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Abstract. We propose a structural model with a joint process of tangible assets (marker) and firm status for the pricing of corporate securities. The firm status is assumed to be latent or unobservable, and default occurs when the firm status process reaches a default threshold at the first time. The marker process is observable and assumed to be correlated with the latent firm status. The recovery upon default is a fraction of tangible assets at the time of default. Our model can evaluate both the corporate debt and equity to fit their market prices in a unified framework. When the two processes are perfectly correlated, our model is reduced to the seminal Black-Cox model. Numerical examples are given to support the usefulness of our model.

Keywords: Structural model, Latent process, Black-Cox model, credit spread

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1 Introduction

A large number of studies have been conducted for the evaluation of credit risk of a corporate firm based on the structural model. The structural model proposed by Merton (1974) and Black and Cox (1976) provides an intuitive understanding of credit risk by modeling equity and corporate debt as contingent claims written on the firm value.[1] For example, CreditMetrics™ (1997) calculates VaR (Value-at-Risk) of a credit portfolio based on the Merton model (1974).

An attractive feature of this approach is that we can analyze how firm-specific variables such as debt ratio influence debt values. For example, Leland (1994) examines the optimal capital structure of a firm by introducing firm's endogenous bankruptcy. See also Mella-Barral and Perraudin (1997) and Chen and Kou (2005) for extensions of the Leland model. Also, it can treat complex contingent claims written on the firm's asset value.[2]

Since then, many attempts have been made to extend the structural model. Among them, Longstaff and Schwartz (1995) and Briys and de Varenne (1994) proposed structural models with stochastic interest rates. This extension is particularly important for valuing corporate debts with longer maturity. On the other hand, Zhou (2001) and Kijima and Suzuki (2001) assumed that the firm value process follows a jump-diffusion process of Merton (1976). This extension makes it possible to generate more realistic and flexible credit spreads.[3]

However, it is well known that the firm value itself cannot explain actual credit spreads observed in the market. Also, the key assumption that the firm value is observable is problematic. In this paper, we extend the classic structural model by taking these points into account. Namely, we consider a bivariate process of tangible assets (marker) and firm status (or actual firm value) for the pricing of corporate securities. The actual firm value is assumed to be latent or unobservable and determines default of the firm.[4] The marker process is observable and assumed to be correlated with the latent firm status. The recovery upon default is a fraction of tangible assets at the time of default.[5] When the two processes are perfectly correlated, our model is reduced to the seminal Black–Cox model.

Latent process models have been used in the medical science to specify health status.

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[1] The term structure models of credit spreads are classified into three approaches in the finance literature, i.e. the structural approach, the reduced-form approach and the hybrid approach. We refer to Longstaff and Schwartz (1995), Jarrow and Turnbull (1995) and Madan and Unal (2000) for these approaches.

[2] See, e.g., Black and Cox (1976) and Kijima and Suzuki (2001).

[3] If the firm value is assumed to follow a diffusion process in the classic structural model, it takes time to reach the default boundary so that the firm never defaults unexpectedly (i.e., by surprise). This means that the short-term debt should have zero credit spreads, whence the model generates only unrealistic short-term credit spreads.

[4] As a similar idea, Giesecke (2004) assumed that the default boundary is not prespecified but given as a random variable.

[5] In our model, the recovery rate of debt is random, consistent with the empirical findings reported by, e.g., Fons (1994). In the Black–Cox model, however, the recovery is constant and equal to the prespecified default boundary.

In general, health status is latent or unobservable while some markers checked by a medical doctor are observable. The seminal paper by Whitmore, Crowder and Lawless (1998) proposed a bivariate model in which a latent health status process determines the failure (death or onset of disease) time, as the first hitting time to a failure threshold level, and the observable marker process is correlated with the health status process.[6]

The health status model has an apparent similarity to default of a firm. Namely, the actual firm status is latent or unobservable while some markers (analyst reports, credit rating, etc.) checked by experts are observable. In this paper, we intend to develop a latent process model for the pricing of corporate securities (debt and equity). As a marker process, we take the value of tangible assets whose market value is evaluated by experts. The actual firm status (latent variable) can be imputed from the market prices of debt and equity. The firm status value is then used to be a proxy of credit quality of the firm.

This paper is organized as follows. In the next section, we formally define the latent process model for the pricing of corporate securities. While Section 3 discusses how to derive the pricing formulas in closed form using the 'change of measure' technique, Section 4 is devoted to numerical examples. We will examine the impact of model parameters on the debt value. Also, the firm status values are calculated from the actual market data for some Japanese firms. The model will prove useful in practice for the evaluation of credit risk. Section 5 concludes this paper. Some mathematical results are given in the appendix.

Throughout the paper, we fix the probability space $(\Omega, \mathcal{F}, Q)$ and denote the expectation operator by E^Q. The probability measure Q is the *risk-neutral measure* and we assume that such Q exists, since we are interested in the pricing of financial instruments.

2 The Model Description

In this section, we describe our model and discuss the basic assumptions. Some of them are parallel to those in Black and Cox (1976), and we refer to the seminal paper for detailed discussions about the assumptions. In particular, we assume that the capital market is frictionless and free of informational asymmetry. The risk-free instantaneous interest rate is constant and denoted by r.

Consider a corporate firm and let $V(t)$ denote the time-t market value of tangible assets of the firm. We assume that $V(t)$ is governed by the following stochastic differential equation (SDE for short) under Q:

$$\frac{\mathrm{d}V(t)}{V(t)} = r\mathrm{d}t + \sigma_v \mathrm{d}z_v(t), \qquad t \geq 0, \tag{2.1}$$

[6]See also Lee, Degfuttola and Schoenfeld (2000) for further discussions of the latent health status model. They developed a generalized linear regression model to estimate the parameters from censored survival data and marker measurement.

where the volatility σ_v is constant and $z_v(t)$ denotes a standard Brownian motion. The tangible assets are assumed to be traded in the market, whence the instantaneous rate of return of the asset $V(t)$ is equal to the risk-free interest rate r under Q. The tangible assets can be collateral when the firm is liquidated. Note however that the value of tangible assets does not coincide with the actual firm value in general.

Suppose that there exists a process $A(t)$ that represents the actual firm value at time t. It is assumed that the process $A(t)$ is latent or unobservable (and so it cannot be traded in the market), and that the *risk-adjusted* $A(t)$ follows the SDE

$$\frac{\mathrm{d}A(t)}{A(t)} = \mu_a \mathrm{d}t + \sigma_a \mathrm{d}z_a(t), \qquad t \geq 0, \tag{2.2}$$

under Q, where the volatility σ_a is constant and $z_a(t)$ is a standard Brownian motion. Of course, because the value of tangible assets contributes to the actual firm value, we assume that $A(t)$ is correlated to $V(t)$ with correlation coefficient ρ. In other words, we assume that $\mathrm{d}z_v(t)\mathrm{d}z_a(t) = \rho \mathrm{d}t$, where ρ is a constant. It should be noted that the actual firm value $A(t)$ can be less than the value $V(t)$ of tangible assets, meaning that intangible assets of the firm have a negative value.

For notational convenience, we define $X(t) = \log A(t)$ and $Y(t) = \log V(t)$. We call $X(t)$ the *firm status process*. Let $(w_1(t), w_2(t))$ be a two-dimensional standard Brownian motion such that

$$\begin{cases} z_v(t) = w_1(t), \\ z_a(t) = \rho w_1(t) + \sqrt{1-\rho^2} w_2(t), \end{cases}$$

since $\mathrm{d}z_v(t)\mathrm{d}z_a(t) = \rho \mathrm{d}t$. It follows from (2.1) and (2.2) that we have

$$\mathrm{d}X(t) = \mu_x \mathrm{d}t + \sigma_a \mathrm{d}w_1(t), \qquad \mu_x = \mu_a - \frac{1}{2}\sigma_a^2, \tag{2.3}$$

and

$$\mathrm{d}Y(t) = \mu_y \mathrm{d}t + \sigma_v \left(\rho \mathrm{d}w_1(t) + \sqrt{1-\rho^2}\mathrm{d}w_2(t)\right), \qquad \mu_y = r - \frac{1}{2}\sigma_v^2, \tag{2.4}$$

respectively.

The basic idea behind the firm status process is the following. In the static setting, Altman (1968) developed a scoring model to identify bankrupt firms from surviving firms using a discriminant analysis. For explanatory variables x_j, suppose that the parameters β_j are estimated so that the *score* of a firm is determined as

$$Z = \beta_0 + \sum_{j=1}^{n} \beta_j x_j.$$

The firm is classified into a bankrupt group when the Z-score is negative. Of course, the score is subject to an error, so that it should be considered as a random variable. Note that, very often, one of the explanatory variables is the value of tangible assets or its variant.

Hence, our model described above seems a natural extension of the scoring model to the continuous-time setting.

Now, suppose that the firm issues two classes of claims, a single homogeneous class of debt and the residual claim (equity). As in Black and Cox (1976), the corporate debt has maturity T and default occurs when the firm status process $X(t)$ reaches the default threshold x_B before the maturity. The face value of the debt is denoted by F. It is assumed that $F \geq \Gamma \equiv \mathrm{e}^{x_B}$. Hence, the firm can survive until the maturity even though the actual firm value $A(t)$, $t < T$, is less than the face value F. At the maturity, however, default occurs when $A(T) < F$, because the actual firm value is less than the face value.

For notational convenience, we introduce the following stopping time:

$$\tau = \inf\{t \geq 0 : A(t) \leq \Gamma\} = \inf\{t \geq 0 : X(t) \leq x_B\}, \qquad x_B = \log \Gamma. \tag{2.5}$$

When $\tau < T$, default occurs before the maturity. When $\tau \geq T$ and $A(T) < F$, default occurs at the maturity.

Denote the debt holders' payoff by D and the equity holders' payoff by E. The payoffs in our model are divided into the following five cases:

1. $\{\tau > T,\, A(T) \geq F,\, V(T) \geq F\}$ or $\{\tau > T,\, A(T) \geq F,\, V(T) < F\}$: In the former case, no default occurs and the value of tangible assets is high enough to repay the debt. Then, debt holders receive the face value and equity holders will receive the residual. That is, $D = F$ and $E = A(T) - F$. On the other hand, the latter case assumes no default, but the value of tangible assets is not enough to repay the debt. In this case, because the firm does not default, we assume that the firm (equity holders) repays the debt by increasing the capital. In other words, equity holders issue new stocks to repay the debt as in Leland (1994). Then, as in the former case, we have $D = F$ and $E = A(T) - F$.

2. $\{\tau > T,\, A(T) < F,\, V(T) \geq F\}$: Default occurs at the maturity, while the value of tangible assets is high enough to repay the debt. In this case, equity holders *liquidate* the firm to repay the debt and they receive the residual after paying a liquidation cost. That is, $D = F$ and $E = \alpha_2(V(T) - F)$, where $1 - \alpha_2$ represents the rate of liquidation cost.

3. $\{\tau > T,\, A(T) < F,\, V(T) < F\}$: Default occurs at the maturity, and the tangible assets are not enough to repay the debt. In this case, debt holders can receive only the tangible assets after paying a default cost while equity holders will receive nothing. That is, $D = \alpha_1 V(T)$ and $E = 0$, where $1 - \alpha_1$ represents the rate of default cost.

4. $\{\tau \leq T,\, V(\tau) \geq F\}$: Default occurs before the maturity, while the value of tangible assets at the time of default is high enough to repay the debt. In this case, as in Case 2, equity holders liquidate the firm to repay the debt and they receive the residual after paying a liquidation cost. That is, $D = F$ and $E = \alpha_2(V(\tau) - F)$.

5. $\{\tau \le T,\, V(\tau) < F\}$: Default occurs before the maturity, and the tangible assets are not enough to repay the debt. In this case, as in Case 3, debt holders can receive only the tangible assets after paying a default cost while equity holders will receive nothing. That is, $D = \alpha_1 V(\tau)$ and $E = 0$.

In the following, we denote the value of debt by $D(T)$ and the value of equity by $E(T)$, where T is the maturity of the debt. From the above discussions about the payoffs, we obtain the following:

$$\begin{aligned} D(T) &= E^Q[\mathrm{e}^{-rT} F 1_{\{\tau > T,\, A(T) \ge F\}}] \\ &+ E^Q[\mathrm{e}^{-rT} F 1_{\{\tau > T,\, A(T) < F,\, V(T) \ge F\}}] \\ &+ E^Q[\mathrm{e}^{-rT} \alpha_1 V(T) 1_{\{\tau > T,\, A(T) < F,\, V(T) < F\}} \\ &+ E^Q[\mathrm{e}^{-r\tau} F 1_{\{\tau \le T,\, V(\tau) \ge F\}}] \\ &+ E^Q[\mathrm{e}^{-r\tau} \alpha_1 V(\tau) 1_{\{\tau \le T,\, V(\tau) < F\}}] \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} E(T) &= E^Q[\mathrm{e}^{-rT} (A(T) - F) 1_{\{\tau > T,\, A(T) \ge F\}}] \\ &+ E^Q[\mathrm{e}^{-rT} \alpha_2 (V(T) - F) 1_{\{\tau > T,\, A(T) < F,\, V(T) \ge F\}}] \\ &+ E^Q[\mathrm{e}^{-r\tau} \alpha_2 (V(\tau) - F) 1_{\{\tau \le T,\, V(\tau) \ge F\}}]. \end{aligned} \tag{2.7}$$

The credit spread is defined by

$$s(T) = -\frac{1}{T} \log \frac{D(T)}{F} - r, \qquad T > 0. \tag{2.8}$$

3 Change of Measure Formulas

In order to derive the pricing formulas in closed form for both debt and equity, we define

$$\begin{cases} \text{(i)} & f_1 \equiv \mathrm{e}^{-rT} E^Q\left[1_{\{\tau > T,\, A(T) \ge F\}}\right], \\ \text{(ii)} & f_2 \equiv \mathrm{e}^{-rT} E^Q\left[1_{\{\tau > T,\, A(T) < F,\, V(T) \ge F\}}\right], \\ \text{(iii)} & f_3 \equiv \mathrm{e}^{-rT} E^Q\left[V(T) 1_{\{\tau > T,\, A(T) < F,\, V(T) < F\}}\right], \\ \text{(iv)} & f_4 \equiv E^Q\left[\mathrm{e}^{-r\tau} 1_{\{\tau \le T,\, V(\tau) \ge F\}}\right], \\ \text{(v)} & f_5 \equiv E^Q\left[\mathrm{e}^{-r\tau} V(\tau) 1_{\{\tau \le T,\, V(\tau) < F\}}\right], \end{cases} \tag{3.1}$$

and

$$\begin{cases} \text{(i}'\text{)} & g_1 \equiv \mathrm{e}^{-rT} E^Q\left[A(T) 1_{\{\tau > T,\, A(T) \ge F\}}\right], \\ \text{(ii}'\text{)} & g_2 \equiv \mathrm{e}^{-rT} E^Q\left[V(T) 1_{\{\tau > T,\, A(T) < F,\, V(T) \ge F\}}\right], \\ \text{(iv}'\text{)} & g_4 \equiv E^Q\left[\mathrm{e}^{-r\tau} V(\tau) 1_{\{\tau \le T,\, V(\tau) \ge F\}}\right]. \end{cases} \tag{3.2}$$

Then, from (2.6) and (2.7), we have

$$D(T) = F[f_1 + f_2 + f_4] + \alpha_1[f_3 + f_5]$$

and

$$E(T) = [g_1 - F f_1] + \alpha_2[g_2 - F f_2] + \alpha_2[g_4 - F f_4],$$

respectively. In what follows, we evaluate these expectations using appropriate change of measures.

Consider the two-dimensional process $(X(t), Y(t))$ defined by (2.3) and (2.4), respectively. That is,

$$\begin{cases} \mathrm{d}X(t) &= \mu_x \mathrm{d}t + \sigma_x \mathrm{d}w_1(t), \qquad X(0) = x_0, \\ \mathrm{d}Y(t) &= \mu_y \mathrm{d}t + \sigma_y \left(\rho \mathrm{d}w_1(t) + \sqrt{1-\rho^2}\mathrm{d}w_2(t)\right), \qquad Y(0) = y_0, \end{cases} \tag{3.3}$$

where $\sigma_x = \sigma_a$ and $\sigma_y = \sigma_v$. Also, we introduce the following variables:

$$Z(t) = \frac{X(t) - x_0}{\sigma_x}, \qquad \mu_z = \frac{\mu_x}{\sigma_x}, \qquad \beta = \mu_y - \rho\sigma_y\mu_z, \tag{3.4}$$

for notational simplicity.

Now, define new probability measure P by

$$\frac{\mathrm{d}P}{\mathrm{d}Q}\Big|_{\mathcal{F}_t} = \eta(t) \equiv \exp\left\{-\mu_z w_1(t) - \frac{1}{2}\mu_z^2 t\right\}, \tag{3.5}$$

and let

$$\begin{cases} \mathrm{d}W_1(t) &= \mathrm{d}w_1(t) + \mu_z \mathrm{d}t, \\ \mathrm{d}W_2(t) &= \mathrm{d}w_2(t). \end{cases} \tag{3.6}$$

By Girsanov's theorem, the process $(W_1(t), W_2(t))$ is a two-dimensional standard Brownian motion under P. It follows from (3.3), (3.4) and (3.6) that $Z(t) = W_1(t)$ and

$$\begin{cases} X(t) &= x_0 + \sigma_x Z(t), \\ Y(t) &= y_0 + \beta t + \sigma_y \left(\rho Z(t) + \sqrt{1-\rho^2} W_2(t)\right). \end{cases} \tag{3.7}$$

The default epoch (2.5) can be rewritten as

$$\tau = \inf\{t \geq 0 : Z(t) \leq z_B\}, \qquad z_B = \frac{x_B - x_0}{\sigma_x}. \tag{3.8}$$

In the following, we derive the formulas for f_j in (3.1) only. The formulas for g_j can be obtained similarly.[7] We denote the probability density function (PDF for short) of a standard normal distribution by $\phi(x)$, and the joint PDF of a standard bivariate normal distribution with correlation coefficient ρ by $\phi_2(x, y; \rho)$. Also, the expectation operator under the new probability measure P is denoted simply by E.

[7] The formulas for g_j are available upon request.

First, we derive f_1 using the change of measure formula (3.5). By definition, we obtain

$$
\begin{aligned}
f_1 &= E^Q[\mathrm{e}^{-rT}1_{\{\tau>T,\, A(T)\geq F\}}] \\
&= E\left[\mathrm{e}^{-rT-\frac{1}{2}\mu_z^2T+\mu_zZ(T)}1_{\{\tau>T,\, A(T)\geq F\}}\right] \\
&= \int_{z>z_F} \mathrm{e}^{-rT-\frac{1}{2}\mu_z^2T+\mu_zz}\left\{\phi\left(\frac{z}{\sqrt{T}}\right)-\phi\left(\frac{z-2z_B}{\sqrt{T}}\right)\right\}\frac{\mathrm{d}z}{\sqrt{T}} \qquad \text{(by Lemma 1)} \\
&= \mathrm{e}^{-rT}\int_{z>z_F}\left\{\phi\left(\frac{z-\hat{\mu}_{z1}}{\sqrt{T}}\right)-\mathrm{e}^{2\mu_zz_B}\phi\left(\frac{z-\hat{\mu}_{z2}}{\sqrt{T}}\right)\right\}\frac{\mathrm{d}z}{\sqrt{T}}, \qquad \text{(by Lemma 4)}
\end{aligned}
$$

where

$$
z_F=\frac{\log F-x_0}{\sigma_x},\quad z_B=\frac{x_B-x_0}{\sigma_x},\quad \hat{\mu}_{z1}=\mu_zT,\quad \hat{\mu}_{z2}=\mu_zT+2z_B.
$$

Next, for f_2, we obtain

$$
\begin{aligned}
f_2 &= E^Q[1_{\{\tau>T,\, A(T)<F,\, V(T)\geq F\}}] \\
&= E\left[\mathrm{e}^{-rT-\frac{1}{2}\mu_z^2T+\mu_zZ(T)}1_{\{\tau>T,\, A(T)<F,\, V(T)\geq F\}}\right] \\
&= \int\int_{z_B<z<z_F,\ y>\log F}\mathrm{e}^{-rT-\frac{1}{2}\mu_z^2T+\mu_zz}\left\{\phi_2\left(\frac{z}{\sqrt{T}},\frac{y-(y_0+\beta T)}{\sigma_y\sqrt{T}};\rho\right)\right. \\
&\qquad \left.-\phi_2\left(\frac{z-2z_B}{\sqrt{T}},\frac{y-(y_0+\beta T+2\sigma_y\rho z_B)}{\sigma_y\sqrt{T}};\rho\right)\right\}\frac{\mathrm{d}z}{\sqrt{T}}\frac{\mathrm{d}y}{\sigma_y\sqrt{T}} \qquad \text{(by Lemma 2)} \\
&= \mathrm{e}^{-rT}\int\int_{z_B<z<z_F,\ y>\log F}\left\{\phi_2\left(\frac{z-\hat{\mu}_{z1}}{\sqrt{T}},\frac{y-\hat{\mu}_{y1}}{\sigma_y\sqrt{T}};\rho\right)\right. \\
&\qquad \left.-\mathrm{e}^{2\mu_zz_B}\phi_2\left(\frac{z-\hat{\mu}_{z2}}{\sqrt{T}},\frac{y-\hat{\mu}_{y2}}{\sigma_y\sqrt{T}};\rho\right)\right\}\frac{\mathrm{d}z}{\sqrt{T}}\frac{\mathrm{d}y}{\sigma_y\sqrt{T}}, \qquad \text{(by Lemma 4)}
\end{aligned}
$$

where

$$
\hat{\mu}_{y1}=y_0+\beta T+\mu_z\rho\sigma_yT, \qquad \hat{\mu}_{y2}=\hat{\mu}_{y1}+2\sigma_y\rho z_B.
$$

Third, for f_3, we obtain

$$
\begin{aligned}
f_3 &= E^Q[V(T)1_{\{\tau>T,\, A(T)<F,\, V(T)<F\}}] \\
&= E\left[\mathrm{e}^{-rT-\frac{1}{2}\mu_z^2T+\mu_zZ(T)+Y(T)}1_{\{\tau>T,\, A(T)<F,\, V(T)<F\}}\right] \\
&= \int\int_{z_B<z<z_F,\ y<\log F}\mathrm{e}^{-rT-\frac{1}{2}\mu_z^2T+\mu_zz+y}\left\{\phi_2\left(\frac{z}{\sqrt{T}},\frac{y-(y_0+\beta T)}{\sigma_y\sqrt{T}};\rho\right)\right. \\
&\qquad \left.-\phi_2\left(\frac{z-2z_B}{\sqrt{T}},\frac{y-(y_0+\beta T+2\sigma_y\rho z_B)}{\sigma_y\sqrt{T}};\rho\right)\right\}\frac{\mathrm{d}z}{\sqrt{T}}\frac{\mathrm{d}y}{\sigma_y\sqrt{T}} \qquad \text{(by Lemma 2)} \\
&= \mathrm{e}^{-rT+\tilde{\mu}_{y1}}\int\int_{z_B<z<z_F,\ y<\log F}\left\{\phi_2\left(\frac{z-\tilde{\mu}_{z1}}{\sqrt{T}},\frac{y-\tilde{\mu}_{y1}}{\sigma_y\sqrt{T}};\rho\right)\right. \\
&\qquad \left.-\mathrm{e}^{2(\mu_z+\sigma_y\rho)z_B}\phi_2\left(\frac{z-\tilde{\mu}_{z2}}{\sqrt{T}},\frac{y-\tilde{\mu}_{y2}}{\sigma_y\sqrt{T}};\rho\right)\right\}\frac{\mathrm{d}z}{\sqrt{T}}\frac{\mathrm{d}y}{\sigma_y\sqrt{T}}, \qquad \text{(by Lemma 4)}
\end{aligned}
$$

where

$$\begin{cases} \tilde{\mu}_{z1} = \hat{\mu}_{z1} + \rho\sigma_y T, & \tilde{\mu}_{y1} = \hat{\mu}_{y1} + \sigma_y^2 T, \\ \tilde{\mu}_{z2} = \hat{\mu}_{z2} + \rho\sigma_y T, & \tilde{\mu}_{y2} = \hat{\mu}_{y2} + \sigma_y^2 T. \end{cases}$$

Fourth, to obtain f_4, we invoke the joint PDF $q(y,t)$ given in Lemma 5. Namely,

$$\begin{aligned} f_4 &= E^Q[\mathrm{e}^{-r\tau}1_{\{\tau\le T, V(\tau)>F\}}] \\ &= E\left[\mathrm{e}^{-r\tau}\eta(T)1_{\{\tau\le T, V(\tau)>F\}}\right] \\ &= E\left[E[\mathrm{e}^{-r\tau}\eta(T)1_{\{\tau\le T, V(\tau)>F\}}|\tau]\right] \\ &= E\left[\mathrm{e}^{-r\tau}\eta(\tau)1_{\{\tau\le T, V(\tau)>F\}}\right] \qquad (\eta(t) \text{ is a martingale}) \\ &= \int\int_{\tau<T,\ y>\log F} \mathrm{e}^{-r\tau-\frac{1}{2}\mu_z^2\tau+\mu_z z_B} q(y,\tau)\mathrm{d}y\mathrm{d}\tau. \qquad \text{(by Lemma 5)} \end{aligned}$$

We note that the above formula can be simplified more; however, this formula seems enough for the purpose of numerical calculation.

Finally, similar to f_4, we obtain

$$f_5 = \int\int_{\tau<T,\ y<\log F} \mathrm{e}^{-r\tau-\frac{1}{2}\mu_z^2\tau+\mu_z z_B+y} q(y,\tau)\mathrm{d}y\mathrm{d}\tau$$

by Lemma 5.

4 Numerical Examples

This section provides numerical examples to demonstrate the usefulness of our model. The base parameters used in the examples are listed in Table 1. Recall that F is the face value of debt, $V(0)$ the initial value of tangible assets, $A(0)$ the initial value of actual firm value, Γ the default threshold, r the risk-free interest rate, μ_a the mean rate of return of $A(t)$, σ_v the volatility of $V(t)$, σ_a the volatility of $A(t)$, ρ the correlation coefficient between them, α_1 the rate of default cost, and α_2 the rate of liquidation cost.

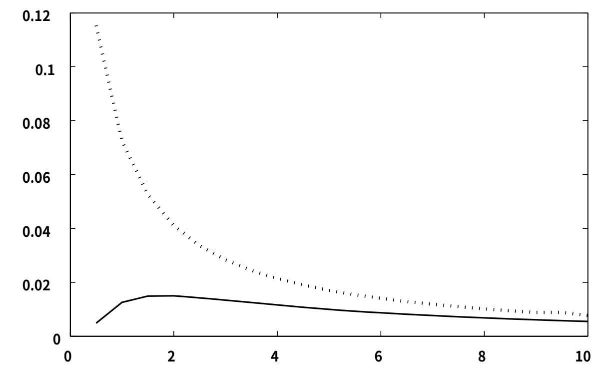
Table 1: Parameter values for the base case

F	$V(0)$	$A(0)$	Γ	r	μ_a	σ_v	σ_a	ρ	α_1	α_2
1.0	1.0	1.4	0.5	0.05	0.05	0.2	0.2	0.7	0.8	0.8

4.1 The impact of parameters on the credit spread

First, we investigate the impact of initial values $A(0)$ and $V(0)$ on the credit spread $s(T)$ with the other parameters being the same as in the base case (see Figures 1 and 2, respectively). As expected, the lower the initial value $A(0)$, the higher the credit spread $s(T)$, in particular, for the short-term debt. This is so, because there is a high possibility of default in the short term when $A(0)$ is low; but, conditional on survival, the effect of the initial value disappears. This phenomena on the term structure of credit spreads is well recognized in the market, and several attempts have been made to explain the empirical result. See, e.g., Fons (1994) and Kijima (1998).

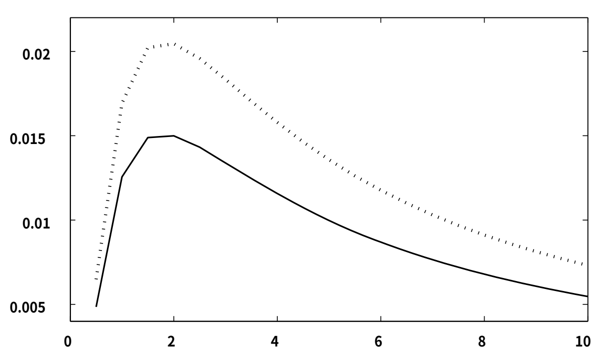
Figure 1: Impact of the initial value $A(0)$



The solid line corresponds to the base case ($A(0) = 1.4$), while the dashed line to the case with $A(0) = 1.1$ and the other parameters being the same as in the base case.

Similarly, the lower the initial value $V(0)$, the higher the credit spread $s(T)$. Note however that the term structure is more or less parallel with respect to the change in $V(0)$. This is so, because the value of tangible assets plays only the role of recovery upon default in our model.

Figure 2: Impact of the initial value $V(0)$

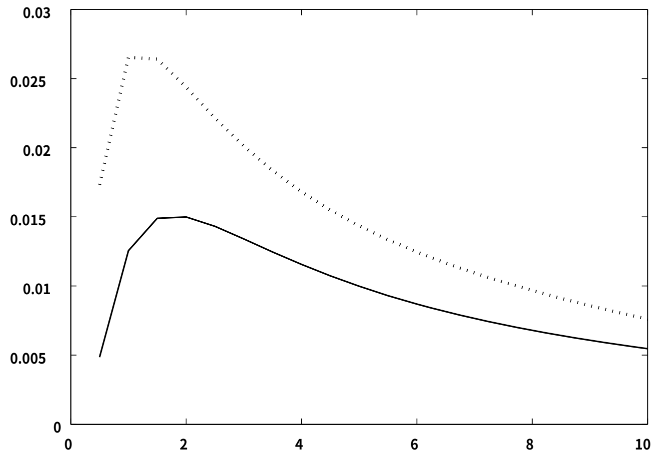


The solid line corresponds to the base case ($V(0) = 1.0$), while the dashed line to the case with $V(0) = 0.8$ and the other parameters being the same as in the base case.

Next, we investigate the impact of volatilities σ_a and σ_v on the credit spread (see Figures 3 and 4, respectively). Recall that σ_a is the volatility of the actual firm (latent) process and σ_v the volatility of the value of tangible assets. The higher the volatility σ_a, the higher the credit spread $s(T)$, because the probability of default becomes higher when the volatility

σ_a becomes higher. Note that, due to the same reasons as above, the impact of volatility σ_a appears significantly for the short-term debt and the effect disappears as the maturity becomes longer.

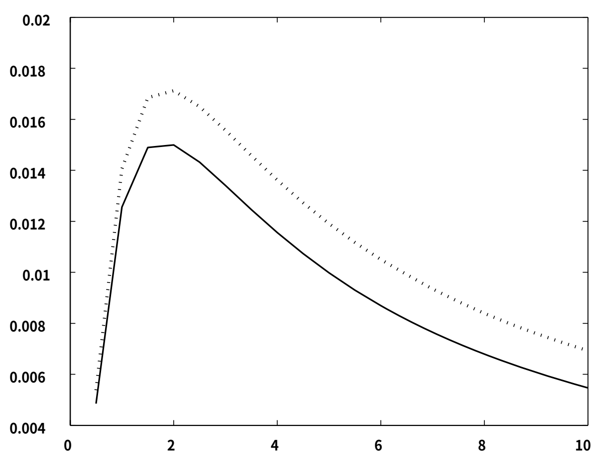
Figure 3: Impact of the volatility σ_a



The solid line corresponds to the base case ($\sigma_a = 0.2$), while the dashed line to the case with $\sigma_a = 0.25$ and the other parameters being the same as in the base case.

On the other hand, the higher the volatility σ_v, the higher the credit spread $s(T)$, because the possibility of large loss upon default becomes higher when the volatility σ_v becomes higher. Note that the term structure of credit spreads is more or less parallel with respect to the change in σ_v.

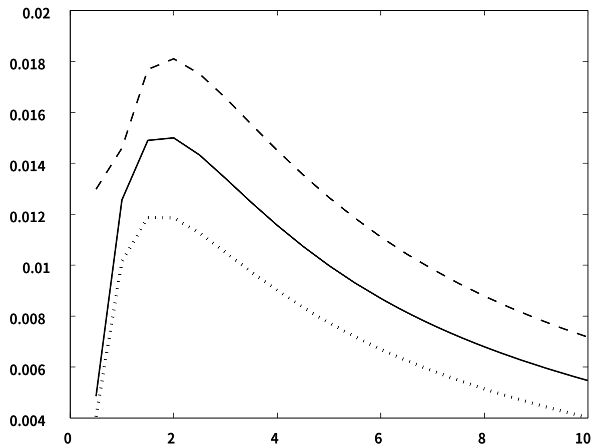
Figure 4: Impact of the volatility σ_v



The solid line corresponds to the base case ($\sigma_v = 0.2$), while the dashed line to the case with $\sigma_v = 0.25$ and the other parameters being the same as in the base case.

Finally, the impact of the correlation coefficient ρ on the credit spread $s(T)$ is investigated (see Figure 5). As expected, the higher the correlation coefficient ρ, the higher the credit spread $s(T)$, because the joint probability of default and large loss becomes higher as the correlation coefficient ρ becomes higher.

Figure 5: Impact of the correlation coefficient ρ



The solid line corresponds to the base case ($\rho = 0.7$), while the dashed and dotted line to the cases with $\rho = 0.5$ and $\rho = 0.9$, respectively, and the other parameters being the same as in the base case.

4.2 Calibration of the actual firm value

In this subsection, we calibrate the actual firm value $A(0)$ from the market data (debt and equity). There may be many other ways to compute the value $A(0)$; however, we take the simplest means to avoid technical difficulties. That is, consider a firm and suppose that the firm value $A(0) = A$ and its volatility σ_a are unknown and other parameters are all estimated correctly. Then, the debt and equity values are functions of variables A and σ_a, say $D = f(A, \sigma_a)$ and $E = g(A, \sigma_a)$. The firm value A can then be obtained by solving the (non-linear) simultaneous equations.

In this numerical example, we consider four big Japanese companies, Bridgestone, Sony, Nissan and Tokyo Electronic Power (TEPCO). All the data and the results computed from our model are listed in Table 2. The market data are based on the closing prices of Tokyo Stock Exchange and the published financial data on March 2, 2007. The maturity is calculated as the weighted average of straight bonds issued by the firm. All the parameter values not listed in Table 2 are the same as the base case.

Although the credit spread of TEPCO is the lowest among the four companies, its actual firm value normalized by the value of tangible assets, $A(0)/V(0)$, is the lowest. Also, the volatility of the firm value of TEPCO is the lowest. These results are plausible because TEPCO is the biggest electronic power company in Japan. It is interesting to note that the

Table 2: The actual firm value calibrated from the market data

	Bridgestone	Sony	Nissan	TEPCO
$E(T)$	8.243	7.769	6.428	1.024
$s(T)$	17.4	21.6	23.0	12.4
T	5.17	8.25	3.42	5.83
r	1.375	1.547	1.025	1.303
$A(0)$	7.587	7.052	6.374	1.990
σ_a	0.345	0.295	0.385	0.130
$A(0) - V(0)$	6.587	6.052	5.374	0.990

stock prices of Bridgestone, Sony and Nissan are higher than the actual firm values. Such a case can happen when the firm has a very high reputation in the stock market. In fact, these firms have very high values $A(0) - V(0)$ that can be considered as the value of intangible assets.

5 Concluding Remarks

In this paper, we proposed a structural model with a joint process of tangible assets (marker) and firm status for the pricing of corporate securities (debt and equity). The firm status is assumed to be latent or unobservable and determines default of the firm. The marker process is observable and used to repay the debt, which is assumed to be correlated with the latent firm status. When the two processes are perfectly correlated, our model is reduced to the seminal Black–Cox model.

Using an appropriate change of measure technique, we obtain the pricing formulas for both corporate debt and equity in closed form. Numerical examples are given to support the usefulness of our model. A desirable feature in our model is that the recovery rate is random, consistent with the empirical findings. Also, debt and equity are evaluated in the unified framework. Based on the model, the actual firm value (latent variable) can be imputed from the market prices of debt and equity. The firm status value is then used to be a proxy of credit quality of the firm.

However, there remain many issues to be addressed. For example, when pricing the equity issued by a firm, we assume that equity has the same maturity as debt. Although this is the common assumption in the structural approach (see Merton (1974) and Black and Cox (1976)), equity has no maturity in general. A unified model for the pricing of both equity and debt without assuming the maturity for equity is required. Also, our model cannot generate a realistic term structure of credit spreads. One way to overcome the deficit is to introduce a jump process. Another way seems to use the information structure available for investors explicitly. In our model, we assume that the firm defaults when the firm status process reaches a default threshold. However, the process is not observed by investors and default occurs before maturity by surprise. These problems are our future works.

A Some Lemmas

First note that the process $Z(t)$ is a standard Brownian motion under P. It is well known (see, e.g., Karatzas and Shreve (1988)) that the first passage time τ defined by (3.8) has the PDF

$$P\{\tau \in \mathrm{d}t\} \equiv \frac{\partial}{\partial t} P\{\tau \le t\}\mathrm{d}t = \frac{z_B}{\sqrt{2\pi t^3}} \exp\left(-\frac{z_B^2}{2t}\right) \mathrm{d}t$$

and its Laplace transform is given by

$$E[\mathrm{e}^{-\alpha\tau}] = \exp\left(-z_B\sqrt{2\alpha}\right), \qquad \alpha > 0.$$

Let $m_Z(t) = \min_{0\le s\le t} Z(s)$ and define

$$p_0(x|m) \equiv P\{Z(t) \in \mathrm{d}x, m_Z(t) > m\}, \qquad m < 0, \quad x > m,$$

under the new probability measure P. The next result is well known and the proof is omitted.

Lemma 1 *we have*

$$p_0(x|m) = \frac{1}{\sqrt{t}} \left\{ \phi\left(\frac{x}{\sqrt{t}}\right) - \phi\left(\frac{x-2m}{\sqrt{t}}\right) \right\} \mathrm{d}x.$$

The next result is an extension of Lemma 1 to the bivariate case. Define

$$q_0(x, y|m) \equiv P\{Z(t) \in \mathrm{d}x, Y(t) \in \mathrm{d}y, m_Z(t) > m\}, \qquad m < 0, \quad x > m.$$

Lemma 2 *we have*

$$\begin{aligned} q_0(x, y|m) &= \left\{ \phi_2\left(\frac{x}{\sqrt{t}}, \frac{y-(y_0+\beta t)}{\sigma_y\sqrt{t}}; \rho\right) \right. \\ &\qquad \left. - \phi_2\left(\frac{x-2m}{\sqrt{t}}, \frac{y-(y_0+\beta t+2\sigma_y\rho m)}{\sigma_y\sqrt{t}}; \rho\right) \right\} \frac{\mathrm{d}x}{\sqrt{t}} \frac{\mathrm{d}y}{\sigma_y\sqrt{t}}. \end{aligned}$$

Proof. By definition, we obtain

$$\begin{aligned} q_0(x, y|m) &= P\left\{ m_Z(t) > m,\, W_1(t) \in \mathrm{d}x,\, W_2(t) \in \frac{\mathrm{d}y - (y_0+\beta t+\sigma_y\rho x)}{\sigma_y\sqrt{(1-\rho^2)t}} \right\} \\ &= p_0(x|m)\mathrm{d}x P\left\{ W_2(t) \in \frac{\mathrm{d}y - (y_0+\beta t+\sigma_y\rho x)}{\sigma_y\sqrt{(1-\rho^2)t}} \right\}, \end{aligned}$$

because $W_1(t)$ and $W_2(t)$ are mutually independent under P. It follows from Lemma 1 that

$$\begin{aligned} q_0(x,y|m) &= \frac{1}{\sqrt{t}}\left\{\phi\left(\frac{x}{\sqrt{t}}\right)-\phi\left(\frac{x-2m}{\sqrt{t}}\right)\right\} \\ &\qquad \times \frac{1}{\sigma_y\sqrt{(1-\rho^2)t}}\phi\left(\frac{y-(y_0+\beta t+\sigma_y\rho x)}{\sigma_y\sqrt{(1-\rho^2)t}}\right)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

The lemma follows since

$$\phi_2(x,y;\rho)=\phi(x)\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right).$$

Let us define

$$q_1(x,y|m)\equiv P\{Z(t)\in \mathrm{d}x, Y(t)\in \mathrm{d}y, m_Z(t)<m\}.$$

Then, since

$$q_1(x,y|m)=P\{Z(t)\in \mathrm{d}x, Y(t)\in \mathrm{d}y\}-P\{Z(t)\in \mathrm{d}x, Y(t)\in \mathrm{d}y, m_Z(t)>m\},$$

the next result follows from Lemma 2 at once.

Lemma 3 *we have*

$$q_1(x,y|m)=\phi_2\left(\frac{x-2m}{\sqrt{t}},\frac{y-(y_0+\beta t+2\sigma_y\rho m)}{\sigma_y\sqrt{t}};\rho\right)\frac{\mathrm{d}x}{\sqrt{t}}\frac{\mathrm{d}y}{\sigma_y\sqrt{t}}$$

The next result seems well known and the proof is omitted.

Lemma 4 *we have*

$$\begin{aligned} &\mathrm{e}^{\lambda_1 x+\lambda_2 y}\phi_2\left(\frac{x-\mu_y}{\sigma_y},\frac{y-\mu_y}{\sigma_y};\rho\right) \\ =\ &\exp\left(\lambda_1\mu_x+\lambda_2\mu_y+\frac{1}{2}(\lambda_1^2\sigma_x^2+2\lambda_1\lambda_2\rho\sigma_x\sigma_y+\lambda_2^2\sigma_y^2)\right) \\ &\quad\times\phi_2\left(\frac{x-(\mu_x+\lambda_1\sigma_x^2+\lambda_2\rho\sigma_x\sigma_y)}{\sigma_x},\frac{y-(\mu_y+\lambda_2\sigma_y^2+\lambda_1\rho\sigma_x\sigma_y)}{\sigma_y};\rho\right) \end{aligned}$$

Finally, let

$$q(y,t)\equiv P\{Y(\tau)\in \mathrm{d}y,\ \tau\in \mathrm{d}t\}.$$

We then have the following.

Lemma 5 *We have*

$$q(y,t)=\frac{1}{\sigma_y\sqrt{(1-\rho^2)t}}\phi\left(\frac{y-(y_0+\beta t+\sigma_y\rho z_B)}{\sigma_x\sqrt{t}}\right)\times\frac{z_B}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{z_B^2}{2t}}\mathrm{d}y\mathrm{d}t$$

Proof. By definition, we have

$$q(y,t) = P\left\{y_0 + \beta t + \sigma_y\Big(\rho W_1(t) + \sqrt{1-\rho^2}W_2(t)\Big) \in \mathrm{d}y \Big| \tau \in \mathrm{d}t\right\} P\{\tau \in \mathrm{d}t\}.$$

But, since

$$W_1(\tau) = Z(\tau) = \frac{x_B - x}{\sigma_x}$$

from (3.8), and since τ and $W_2(t)$ are independent, we obtain

$$q(y,t) = P\left\{y_0 + \beta t + \sigma_y\Big(\rho\frac{x_B - x_0}{\sigma_x} + \sqrt{1-\rho^2}W_2(t)\Big) \in \mathrm{d}y\right\} P\{\tau \in \mathrm{d}t\}$$

and the result follows at once.

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