Research Paper Series

No. 31

Uniform, Equal Division, and the Other Envy-free Rules Between the Two

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April 2007

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Uniform, equal division, and other envy-free rules between the two *

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April 23, 2007

Abstract

This paper studies the problem of fairly allocating an amount of a divisible resource when preferences are single-peaked. We first show that, given any preference profile, the set of allocations chosen by *envy-free* and *peak-only* rules is linearly ordered by the Pareto dominance relation, where the uniform allocation is the top and the equal division is the bottom. We then establish the complete lattice structure of the set of *envy-free* and *peak-only* rules with respect to a dominance relation induced by the Pareto dominance relation. The greatest and least elements of the lattice are the uniform rule and the equal division rule, respectively. Therefore, in the choice of *envy-free* and *peak-only* rules, there is no conflict among individual interests, and the uniform rule is unanimously considered to be best, while the equal division rule is worst.

Keywords: Uniform rule, Choice of rules, Lattice, Pareto dominance, Single-peaked preference, Fair allocation.

JEL codes: C72, D63, D61, C78, D71.

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*T. Sakai thanks the financial support from the project on "Analysis of resource allocation by game theory and matching theory" of Yokohama National University.

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1 Introduction

This paper studies the problem of fairly allocating an amount of a divisible resource among agents whose preferences are single-peaked (Sprumont, 1991). An allocation rule, or simply a *rule*, is a function which maps each single-peaked preference profile to an allocation. The fairness property of the rules we are interested in is *envy-freeness*, which states that, at any chosen allocation, no one should prefer anyone else's consumption to her own (Foley, 1967). The practicality condition we are interested in is *peak-onliness*, which states that the choice of allocations should only depend on the peaks of preferences. We say "practical", since the user of any *peak-only* rule only needs information on peak amounts of individual preferences, instead of all complicated details.

Our purpose is to study various *envy-free* and *peak-only* rules and the structure of the set of those rules. We do not impose *efficiency*, although our main results are deeply related to it. The aim is to extract pure implications of *envy-freeness* and *peak-onliness* as much as possible. However, it will turn out that the absence of *efficiency* does clarify the role of *efficiency* in some existing results in the literature, and in this sense, we are studying *efficiency*.

Since the class of *envy-free* and *peak-only* rules is quite large, we try to somehow compare the desirability of those rules. In Theorem 1, we show that, for every two *envy-free* and *peak-only* rules and every preference profile, the allocation chosen by one rule weakly Pareto dominates the allocation chosen by the other, and all agents are indifferent between the two allocations if and only if they are the same. Thus, given a preference profile, the Pareto dominance relation on the set of allocations chosen by *envy-free* and *peak-only* rules is a linear ordering. Furthermore, the unique greatest, least elements of the ordering are the uniform, equal division allocations, respectively.

We next consider ranking over rules. We say that a rule *dominates* another one if the allocation chosen by the former rule weakly Pareto dominates the allocation chosen by the latter at each and every preference profile. By definition, this dominance relation is a partial ordering over the set of *envy-free* and *peak-only* rules. In Theorem 2, we show that the set of *envy-free* and *peak-only* rules is a complete lattice with respect to this dominance relation, whose greatest, least elements are the uniform rule, and the equal division rule, respectively. Thus, as the title of this paper says, all *envy-free* and *peak-only* rules are "between" the uniform rule and the equal division rule. An immediate implication obtained from Theorem 2 is that, in the choice of *envy-free* and *peak-only* rules, there is no conflict among individual interests, and the uniform rule is unanimously considered to be best, while the equal division rule is worst.

We also show that, for every *envy-free* and *peak-only* rule and every preference profile, if the chosen allocation is neither the uniform allocation nor the equal division, its variance is more than the variance of the equal division and is less than the variance of the uniform allocation. This result greatly contrasts with Schummer and Thomson's (1997, Proposition 2) result whereby the variance of the uniform allocation is always less than the variance of any other *efficient* allocation.

The rest of the paper proceeds as follows: Section 2 offers the model. Section 3 presents main results. Section 4 provides discussions. Section 5 concludes the paper. Proofs of the main results are relegated to the Appendix.

2 Model

2.1 Basic definitions

Let $N \equiv \{1, 2, \ldots, n\}$ be the finite set of *agents*. There is a fixed amount of an infinitely divisible resource $\Omega > 0$ to be allocated. An *allotment* for $i \in N$ is $x_i \in [0, \Omega]$. An *allocation* is a vector of allotments $x \equiv (x_1, x_2, \ldots, x_n) \in [0, \Omega]^N$ such that $\sum_{i \in N} x_i = \Omega$. Let X be the set of allocations.

A *single-peaked preference relation* is a transitive, complete, and continuous binary relation R_i over $[0, \Omega]$ for which there exists a unique point $p_i \in [0, \Omega]$ such that for each $x_i, x'_i \in [0, \Omega]$,

$$[x'_i < x_i \leq p_i \text{ or } p_i \leq x_i < x'_i] \Longrightarrow x_i \ P_i \ x'_i,$$

where the symmetric and asymmetric parts of R_i are denoted by I_i and P_i, respectively. The point p_i is called the *peak* of R_i. Let $\mathscr{R}$ be the set of single-peaked preferences and $\mathscr{R}^N$ the set of single-peaked preference profiles $R \equiv (R_1, R_2, \ldots, R_n)$.

2.2 Rules and axioms

A *rule* is a function $f : \mathscr{R}^N \to X$ which maps a preference profile $R \in \mathscr{R}^N$ to an allocation $f(R) \equiv (f_1(R), f_2(R), \ldots, f_n(R)) \in X$. We are interested in the rules satisfying the following two requirements. The first condition is a central fairness property, which states that no one should prefer anyone else's allotment to her own at any chosen allocation (Foley, 1967). The next one is a practicality property, which states that the choice of allocations only needs information on peak amounts of individual preferences, instead of full information on preferences (Thomson, 1994).

***Envy-freeness*:** For each $R \in \mathscr{R}^N$ and each $i, j \in N$, $f_i(R)\ R_i\ f_j(R)$.

***Peak-onliness*:** For each $R, R' \in \mathscr{R}^N$, if $p_i = p'_i$ for each $i \in N$, then $f(R) = f(R')$.

Let $\mathscr{F}$ be the set of *envy-free* and *peak-only* rules. The following rules are example of rules that belong to $\mathscr{F}$:

***Uniform rule* (Benassy, 1982; Sprumont, 1991), *U*:** For each $R \in \mathscr{R}^N$ and each $i \in N$,

$$U_i(R) = \begin{cases} \min\{p_i, \lambda\} & \text{if } \sum_{j \in N} p_j \leq \Omega, \\ \max\{p_i, \lambda\} & \text{if } \sum_{j \in N} p_j \geq \Omega, \end{cases}$$

where λ solves $\sum_{j \in N} U_j(R) = \Omega$.

***Equal division rule, E*:** For each $R \in \mathscr{R}^N$ and each $i \in N$,

$$E_i(R) = \frac{\Omega}{n}.$$

Since the seminal work by Sprumont (1991), the uniform rule has played the central role in the literature. [1] The equal division rule is characterized by Bochet and Sakai (2007) on the basis of a strong implementability condition. A notable difference between the uniform rule and the equal division rule is that the former allocates resources efficiently but the latter does not, where *efficiency* is simply defined in this context as:

***Efficiency*:** For each $R \in \mathscr{R}^N$,

$$\Omega \leq \sum_{j \in N} p_j \Longrightarrow [f_i(R) \leq p_i \text{ for each } i \in N]\,,$$

$$\sum_{j \in N} p_j \leq \Omega \Longrightarrow [p_i \leq f_i(R) \text{ for each } i \in N]\,.$$

Thomson (1994, Lemma 1) shows that the uniform rule is the only *efficient*, *envy-free*, and *peak-only* rule. The next basic lemma clarifies what happens if *efficiency* is dropped from the list of Thomson's axioms.

[1] We refer to Thomson (2005) for a survey on various characterizations of the uniform rule.

Lemma 1. *Let f be a peak-only rule. Then, it is envy-free if and only if for each $R \in \mathscr{R}^N$ and each $i \in N$,*

$$f_i(R) < p_i \Longrightarrow f_i(R) = \max_{j \in N} f_j(R),$$
$$p_i < f_i(R) \Longrightarrow f_i(R) = \min_{j \in N} f_j(R).$$

Proof. We first show the "only if" part. Let $R \in \mathscr{R}^N$, $i \in N$, and $x \equiv f(R)$. We only consider the case $x_i < p_i$, since the opposite case can be dealt with by a parallel way. Suppose, by contradiction, that there exists $j \in N$ such that $x_i < x_j$. Then there exists $R'_i \in \mathscr{R}$ such that $p(R'_i) = p_i$ and $x_j \; P'_i \; x_i$. By *peak-onliness*, $x = f(R'_i, R_{-i})$, a contradiction to *envy-freeness*.

We next show the "if" part. Let $R \in \mathscr{R}^N$, $i \in N$, and $x \equiv f(R)$. If $x_i = p_i$, then obviously i envies no one. If $x_i < p_i$, then since $x_i = \max_{j \in N} x_j$, i envies no one. Similarly, if $p_i < x_i$, then since $x_i = \min_{j \in N} x_j$, i envies no one. □

Although we do not explicitly study *efficiency*, this does not mean that we are not interested in *efficiency*. Indeed, the absence of *efficiency* can clarify how significant the role of *efficiency* is in existing results that depend on *efficiency*. Also, one may obtain characterizations of some interesting rules without using *efficiency*. [2] There is another merit of not imposing *efficiency*. When a rule itself is to be selected among a class of rules, an *efficient* rule may not be chosen because some inefficient rule is supported by a large majority. However, if the *efficient* rule Pareto dominates other rules, we can conclude that the rule can obtain the unanimous support. Indeed we will establish such results.

2.3 Definitions on binary relations

We introduce some standard definitions on binary relations.

Definition 1 (Partial ordering). A binary relation $\succsim$ on a set A is a *partial ordering* if it satisfies:

- *Reflexivity*: For each $a \in A$, $a \succsim a$,
- *Transitivity*: For each $a, b, c \in A$, $[a \succsim b$ and $b \succsim c] \Longrightarrow a \succsim c$,

[2] Characterizations of the uniform rule without *efficiency* can be found in Sönmez (1994), Ehlers (2002), and Chun (2003, 2006).

- *Anti-symmetry*: For each $a, b \in A$, $[a \succsim b$ and $b \succsim a] \Longrightarrow a = b$.

Then a pair $(A, \succsim)$ is called a *partially ordered set*.

Definition 2 (Linear ordering). A binary relation $\succsim$ on a set A is a *linear ordering* if it is a partial ordering that satisfies:

- *Completeness*: For each $a, b \in A$, $a \succsim b$ or $b \succsim a$.

Then a pair $(A, \succsim)$ is called a *linearly ordered set*.

Definition 3 (Lattice theoretic notions). Consider a partial ordering $\succsim$ on a set A.

- *Join*: Given $B \subseteq A$, an element $a \in A$ is the *join* of B for $\succsim$ if it is the least maximal of B according to $\succsim$; that is, (i) for each $b \in B$, $a \succsim b$ and (ii) for each $a' \in A$, $[a' \succsim b$ for each $b \in B] \Longrightarrow a' \succsim a$.

- *Meet*: Similarly, an element $a \in A$ is the *meet* of B for $\succsim$ if it is the greatest minimal of B; that is, (i) for each $b \in B$, $b \succsim a$ and (ii) for each $a' \in A$, $[b \succsim a'$ for each $b \in B] \Longrightarrow a \succsim a'$.

- *Lattice*: A partially ordered set $(A, \succsim)$ is a *lattice* if for each $a, b \in A$, there exist the join and meet of $\{a, b\}$ for $\succsim$.

- *Complete lattice*: A partially ordered set $(A, \succsim)$ is a *complete lattice* if for each $B \subseteq A$, there exist the join and meet of B for $\succsim$.

If they exist, the join and the meet of B are uniquely determined by anti-symmetry of $\succsim$.

3 Main results

Our main results clarify Pareto ordered structures of the set of allocations chosen by *envy-free* and *peak-only* rules and the set of the rules.

Given $R \in \mathscr{R}^N$, let $\mathscr{F}(R) \equiv \{x \in X : \exists f \in \mathscr{F},\ f(R) = x\}$ be the set of allocations chosen by some *envy-free* and *peak-only* rule at R. Then the *dominance relation* on $\mathscr{F}(R)$, $dom[R]$, is defined to be the binary relation on $\mathscr{F}(R)$ such that for each $x, y \in X$,

$$x \; dom[R] \; y \iff [x_i \; R_i \; y_i \text{ for each } i \in N].$$

Our first main theorem ensures that, given any preference profile, all allocations chosen by some *envy-free* and *peak-only* rules are linearly ordered by the dominance relation:

Theorem 1. *For each $R \in \mathscr{R}^N$, $(\mathscr{F}(R), dom[R])$ is a linearly ordered set such that for each $f \in \mathscr{F}$,*

$$U(R)\ dom[R]\ f(R)\ dom[R]\ E(R).$$

Proof. See, the Appendix. □

We next analyze the order structure of the set $\mathscr{F}$. The *dominance relation* on $\mathscr{F}$, *dom*, is defined by, for each $f, g \in \mathscr{F}$,

$$f\ dom\ g \iff [f(R)\ dom[R]\ g(R) \text{ for each } R \in \mathscr{R}^N].$$

Note that *dom* is a partial ordering on $\mathscr{F}$. The next theorem shows that this ordering in fact establishes the complete lattice structure of $\mathscr{F}$:

Theorem 2. *The partially ordered set $(\mathscr{F}, dom)$ is a complete lattice whose greatest, least elements are the uniform rule, the equal division rule, respectively.*

Proof. See, the Appendix. □

This theorem implies that, under *envy-freeness* and *peak-onliness*, the uniform rule can be selected without caring who gains or loses from the choice of rules, since everyone gains by the use of the uniform rule independently of their preferences.

4 Discussions

4.1 Variance

The *variance function* is the function *var*: $\mathbb{R}^N_+ \to \mathbb{R}_+$ defined by, for each $x \in X$,

$$var(x) \equiv \frac{1}{n} \sum_{i \in N} \left(x_i - \frac{\Omega}{n} \right)^2.$$

The following result by Schummer and Thomson (1997) states that the variance of the uniform allocation is always smaller than the variance of any other *efficient* allocation:

Proposition 1. *For each efficient rule f and each $R \in \mathscr{R}^N$,*

$$var(U(R)) \leq var(f(R)).$$

Proof. See, Schummer and Thomson (1997, Proposition 2). □

This proposition ensures that the variance of the uniform allocation is the smallest among all *efficient* allocations. Our next proposition establishes an exactly converse implication for *envy-freeness* and *peak-onliness*.

Proposition 2. *For each* envy-free *and* peak-only *rule f and each $R \in \mathscr{R}^N$,*

$$var(E(R)) \leq var(f(R)) \leq var(U(R)).$$

Proof. See, the Appendix. □

Propositions 1 and 2 together suggest that, under *peak-onliness*, the uniform allocation is the "threshold" that separates the set of *efficient* allocations and the set of *envy-free* allocations in view of variance. This implication also explains why there is no *efficient*, *envy-free*, and *peak-only* rule other than the uniform rule.

4.2 Lattice structures in other models

In the context of two-sided matching problems such as marriage problems, job-matching problems, or assignment games,[3] the set of stable outcomes often forms a complete lattice (e.g., Roth, 1984; Sotomayor, 1999, 2000). A notable feature of two-sided matching problems is the presence of indivisibilities such as people or firms. On the other hand, there seems no lattice-like result in various allocation problems of divisible resources. To the best of our knowledge, the present study is the first one that finds lattice structures in divisible resource allocation problems.

4.3 Characterizations without efficiency

The role of *efficiency* in some characterization results in other social choice environments is often quite weak. For example, in Arrow's impossibility theorem, the role of *efficiency* (the Pareto principle) is only to eliminate the inversely dictatorial social welfare function (e.g., Murakami, 1968). Also, its role is only to eliminate the disagreement solution in characterizations of Nash's bargaining solution (Roth, 1977;

[3]For a survey of two-sided matching problems, see, Roth and Sotomayor (1990).

Thomson and Lensberg, 1988) and the no-trade rule in a characterization of the core in house allocation (Miyagawa, 2002). Our Lemma 1 contrasts with such results, since it ensures that there are indeed many *envy-free* and *peak-only* rules, although the uniform rule is the unique *efficient* rule among them.

5 Conclusion

We established the Pareto dominance relation over the set of allocations and the lattice structure over the set of rules under *envy-freeness* and *peak-onliness*. They suggest how strong the position of the uniform rule is and how weak the position of the equal division rule is in the division problem with single-peaked preferences. In general, this kind of easy-to-compare relations is rarely observed, except for two-sided matching problems. Thus results like ours are quite infrequent. Since neither the set of *envy-free* rules nor the set of *peak-only* rules exhibits any lattice structure, our results do depend on the pair of the two properties. Finding interesting sublattices of the set of *envy-free* and *peak-only* rules is an interesting future research topic.

Appendix: Proofs

The proofs proceed in several lemmas. To simplify notation, given $x \in X$, we write $\underline{x} \equiv \min_{i \in N} x_i$ and $\overline{x} \equiv \max_{i \in N} x_i$.

Lemma 2. *Let f be an envy-free and peak-only rule. For each $R \in \mathscr{R}^N$, if $\underline{f(R)} < \overline{f(R)}$, then for each $i \in N$,*

$$f_i(R) = \underline{f(R)} \iff p_i \leq \underline{f(R)}.$$

Proof. Let $x \equiv f(R)$. Let us first show ($\Longrightarrow$). If $x_i = \underline{x}$ but $\underline{x} < p_i$, then $x_i < p_i$. By Lemma 1, $x_i = \overline{x}$, a contradiction to $\underline{x} < \overline{x}$. Next let us show ($\Longleftarrow$). If $p_i \leq \underline{x}$ but $\underline{x} < x_i$, then $p_i < x_i$, a contradiction to Lemma 1. □

Lemma 3. *Let f be an envy-free and peak-only rule. For each $R \in \mathscr{R}^N$, if $\underline{f(R)} < \overline{f(R)}$, then for each $i \in N$,*

$$f_i(R) = \overline{f(R)} \iff \overline{f(R)} \leq p_i.$$

Proof. Similarly shown as Lemma 2. □

Lemma 4. *Let f be an envy-free and peak-only rule. For each $R \in \mathscr{R}^N$ and each $i \in N$,*

(i) $\underline{f(R)} \leq p_i \leq \overline{f(R)} \Longrightarrow p_i = f_i(R)$,

(ii) $\underline{f(R)} < f_i(R) < \overline{f(R)} \Longrightarrow p_i = f_i(R)$.

Proof. Let $x \equiv f(R)$. Part (i): By a contraposition argument, suppose that $p_i \neq x_i$. Consider the case $p_i < x_i$. Then by Lemma 1, $x_i = \underline{x}$, so $p_i < \underline{x}$. Next consider the case $x_i < p_i$. Then by Lemma 1, $x_i = \overline{x}$, so $\overline{x} < p_i$.

Part (ii): Similarly shown as Part (i). □

Lemma 5. *Let f, g be envy-free and peak-only rules. For each $R \in \mathscr{R}^N$, if $\underline{f(R)} = \underline{g(R)}$, then $f(R) = g(R)$.*

Proof. Let $x \equiv f(R)$, $y \equiv g(R)$, and $\underline{x} = \underline{y}$. If $\underline{x} = \overline{x}$ or $\underline{y} = \overline{y}$, then by feasibility, $x = E(R) = y$. Hence, let us consider the case $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$, Without loss of generality, we can assume $\overline{x} \leq \overline{y}$. Let

$$
\begin{aligned}
N(\underline{x}) &\equiv \{i \in N : p_i \leq \underline{x}\}, \\
N(\underline{y}) &\equiv \left\{i \in N : p_i \leq \underline{y}\right\}, \\
N(\overline{x}) &\equiv \{i \in N : \overline{x} \leq p_i\}, \\
N(\overline{y}) &\equiv \{i \in N : \overline{y} \leq p_i\}.
\end{aligned}
$$

Note that $N(\overline{y}) \subseteq N(\overline{x})$.

Since $\underline{x} = \underline{y}$ is assumed, we have $N(\underline{x}) = N(\underline{y})$, and by Lemma 2,

$$x_i = \underline{x} = \underline{y} = y_i \text{ for each } i \in N(\underline{x}). \tag{1}$$

By Lemma 4,

$$x_i = p_i = y_i \text{ for each } i \in N \setminus (N(\underline{x}) \cup N(\overline{x})). \tag{2}$$

By Lemmas 3 and 4,

$$x_i = \overline{x} \leq p_i = y_i \text{ for each } i \in N(\overline{x}) \setminus N(\overline{y}). \tag{3}$$

By Lemma 3,

$$x_i = \overline{x} \leq \overline{y} = y_i \text{ for each } i \in N(\overline{y}). \tag{4}$$

Since $\sum_{i \in N} x_i = \sum_{i \in N} y_i$, (1)–(4) together imply $x = y$. □

Lemma 6. *Let f, g be envy-free and peak-only rules. For each $R \in \mathscr{R}^N$, if $\underline{f(R)} < \underline{g(R)}$, then $\overline{g(R)} < \overline{f(R)}$.*

Proof. Suppose, by contradiction, that there exist $f, g \in \mathscr{F}$ and $R \in \mathscr{R}^N$ such that, whenever $x \equiv f(R)$ and $y \equiv g(R)$,

$$\underline{x} < \underline{y} \text{ and } \overline{x} \leq \overline{y}.$$

By feasibility,

$$\underline{x} < \underline{y} \leq \overline{x} \leq \overline{y} \text{ and } \underline{y} < \overline{y}.$$

By Lemmas 2–4,

$$\begin{array}{ll} x_i = \underline{x} < \underline{y} = y_i & \text{if } p_i \leq \underline{x}, \\ x_i = p_i \leq \underline{y} = y_i & \text{if } \underline{x} < p_i \leq \underline{y}, \\ x_i = p_i = y_i & \text{if } \underline{y} < p_i \leq \overline{x}, \\ x_i = \overline{x} < p_i = y_i & \text{if } \overline{x} < p_i \leq \overline{y}, \\ x_i = \overline{x} \leq \overline{y} = y_i & \text{if } \overline{y} < p_i. \end{array}$$

For $j \in N$ such that $x_j = \underline{x}$, Lemma 2 implies $p_j \leq \underline{x}$, so $x_j < y_j$. Hence, the above five relations together imply $\sum_{i \in N} x_i < \sum_{i \in N} y_i$, a contradiction. □

Lemma 7. *Let f, g be envy-free and peak-only rules. For each $R \in \mathscr{R}^N$, if $\underline{f(R)} < \underline{g(R)}$, then $f(R)$ dom$[R]$ $g(R)$ and not $g(R)$ dom$[R]$ $f(R)$.*

Proof. Immediately follows from Lemmas 2, 3, 4, and 6. □

Proof of Theorem 1. Let $R \in \mathscr{R}^N$. Obviously, *dom*$[R]$ is reflexive and transitive. Completeness follows from Lemmas 5 and 7. It remains to check anti-symmetry. If f, g, R are such that $f(R)$ *dom*$[R]$ $g(R)$ and $f(R)$ *dom*$[R]$ $g(R)$, then Lemma 7 implies $\underline{f(R)} = \underline{g(R)}$, which in turn implies by Lemma 5 $f(R) = g(R)$. Thus *dom*$[R]$ is anti-symmetric.

It remains to see that, given $f \in \mathscr{F}$ and $R \in \mathscr{R}^N$, $U(R)$ *dom*$[R]$ $f(R)$ *dom*$[R]$ $E(R)$. Obviously, $f(R)$ *dom*$[R]$ $E(R)$. Lemmas 1 and 7 together imply $U(R)$ *dom*$[R]$ $f(R)$. □

Proof of Theorem 2. We first show that $(\mathscr{F}, dom)$ is a lattice. Let $f, g \in \mathscr{F}$. Define the rule $f \vee g$ by, for each $R \in \mathscr{R}^N$,

$$f \vee g(R) \equiv \begin{cases} f(R) & \text{if } f(R) \; dom[R] \; g(R), \\ g(R) & \text{if } g(R) \; dom[R] \; f(R). \end{cases}$$

By Theorem 1, $f \vee g$ is well-defined. By Theorem 1, $f \vee g$ is *envy-free* and *peak-only*, so $f \vee g \in \mathscr{F}$. Obviously, $f \vee g$ is the join of f and g. The meet $f \wedge g$ can be found by a parallel way. Thus $(\mathscr{F}, dom)$ is a lattice.

We next show that $(\mathscr{F}, dom)$ is complete. Let $\mathscr{G} \subseteq \mathscr{F}$. Define the rule $\vee\mathscr{G}$ by, for each $R \in \mathscr{R}^N$,

$$\vee\mathscr{G}(R) \equiv x,$$

where x is such that $x \in \mathscr{F}(R)$ and $\underline{x} = \inf_{f \in \mathscr{G}} \min_{i \in N} f_i(R)$. Note that the existence of x follows from the compactness of $\mathscr{F}(R)$ and the uniqueness of x follows from Lemma 5. Thus $\vee\mathscr{G}$ is well-defined. Obviously, $\vee\mathscr{G}$ is the unique least upper bound of $\mathscr{G}$. The unique greatest lower bound of $\mathscr{G}$ can be parallely found. Thus $(\mathscr{F}, dom)$ is a complete lattice.

The fact that the uniform, the equal division rules are the greatest, least elements of $(\mathscr{F}, dom)$, respectively, immediately follows from Theorem 1. □

Proof of Proposition 2. Let $f \in \mathscr{F}$ and $R \in \mathscr{R}^N$. Obviously, $var(E(R)) \leq var(f(R))$. Lemmas 2–6 together imply $var(f(R)) \leq var(U(R))$. □

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