6.12	6.13	6.13	6.12	6.24	6.20	6.18	6.14
ATMF receiver's swaption price (bp)												
1	32.1	84.1	138.3	230.7	20.8	41.8	53.3	65.6	24.9	58.2	77.7	98.3
3	54.5	153.7	240.3	379.9	24.2	51.5	67.0	83.6	30.3	71.0	94.8	120.0
5	67.4	186.0	284.8	442.1	23.2	50.2	65.7	82.2	28.4	66.8	89.3	112.9
10	73.8	199.9	301.6	463.0	18.0	39.3	51.5	64.6	20.7	48.7	65.2	82.5
Yield volatility (pct)												
1	202	89.5	64.4	36.7	10.1	6.94	5.55	3.87	10.5	9.00	7.86	5.98
3	73.9	55.2	44.7	28.9	7.22	5.38	4.44	3.17	8.98	7.65	6.62	4.97
5	48.5	39.6	33.6	23.4	5.94	4.54	3.77	2.72	7.78	6.58	5.67	4.21
10	26.5	23.3	20.8	15.9	4.41	3.40	2.84	2.05	5.81	4.87	4.16	3.04
Absolute volatility (pct)												
1	0.95	0.73	0.72	0.62	0.58	0.41	0.33	0.23	0.80	0.66	0.56	0.41
3	0.86	0.80	0.75	0.62	0.43	0.33	0.27	0.19	0.64	0.53	0.45	0.33
5	0.83	0.77	0.72	0.59	0.36	0.28	0.23	0.17	0.52	0.44	0.37	0.27
10	0.71	0.66	0.61	0.50	0.27	0.21	0.17	0.12	0.37	0.30	0.26	0.18

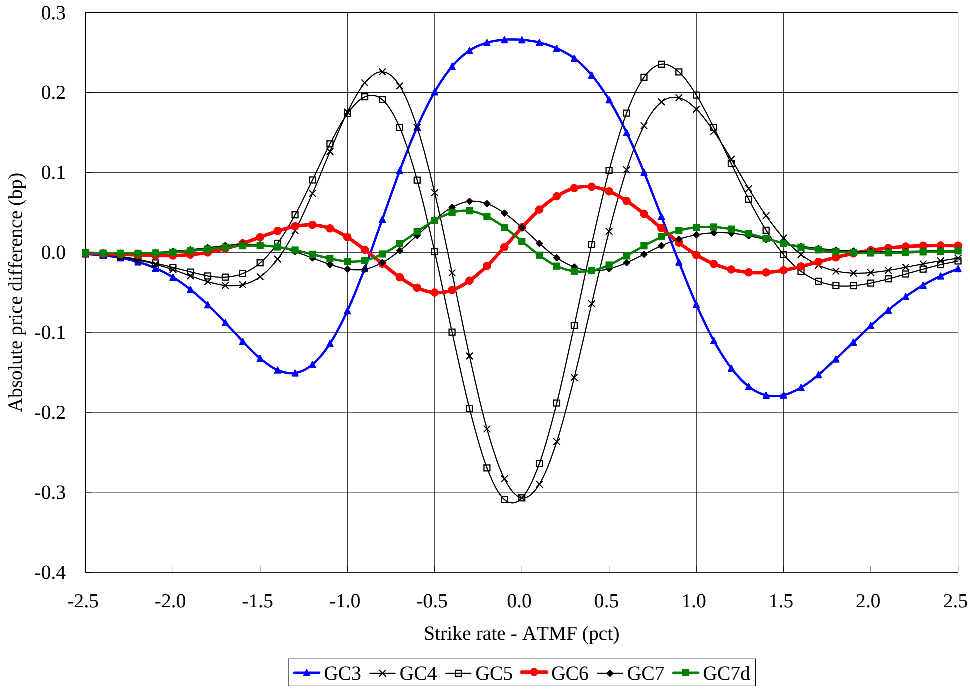

Figure 1: Price differnces GC-MC (1 into 10, Model 1)

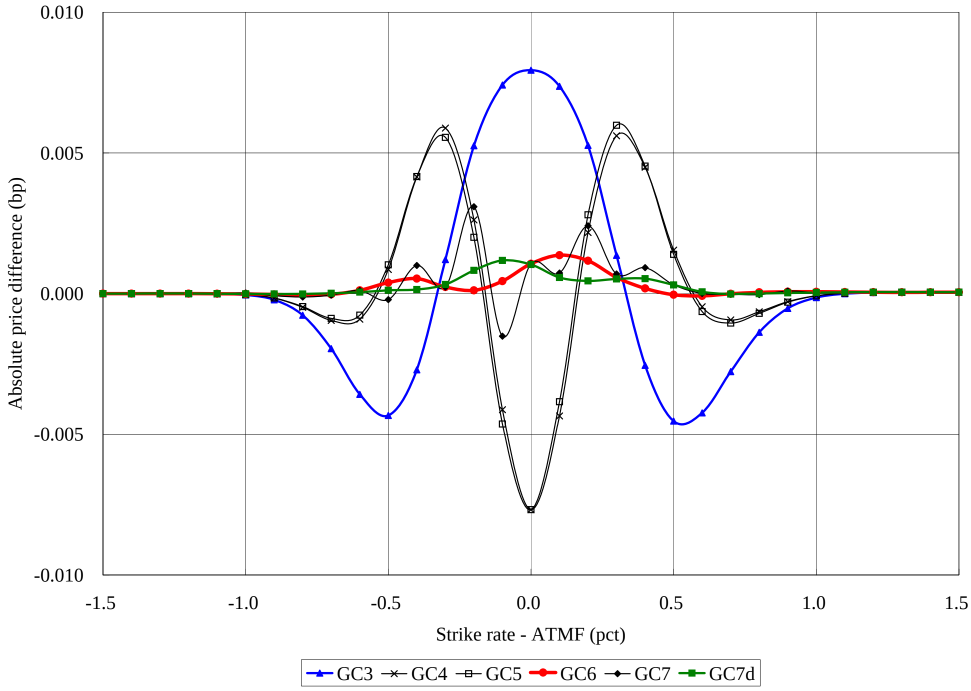

Figure 2: Price differnces GC-MC (1 into 10, Model 2)

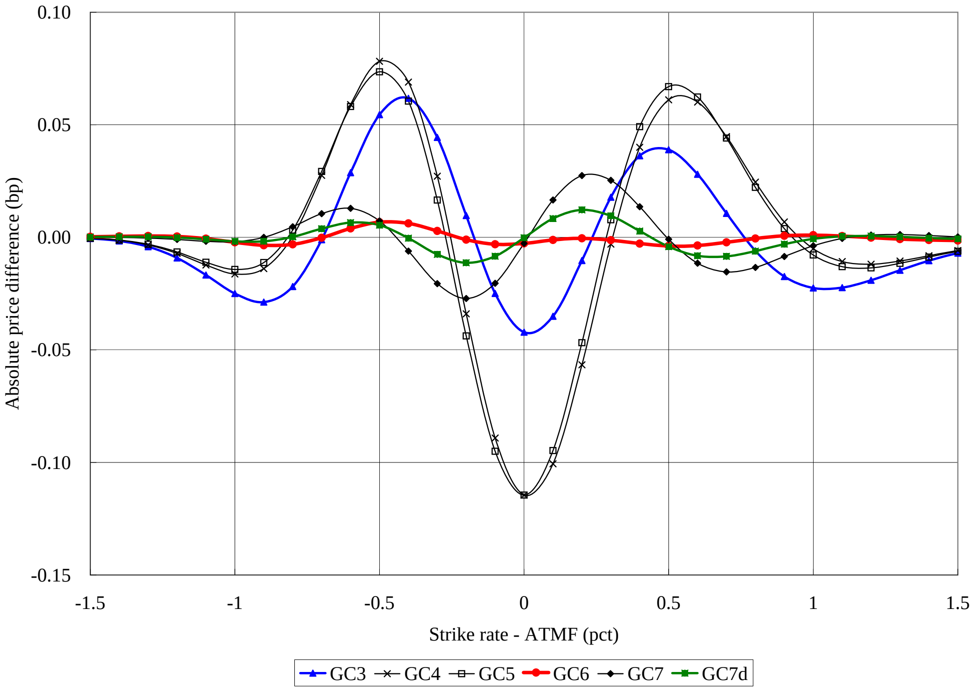

Figure 3: Price differnces GC-MC (1 into 10, Model 3)

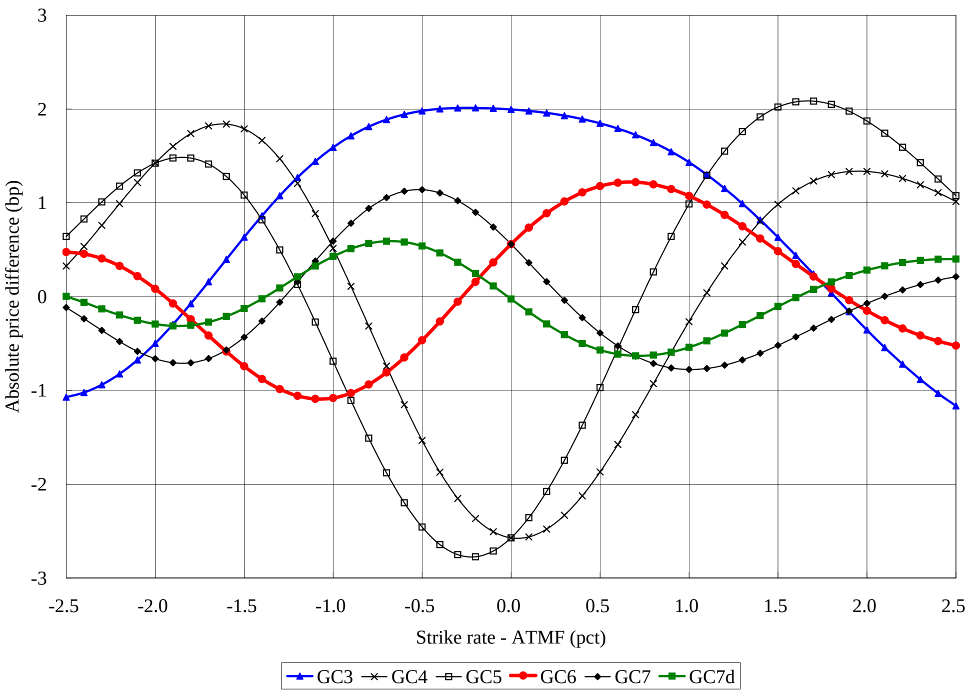

Figure 4: Price differnces GC-MC (5 into 10, Model 1)

Table 3: Absolute difference of swaption price (GC-MC, bp)

	1yr into 10yr						5yr into 10yr	
Strike rate	Model 1		Model 2		Model 3		Model 1	
-ATMF (pct)	GC3	GC6	GC3	GC6	GC3	GC6	GC3	GC6
-2.0	-0.031	-0.004	0.000	0.000	0.000	0.000	-0.499	0.083
-1.5	-0.133	0.019	0.000	0.000	-0.001	0.000	0.634	-0.743
-1.0	-0.073	0.019	0.000	0.000	-0.025	-0.002	1.591	-1.081
-0.5	0.201	-0.050	-0.004	0.000	0.055	0.007	1.980	-0.466
0.0	0.266	0.031	0.008	0.001	-0.042	-0.003	1.996	0.559
0.5	0.191	0.076	-0.005	0.000	0.039	-0.004	1.848	1.178
1.0	-0.065	-0.003	0.000	0.000	-0.023	0.001	1.431	1.074
1.5	-0.179	-0.022	0.000	0.000	-0.007	-0.001	0.632	0.484
2.0	-0.092	0.003	0.000	0.000	-0.001	-0.001	-0.356	-0.151

Table 4: Absolute difference of ATMF swaption price (GC6-MC, bp)

Option	Swap Maturity			
Expiry	1	3	5	10
Model 1 (Gaussian)				
1	-0.000	-0.001	0.000	0.001
3	-0.001	-0.001	-0.005	-0.052
5	-0.001	-0.001	-0.016	-0.052
10	-0.001	-0.001	-0.010	-0.113
Model 2 (Gaussian)				
1	-0.005	0.002	0.003	0.004
3	0.003	0.003	0.004	0.006
5	-0.001	0.003	0.004	0.005
10	0.003	0.002	0.003	0.004
Model 3 (CIR)				
1	1.200	-0.174	0.018	0.019
3	0.569	-0.060	0.017	0.008
5	0.576	-0.033	0.030	0.004
10	0.819	-0.011	0.078	0.039

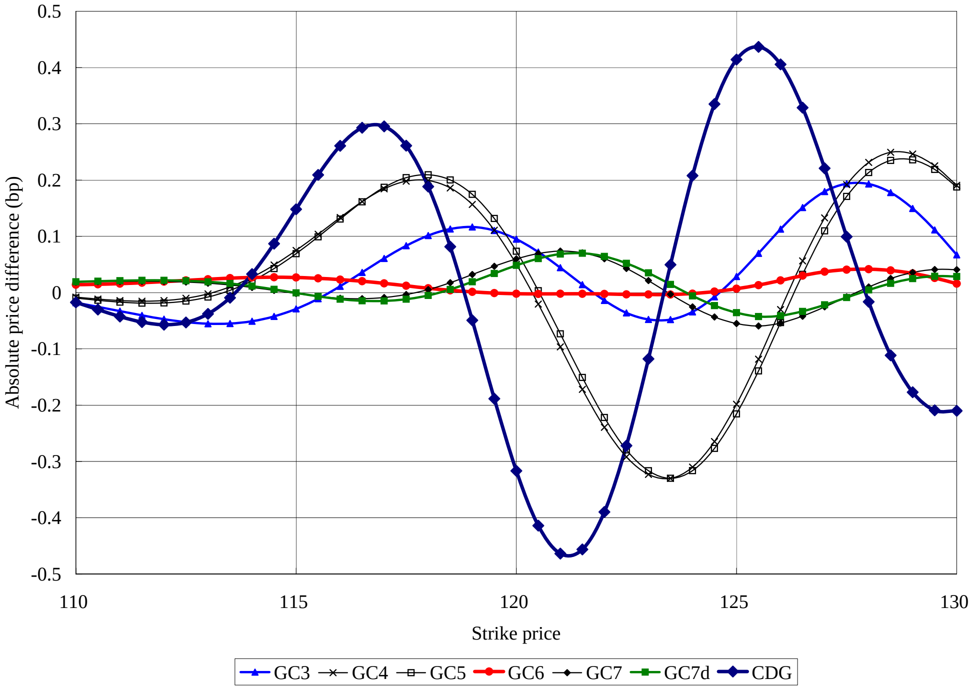

Figure 5: Price differnces for bond options (Model 3)

Table 5: Convexity adjustments

	Model 1 (Gaussian)				Model 2 (Gaussian)				Model 3 (CIR)			
Obser	Swap Maturity				Swap Maturity				Swap Maturity			
-vation	1	5	10	20	1	5	10	20	1	5	10	20
bCA (bp)												
1	0.14	1.18	2.00	2.30	0.07	0.23	0.25	0.22	0.11	0.55	0.64	0.58
3	0.46	3.74	5.81	6.45	0.12	0.47	0.52	0.46	0.23	1.09	1.25	1.13
5	0.76	5.65	8.54	9.51	0.14	0.58	0.64	0.57	0.26	1.26	1.44	1.29
10	1.14	8.08	12.19	13.99	0.16	0.65	0.72	0.65	0.26	1.27	1.44	1.29
nCA (bp)												
1	0.51	1.32	2.10	2.38	0.25	0.33	0.32	0.27	0.37	0.74	0.77	0.67
3	1.47	4.45	6.37	6.85	0.42	0.65	0.64	0.55	0.74	1.46	1.52	1.32
5	2.38	6.86	9.50	10.18	0.49	0.79	0.79	0.67	0.86	1.69	1.75	1.51
10	3.56	9.96	13.66	15.03	0.54	0.88	0.89	0.76	0.87	1.70	1.75	1.51
TA (bp)												
1	-0.37	-0.14	-0.10	-0.08	-0.18	-0.09	-0.06	-0.04	-0.26	-0.19	-0.13	-0.09
3	-1.01	-0.71	-0.56	-0.40	-0.30	-0.17	-0.12	-0.08	-0.49	-0.37	-0.27	-0.19
5	-1.62	-1.22	-0.95	-0.67	-0.35	-0.21	-0.15	-0.10	-0.60	-0.43	-0.29	-0.22
10	-2.43	-1.87	-1.48	-1.04	-0.39	-0.23	-0.16	-0.11	-0.61	-0.43	-0.29	-0.22

Table 6: Price differnces of convexity adjustments

	Model 1 (Gaussian)				Model 2 (Gaussian)				Model 3 (CIR)			
Obser	Swap Maturity				Swap Maturity				Swap Maturity			
-vation	1	5	10	20	1	5	10	20	1	5	10	20
bCA (bp)												
1	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	-0.00	-0.01	-0.01	-0.01
3	0.00	0.01	0.04	0.06	0.00	0.00	0.00	0.00	-0.00	-0.02	-0.02	-0.02
5	0.00	0.03	0.08	0.14	0.00	0.00	0.00	0.00	-0.01	-0.02	-0.03	-0.04
10	0.00	0.05	0.16	0.29	0.00	0.00	0.00	0.00	-0.01	-0.02	-0.04	-0.04
nCA (bp)												
1	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	-0.00	-0.01	-0.01	-0.01
3	0.00	0.01	0.03	0.05	0.00	0.00	0.00	0.00	-0.00	-0.02	-0.02	-0.02
5	0.00	0.02	0.06	0.11	0.00	0.00	0.00	0.00	-0.01	-0.02	-0.03	-0.04
10	0.00	0.04	0.12	0.24	0.00	0.00	0.00	0.00	-0.01	-0.02	-0.04	-0.04
TA (bp)												
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.01	0.02	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.00	0.02	0.04	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 7: Price of vulnerable put option on coupon-bearing bond

Strike price	88.255	93.255	98.255	103.255 (ATMF)	108.255	113.255	118.255
Strike yield (pct)	8.968	7.649	6.414	5.251	4.154	3.115	2.130
MC (bp)	0.386	4.068	30.342	152.419	475.942	930.600	1395.916
GC3	0.099	3.190	31.044	153.559	476.682	929.429	1395.879
(abs. price diff.)	-0.287	-0.878	0.702	1.140	0.739	-1.171	-0.037
GC6	0.338	4.562	29.439	152.840	476.400	930.254	1396.043
(abs. price diff.)	-0.048	0.494	-0.903	0.420	0.458	-0.346	0.127

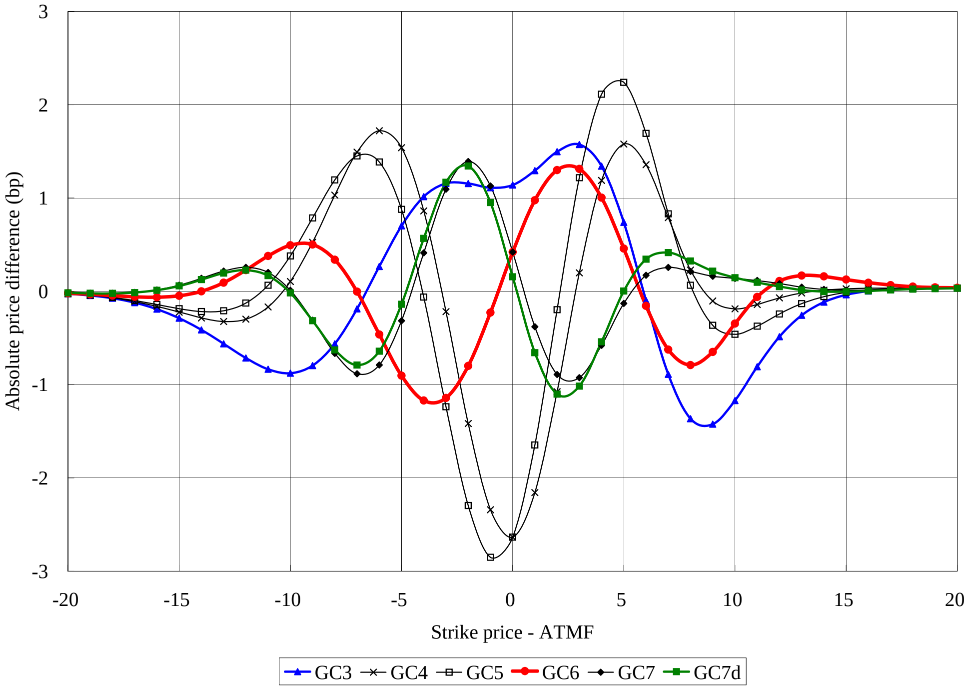

Figure 6: Price differnces of vulnerable put option on coupon-bearing bond

Table 8: Price of vulnerable CDS option (5-yr into 5-yr)

Strike price	110.19	134.19	158.19	182.19 (ATMF)	206.19	230.19	254.19
MC (bp)	423.36	301.55	205.78	136.00	87.73	55.54	34.65
GC3	424.39	305.12	208.66	137.84	89.52	57.49	36.10
(abs. price diff.)	1.03	3.57	2.88	1.84	1.57	1.79	1.95
GC6	423.91	304.94	209.00	137.04	86.09	52.37	31.93
(abs. price diff.)	0.55	3.39	3.23	1.04	-1.64	-3.17	-2.72