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Full-Truthful Implementation in Nash Equilibria

Hideki Mizukami[†] and Takuma Wakayama[‡]

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[†]Faculty of Economics, University of Toyama, 3190 Gofuku, Toyama, 930-8555, JAPAN; mizukami@eco.u-toyama.ac.jp
[‡]Faculty of Urban Liberal Arts, Tokyo Metropolitan University, 1-1 Minami-Osawa, Hachioji, Tokyo 192-0397, JAPAN; t_waka@center.tmu.ac.jp

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Hideki Mizukami†
Faculty of Economics, University of Toyama
mizukami@eco.u-toyama.ac.jp

and

Takuma Wakayama
Faculty of Urban Liberal Arts, Tokyo Metropolitan University
t_waka@center.tmu.ac.jp

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†Corresponding author. Faculty of Economics, University of Toyama, 3190 Gofuku, Toyama 930-8555, Japan.

Abstract

We consider *full-truthful Nash implementation*, which requires truth telling by each agent to be a Nash equilibrium of a direct revelation mechanism, and every Nash equilibrium outcome of the mechanism to be f-optimal. We show that *restricted monotonicity* plus an auxiliary condition is necessary and sufficient for full-truthful Nash implementation, and that full-truthful Nash implementation is equivalent to secure implementation (Saijo et al. (2005)). We also identify a condition both necessary and sufficient for ex-post implementation by the associated direct revelation mechanisms, which closes the gap between Bergemann and Morris (2007)'s necessary and sufficient conditions.

Keywords: Restricted Monotonicity, A Direct Revelation Mechanism, Nash Implementation, Secure Implementation, Ex-Post Implementation.

JEL Classification Numbers: C72, D71.

1 Introduction

The implementation problem is that a mechanism designer, who cannot observe the true preferences of each agent, devises a mechanism whose equilibrium outcomes always coincide with the social goal given by a social choice correspondence. The Nash equilibrium concept has often been used as an equilibrium concept in complete information environments, where each agent knows not only own true preferences but also the true preferences of every other agent, while the mechanism designer cannot observe agents' true preferences. In the seminal paper on Nash implementation, Maskin (1999) showed that *monotonicity* is necessary for Nash implementation, and that monotonicity plus *no veto power* is sufficient for Nash implementation when there are three or more agents. The gap between necessity and sufficiency for Nash implementation has subsequently been closed by Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991), etc.

However, the positive results listed above rely on *complicated* mechanisms used in the constructive proofs, where agents are often forced to announce an outcome, an integer, etc. in addition to a preference profile. Such mechanisms have been criticized not only for practicability but also for a theoretical reason. For example, Jackson (1992) criticized such mechanisms for employing an integer game, because mechanisms involving the integer game fail to satisfy the *best response property* regardless of the use of the Nash equilibrium concept.

In response to the criticisms, we consider Nash implementation by a *direct revelation mechanism*, i.e., a *simple* mechanism where agents are required to report own preferences only. Much attention has recently focused on direct revelation mechanisms from a practical perspective as well as a theoretical viewpoint. For example, Roth (1984) analyzed a direct revelation mechanism used for the National Resident Matching Program, Abdulkadiroğlu and Sönmez (2003) studied direct revelation mechanisms for school choice, and Roth et al. (2004) proposed direct revelation mechanisms for kidney exchange. Moreover, direct revelation mechanisms satisfy *self-relevancy* (Hurwicz (1960)), the requirement that each agent should be asked to reveal information about only herself.[1] So, the mechanisms seem attractive from the viewpoint of informational decentralization. However, few would study Nash implementation by direct revelation mechanisms. This would be partly because the revelation principle for Nash implementation does not hold in general.

Nevertheless, in this paper, we restrict attention to full Nash implementation by a direct revelation mechanism where truth telling by each agent is a Nash equilibrium of the mechanism, which we call *full-truthful Nash implementation.* The restriction narrows the class of Nash implementable social choice functions, since the revelation principle for Nash implementation cannot hold.

[1] Tatamitani (2001) considered Nash implementation by self-relevant mechanisms, where each agent is required to announce own preferences, an outcome, and an agent index.

Indeed, the class of fully-truthfully Nash implementable social choice functions is limited to the class smaller than that of truthfully Nash implementable social choice functions, which is equivalent to that of truthfully dominant strategy implementable social choice functions.

However, the requirement that truthful revelation by each agent should be a Nash equilibrium of a direct revelation mechanism would be attractive from a practical standpoint. If there are multiple equilibria in a direct revelation mechanism violating the requirement, then it would be hard for agents to predict each others' actions, which could lead to miscoordination (see Example 1 in Section 3 for more details). But, since each agent knows the other agents' preferences, if truthful reporting by each agent is a Nash equilibrium of the mechanism, then the *truthful* Nash equilibrium would become a *focal point* (Schelling (1960)), and so agents would be able to coordinate their actions.[2] Moreover, the requirement would be acceptable from a theoretical point of view: if a direct revelation mechanism satisfies the requirement, then it satisfies the best response property. Thus, this paper studies Nash implementation by a "nice" mechanism, which is in contrast to the one devised by Maskin (1999).

To make practical use of a mechanism that is theoretically constructed for Nash implementation, it is important to pay attention to the possibility that agents fail to coordinate their actions. This is because coordination failure could arise if the mechanism possesses multiple equilibria, as mentioned above and demonstrated in coordination game experiments (e.g., see Camerer (2003)). In a mechanism with a focal point, however, agents would be able to coordinate their actions even if the mechanism has multiple equilibria. This paper thinks about Nash implementation by a mechanism designed for the purpose of preventing miscoordination.[3]

This paper relates to one by Saijo et al. (2005), who identified a condition necessary and sufficient for *secure implementation*,[4] i.e., double implementation in dominant strategy equilibria and Nash equilibria. In Theorem 2, we show that full-truthful Nash implementation is equivalent to secure implementation, which gives us an alternative characterization of securely implementable social

[2]One might guess that it is enough to consider truthful Nash implementation if the truthful Nash equilibrium of a direct revelation mechanism becomes a focal point. However, when considering truthful Nash implementation, the truthful Nash equilibrium of the mechanism could not serve as a focal point (see footnote 9 for more details). This implies that the truthful Nash equilibrium outcome could not be achieved in practice even if truthful Nash implementation is possible. So, *full-truthful* Nash implementation needs to be studied.

[3]In addition to thinking of Nash implementation using the idea of a focal point, another way of avoiding miscoordination is to consider implementation using refinements of Nash equilibrium. For example, implementation in undominated Nash equilibria was considered by Palfrey and Srivastava (1991), Jackson et al. (1994), Sjöström (1994), etc. However, miscoordination could still occur if there are multiple undominated Nash equilibria in a mechanism constructed for undominated Nash implementation.

[4]Cason et al. (2006) ran experiments to compare the performance of secure and non-secure mechanisms. They reported experimental results which suggest that secure implementation is appealing from a practical application standpoint.

choice functions. The equivalence derives mostly from the result of Dasgupta et al. (1979) stating that a direct revelation mechanism truthfully Nash implements a social choice function if and only if the mechanism truthfully dominant strategy implements the social choice function. The equivalence indicates that a direct revelation mechanism satisfying the requirement of truthful revelation by each agent being a Nash equilibrium of the mechanism is robust with respect to the equilibrium concept: the mechanism can implement a social choice function not only in Nash equilibria but also in dominant strategy equilibria.

In Section 5, contrary to the other sections, ex-post implementation by the associated direct revelation mechanisms is considered in incomplete information environments with interdependent values. So, this paper is also related to one by Bergemann and Morris (2007), who identified necessary conditions and, separately in general environments and single crossing environments, sufficient conditions for ex-post implementation by the associated direct revelation mechanisms. In Theorem 3, we derive a complete characterization of the class of social choice functions which are ex-post implemented by the associated direct revelation mechanisms. Our characterization result fills the gap between Bergemann and Morris (2007)'s necessary and sufficient conditions for ex-post implementation by the associated direct revelation mechanisms. Furthermore, in contrast to their results, our result holds both in general environments and single crossing environments.

This paper is organized as follows. Section 2 provides notation and definitions. We identify a necessary and sufficient condition for full-truthful Nash implementation in Section 3. In Section 4, we examine the relationship of full-truthful Nash implementation to secure implementation. Section 5 presents a result identifying a condition both necessary and sufficient for ex-post implementation by the associated direct revelation mechanisms. Section 6 contains some concluding remarks.

2 Notation and Definitions

Let $N := \{1, 2, \ldots, n\}$ be the set of *agents*, where $2 \leq n < +\infty$. Let A be the set of feasible *outcomes*.

Each agent $i \in N$ has *preferences* over A, which are represented by a complete and transitive binary relation R_i. The strict preference relation and indifference relation associated with R_i are denoted by P_i and I_i, respectively. Let $\mathscr{R}_i$ denote the set of possible preferences for agent $i \in N$. The *domain* is denoted by $\mathscr{R} := \mathscr{R}_1 \times \mathscr{R}_2 \times \cdots \times \mathscr{R}_n$. A *preference profile* is a list $R = (R_1, R_2, \ldots, R_n) \in \mathscr{R}$. It is assumed that each agent can observe not only her own preferences but also all other agents' preferences.[5]

An *environment* is a collection $(N, A, \mathscr{R})$.

[5] In Section 5, we consider incomplete information environments with interdependent values.

Let $LC_i(a;R_i) := \{b \in A \mid a\, R_i\, b\}$ be agent i's *lower contour set* of $a \in A$ at $R_i \in \mathscr{R}_i$. For each $i \in N$, let $ME_i(\bar{A};R_i) := \left\{a \in \bar{A} \,\middle|\, a\, R_i\, b \text{ for all } b \in \bar{A}\right\}$ be the set of *maximal elements* in $\bar{A} \subseteq A$ at $R_i \in \mathscr{R}_i$.[6]

A *social choice function* is a single-valued function $f\colon \mathscr{R} \to A$ that assigns a feasible outcome $a \in A$ to each preference profile $R \in \mathscr{R}$. Given a social choice function f, let $O_i(R) := \left\{a \in A \,\middle|\, a = f(R'_i, R_{-i}) \text{ for some } R'_i \in \mathscr{R}_i\right\}$ be agent i's *option set* at $R \in \mathscr{R}$. Note that $O_i(R) = O_i(R'_i, R_{-i})$ for all $R \in \mathscr{R}$, all $i \in N$, and all $R'_i \in \mathscr{R}_i$.

Let M_i denote a *message space* of agent $i \in N$. We call $m_i \in M_i$ a *message* of agent $i \in N$. A *mechanism* is a pair $\Gamma = (M, g)$, where $M := M_1 \times M_2 \times \cdots \times M_n$ and $g\colon M \to A$ is an *outcome function.* A mechanism (M, g) is called a *direct revelation mechanism* if $M_i = \mathscr{R}_i$ for all $i \in N$. Given a social choice function f, a mechanism (M, g) is called the *associated direct revelation mechanism* if $M_i = \mathscr{R}_i$ for all $i \in N$ and $g = f$. A *message profile* is denoted by $m = (m_1, m_2, \ldots, m_n) \in M$.

A message profile $m^* = (m^*_1, m^*_2, \ldots, m^*_n) \in M$ is a *Nash equilibrium* of a mechanism (M, g) at $R \in \mathscr{R}$ if, for any $i \in N$, $g(m^*_i, m^*_{-i})\, R_i\, g(m'_i, m^*_{-i})$ for any $m'_i \in M_i$. Let $NE^\Gamma(R) \subseteq M$ denote the set of Nash equilibria of a mechanism $\Gamma = (M, g)$ at $R \in \mathscr{R}$. A message profile $m^* = (m^*_1, m^*_2, \ldots, m^*_n) \in M$ is a *dominant strategy equilibrium* of a mechanism (M, g) at $R \in \mathscr{R}$ if, for any $i \in N$, $g(m^*_i, m'_{-i})\, R_i\, g(m'_i, m'_{-i})$ for any $m'_i \in M_i$ and any $m'_{-i} \in M_{-i}$. Let $DSE^\Gamma(R) \subseteq M$ be the set of dominant strategy equilibria of a mechanism $\Gamma = (M, g)$ at $R \in \mathscr{R}$.

Let $\mathscr{E}$*-equilibrium* be a game theoretic equilibrium concept. Let $g(\mathscr{E}^\Gamma(R)) := \left\{a \in A \,\middle|\, a = g(m) \text{ for some } m \in \mathscr{E}^\Gamma(R)\right\}$ denote the set of $\mathscr{E}$*-equilibrium outcomes* of a mechanism $\Gamma = (M, g)$ at $R \in \mathscr{R}$, where $\mathscr{E}^\Gamma(R) \subseteq M$ denotes the set of $\mathscr{E}$-equilibria of the mechanism at $R \in \mathscr{R}$. A mechanism $\Gamma = (M, g)$ *implements* a social choice function f *in* $\mathscr{E}$*-equilibria* (or $\mathscr{E}$*-implements* f) if $g(\mathscr{E}^\Gamma(R)) = f(R)$ for any $R \in \mathscr{R}$.[7] A social choice function is $\mathscr{E}$*-implementable* (or *implementable in* $\mathscr{E}$*-equilibria*) if there exists a mechanism that $\mathscr{E}$-implements it. A direct revelation mechanism $\Gamma = (\mathscr{R}, g)$ *truthfully implements* a social choice function f *in* $\mathscr{E}$*-equilibria* if $R \in \mathscr{E}^\Gamma(R)$ and $g(R) = f(R)$ for any $R \in \mathscr{R}$.[8] A social choice function is *truthfully* $\mathscr{E}$*-implementable* if there exists a direct revelation mechanism which truthfully $\mathscr{E}$-implements it.

A social choice function is *directly implementable in Nash equilibria* if there exists a direct revelation mechanism which Nash implements it. A direct revelation mechanism $\Gamma = (\mathscr{R}, g)$ *fully-truthfully implements* a social choice function f *in Nash equilibria* if $R \in NE^\Gamma(R)$ and $g(NE^\Gamma(R)) = f(R)$ for any $R \in \mathscr{R}$. A social choice function is *fully-truthfully Nash implementable* if there exists a direct revelation mechanism which fully-truthfully Nash implements it.

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[6]Note that $ME_i(\bar{A};R_i)$ may be empty.

[7]To simplify notation, we write $f(R)$ instead of $\{f(R)\}$.

[8]As we focus on social choice functions, truthful $\mathscr{E}$-implementation can be defined as $R \in \mathscr{E}^\Gamma(R)$ and $g(R) = f(R)$ for any $R \in \mathscr{R}$, instead of as $R \in \mathscr{E}^\Gamma(R)$ and $g(R) \in f(R)$ for any $R \in \mathscr{R}$.

3 Characterizations

In this section, we seek to characterize social choice functions which are fully-truthfully implementable in Nash equilibria.

We begin by proving the *revelation principle for full-truthful Nash implementation*, which says that the class of social choice functions that are fully-truthfully implementable in Nash equilibria is equivalent to that of social choice functions which can be fully-truthfully Nash implemented by the associated direct revelation mechanisms.

Fact 1 (The Revelation Principle for Full-Truthful Nash Implementation). *A social choice function f is fully-truthfully implementable in Nash equilibria if and only if it is fully-truthfully implemented in Nash equilibria by the associated direct revelation mechanism.*

Proof. The if part. This part follows straight from the definition of full-truthful Nash implementation.

The only if part. Let $\Gamma = (\mathscr{R}, f)$ denote the associated direct revelation mechanism. Since f is fully-truthfully implementable in Nash equilibria, there is a direct revelation mechanism $\bar{\Gamma} = (\mathscr{R}, g)$ such that $R \in NE^{\bar{\Gamma}}(R)$ and $g(NE^{\bar{\Gamma}}(R)) = f(R)$ for all $R \in \mathscr{R}$. So, $R \in NE^{\bar{\Gamma}}(R)$ and $g(R) = f(R)$ for all $R \in \mathscr{R}$. Hence, we have $g = f$, implying $\bar{\Gamma} = \Gamma$. Thus, $R \in NE^{\Gamma}(R)$ and $f(NE^{\Gamma}(R)) = f(R)$ for all $R \in \mathscr{R}$. □

The following example demonstrates that if we give up the requirement of truthful revelation by each agent being a Nash equilibrium of a direct revelation mechanism, then there is a social choice function that cannot be Nash implemented by the associated direct revelation mechanism, but which is directly Nash implementable.

Example 1. Consider an environment $(N, A, \mathscr{R})$ such that $\#N = 2$, $A = \{a, b, c\}$, $\mathscr{R} = \{R_1, \bar{R}_1\} \times \{R_2, \bar{R}_2\}$, and $a\ P_i\ b\ P_i\ c$ and $c\ \bar{P}_i\ a\ \bar{P}_i\ b$ for all $i \in N$. A social choice function f is given as follows.

$f =$

	R_2	$\bar{R}_2$
R_1	a	a
$\bar{R}_1$	a	c

Then, f cannot be Nash implemented by the associated direct revelation mechanism $\Gamma = (\mathscr{R}, f)$ (although it can be truthfully implemented in Nash equilibria by Γ). This is because $f(NE^{\Gamma}(\bar{R}_1, \bar{R}_2)) = \{a, c\} \neq \{c\} = f(\bar{R}_1, \bar{R}_2)$, since $NE^{\Gamma}(\bar{R}_1, \bar{R}_2) = \{(R_1, R_2), (\bar{R}_1, \bar{R}_2)\}$.

However, another direct revelation mechanism $\bar{\Gamma} = (\mathscr{R}, g)$ can Nash implement f, where g is defined below.

$g =$

	R_2	$\bar{R}_2$
R_1	c	a
$\bar{R}_1$	a	b

Since $NE^{\bar{\Gamma}}(R_1,R_2) = \{(R_1,\bar{R}_2),(\bar{R}_1,R_2)\}$, $NE^{\bar{\Gamma}}(R_1,\bar{R}_2) = \{(\bar{R}_1,R_2)\}$, $NE^{\bar{\Gamma}}(\bar{R}_1,R_2) = \{(R_1,\bar{R}_2)\}$, and $NE^{\bar{\Gamma}}(\bar{R}_1,\bar{R}_2) = \{(R_1,R_2)\}$, we have $g(NE^{\bar{\Gamma}}(R_1,R_2)) = \{a\} = f(R_1,R_2)$, $g(NE^{\bar{\Gamma}}(R_1,\bar{R}_2)) = \{a\} = f(R_1,\bar{R}_2)$, $g(NE^{\bar{\Gamma}}(\bar{R}_1,R_2)) = \{a\} = f(\bar{R}_1,R_2)$, and $g(NE^{\bar{\Gamma}}(\bar{R}_1,\bar{R}_2)) = \{c\} = f(\bar{R}_1,\bar{R}_2)$, respectively. Thus, f can be Nash implemented by $\bar{\Gamma}$, although it cannot be Nash implemented by the associated direct revelation mechanism.

It should be noted that truth telling by each agent is never a Nash equilibrium of $\bar{\Gamma}$ (whereas it is always a Nash equilibrium of Γ). So, as mentioned in the introduction, coordination failure could occur when $\bar{\Gamma}$ possesses multiple equilibria. In fact, when the true preference profile is R, $\bar{\Gamma}$ has two Nash equilibria, $(R_1,\bar{R}_2)$ and $(\bar{R}_1,R_2)$. It would be difficult for agents to coordinate their actions in $\bar{\Gamma}$ at R, partly because agents are indifferent between Nash equilibrium outcomes $g(R_1,\bar{R}_2)$ and $g(\bar{R}_1,R_2)$ since the equilibrium outcomes are the same by full implementability. However, since each agent knows with certainty every other agent's preferences, and since all Nash equilibrium outcomes are the same by full implementability, if truthful revelation by each agent is a Nash equilibrium of $\bar{\Gamma}$, then the *truthful* Nash equilibrium would be salient, and so it would serve as a focal point.[9] Thus, although $\bar{\Gamma}$ can Nash implement f, it would not be so attractive from a practical viewpoint.[10] ■

Invoking the revelation principle for full-truthful Nash implementation, we restrict attention to the associated direct revelation mechanisms hereafter. We next identify a condition which is necessary for full-truthful Nash implementation by the associated direct revelation mechanisms.

Restricted monotonicity is a version of *monotonicity*[11] (Maskin (1999)), which

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[9]When considering truthful Nash implementation, the truthful Nash equilibrium of a direct revelation mechanism could not be salient, and so it could not be a focal point. This is because, when considering not full-truthful Nash implementation but truthful Nash implementation, the mechanism often has an *untruthful* Nash equilibrium too, whose outcome is not the same as the truthful Nash equilibrium outcome since full-truthful Nash implementation is impossible. Recall that the truthful Nash equilibrium is highlighted since all Nash equilibrium outcomes are the same by full implementability. In fact, the truthful Nash equilibrium would not be highlighted if a direct revelation mechanism has an untruthful Nash equilibrium that Pareto dominates the truthful Nash equilibrium. See also Moore and Repullo (1988) for a similar discussion.

[10]A direct revelation mechanism $\hat{\Gamma} = (\mathscr{R}, g)$ can also Nash implement f, where g is as follows.

$g =$

	R_2	$\bar{R}_2$
R_1	a	c
$\bar{R}_1$	b	a

This is because $NE^{\hat{\Gamma}}(R_1,R_2) = \{(R_1,R_2),(\bar{R}_1,\bar{R}_2)\}$, $NE^{\hat{\Gamma}}(R_1,\bar{R}_2) = \{(\bar{R}_1,\bar{R}_2)\}$, $NE^{\hat{\Gamma}}(\bar{R}_1,R_2) = \{(R_1,R_2)\}$, and $NE^{\hat{\Gamma}}(\bar{R}_1,\bar{R}_2) = \{(R_1,\bar{R}_2)\}$. Similar to $\bar{\Gamma}$, indeed, truthful reporting by each agent is not always a Nash equilibrium of $\hat{\Gamma}$. But, it is a Nash equilibrium whenever $\hat{\Gamma}$ has multiple equilibria. So, since the truthful Nash equilibrium would become a focal point, miscoordination would not arise in $\hat{\Gamma}$ at R, which is in contrast to $\bar{\Gamma}$. Thus, a direct revelation mechanism where truth telling by each agent is a Nash equilibrium of it whenever it has multiple equilibria would be appealing from a practical standpoint. This paper leaves open the question of what condition is necessary and sufficient for Nash implementation by such a mechanism.

[11]A social choice function f satisfies *monotonicity* if, for all $R, R' \in \mathscr{R}$, if $LC_i(f(R); R_i) \subseteq$

requires the following. Suppose a change from $R \in \mathscr{R}$ to $R' \in \mathscr{R}$. Then, for each agent $i \in N$, if any outcome that was weakly worse for her than $f(R)$ *in her option set* at R when her preferences are R_i remains weakly worse for her than $f(R)$ when her preferences are R'_i, then $f(R)$ must still be f-optimal at R'.

Definition 1 (Restricted Monotonicity). A social choice function f satisfies *restricted monotonicity* if, for all $R, R' \in \mathscr{R}$, if $LC_i(f(R); R_i) \cap O_i(R) \subseteq LC_i(f(R); R'_i)$ for all $i \in N$, then $f(R') = f(R)$.

Remark 1. Restricted monotonicity is stronger than monotonicity by definition.

Remark 2. The definition of restricted monotonicity can be rewritten as follows. A social choice function f satisfies *restricted monotonicity* if, for all $R \in \mathscr{R}$ and all $i \in N$, there is a set $C_i(f(R); R) \subseteq O_i(R)$ with $f(R) \in ME_i(C_i(f(R); R); R_i)$ such that for all $R' \in \mathscr{R}$, if $f(R) \in ME_i(C_i(f(R); R); R'_i)$ for all $i \in N$, then $f(R') = f(R)$. This way of defining restricted monotonicity is analogous to that of defining Condition μ (Moore and Repullo (1990)).

Since restricted monotonicity is stronger than monotonicity by Remark 1, it is not clear whether restricted monotonicity is a necessary condition for Nash implementation. The following lemma states that restricted monotonicity is necessary for full-truthful implementation in Nash equilibria by the associated direct revelation mechanisms.

Lemma 1. *If a social choice function f is fully-truthfully implemented in Nash equilibria by the associated direct revelation mechanism, then it satisfies restricted monotonicity.*

Proof. Let $\Gamma = (\mathscr{R}, f)$ denote the associated direct revelation mechanism. Pick any $R, \bar{R} \in \mathscr{R}$ such that $LC_i(f(R); R_i) \cap O_i(R) \subseteq LC_i(f(R); \bar{R}_i)$ for all $i \in N$. Since f is fully-truthfully implemented in Nash equilibria by Γ, we have $R \in NE^{\Gamma}(R)$.

Since $R \in NE^{\Gamma}(R)$, it follows that for all $i \in N$, $f(R)\ R_i\ f(R'_i, R_{-i})$ for all $R'_i \in \mathscr{R}_i$. This implies $f(R) \in ME_i(O_i(R); R_i)$ for all $i \in N$. So, $LC_i(f(R); R_i) \cap O_i(R) = O_i(R)$ for all $i \in N$.

Thus, since $LC_i(f(R); R_i) \cap O_i(R) \subseteq LC_i(f(R); \bar{R}_i)$ for all $i \in N$, we have $O_i(R) \subseteq LC_i(f(R); \bar{R}_i)$ for all $i \in N$. So, for all $i \in N$, $f(R)\ \bar{R}_i\ f(R'_i, R_{-i})$ for all $R'_i \in \mathscr{R}_i$, implying $R \in NE^{\Gamma}(\bar{R})$. Hence, $f(R) \in f(NE^{\Gamma}(\bar{R}))$, whereas $f(NE^{\Gamma}(\bar{R})) = f(\bar{R})$ because f is fully-truthfully Nash implemented by Γ. Thus, $f(R) \in f(NE^{\Gamma}(\bar{R})) = f(\bar{R})$. This implies $f(\bar{R}) = f(R)$, since f is a single-valued function. □

We are now ready to characterize fully-truthfully implementable social choice functions in Nash equilibria. Theorem 1 below says that restricted monotonicity together with an auxiliary condition called *individual maximality* is both necessary and sufficient for full-truthful Nash implementation. It should be noted that Theorem 1 holds even when $n = 2$.

$LC_i(f(R); R'_i)$ for all $i \in N$, then $f(R') = f(R)$. Note that monotonicity is a necessary condition for Nash implementation and is part of the sufficient condition for Nash implementation when there are three or more agents.

Definition 2 (Individual Maximality). A social choice function f satisfies *individual maximality* if, for all $R \in \mathscr{R}$, $ME_i(O_i(R); R_i) \neq \emptyset$ for all $i \in N$.

Remark 3. As we focus on implementation by the associated direct revelation mechanisms, imposing individual maximality on a social choice function is equivalent to requiring the associated direct revelation mechanism to satisfy the *best response property*[12] (Jackson et al. (1994)). As mentioned by Jackson et al. (1994), the best response property would be an appropriate restriction in order for the Nash equilibrium concept to make sense. Theorem 1 shows that the restriction is not only part of the sufficient condition but also part of the necessary condition for a social choice function to be fully-truthfully Nash implementable.

Theorem 1. *A social choice function f is fully-truthfully implementable in Nash equilibria if and only if it satisfies restricted monotonicity and individual maximality.*

Proof. Let $\Gamma = (\mathscr{R}, f)$ denote the associated direct revelation mechanism.

The if part. *Step 1*: $f(R) \in ME_i(O_i(R); R_i)$ for all $R \in \mathscr{R}$ and all $i \in N$.

Suppose to the contrary that $f(R) \notin ME_i(O_i(R); R_i)$ for some $R \in \mathscr{R}$ and some $i \in N$. Let $b \in A$ be such that $b \in ME_i(O_i(R); R_i)$.[13] Then, $b \neq f(R)$. Since $b \in O_i(R)$, $b = f(\bar{R}_i, R_{-i})$ for some $\bar{R}_i \in \mathscr{R}_i$.

Since $f(\bar{R}_i, R_{-i}) = b \in ME_i(O_i(R); R_i)$, we have $LC_i(f(\bar{R}_i, R_{-i}); R_i) \cap O_i(R) = O_i(R)$. So, since $LC_i(f(\bar{R}_i, R_{-i}); \bar{R}_i) \cap O_i(R) \subseteq O_i(R)$, we obtain $LC_i(f(\bar{R}_i, R_{-i}); \bar{R}_i) \cap O_i(R) \subseteq LC_i(f(\bar{R}_i, R_{-i}); R_i) \cap O_i(R)$. Hence, it follows from $O_i(R) = O_i(\bar{R}_i, R_{-i})$ that $LC_i(f(\bar{R}_i, R_{-i}); \bar{R}_i) \cap O_i(\bar{R}_i, R_{-i}) \subseteq LC_i(f(\bar{R}_i, R_{-i}); R_i) \cap O_i(\bar{R}_i, R_{-i})$. So, we have $LC_i(f(\bar{R}_i, R_{-i}); \bar{R}_i) \cap O_i(\bar{R}_i, R_{-i}) \subseteq LC_i(f(\bar{R}_i, R_{-i}); R_i)$, because $LC_i(f(\bar{R}_i, R_{-i}); R_i) \cap O_i(\bar{R}_i, R_{-i}) \subseteq LC_i(f(\bar{R}_i, R_{-i}); R_i)$. Hence, since $LC_j(f(\bar{R}_i, R_{-i}); R_j) \cap O_j(\bar{R}_i, R_{-i}) \subseteq LC_j(f(\bar{R}_i, R_{-i}); R_j)$ for all $j \neq i$, restricted monotonicity implies $f(R) = f(\bar{R}_i, R_{-i})$, which contradicts $f(R) \neq b = f(\bar{R}_i, R_{-i})$.

Step 2: f satisfies *strategy-proofness*.[14]

Since $f(R) \in ME_i(O_i(R); R_i)$ for all $R \in \mathscr{R}$ and all $i \in N$ by Step 1, it follows that $f(R)\ R_i\ f(R'_i, R_{-i})$ for all $R \in \mathscr{R}$, all $i \in N$, and all $R'_i \in \mathscr{R}_i$. Thus, f satisfies strategy-proofness.

Step 3: f is fully-truthfully Nash implementable.

Pick any $R \in \mathscr{R}$. Since f satisfies strategy-proofness by Step 2, $R \in DSE^{\Gamma}(R)$, implying $R \in NE^{\Gamma}(R)$.

Suppose $\bar{R} \in NE^{\Gamma}(R)$. Then, for any $i \in N$, $f(\bar{R})\ R_i\ f(R'_i, \bar{R}_{-i})$ for any $R'_i \in \mathscr{R}_i$. This implies $f(\bar{R}) \in ME_i(O_i(\bar{R}); R_i)$ for all $i \in N$, implying $LC_i(f(\bar{R}); R_i) \cap$

[12] A mechanism (M, g) satisfies the *best response property* if, for all $R \in \mathscr{R}$, all $i \in N$, and all $m_{-i} \in M_{-i}$, there exists $m_i \in M_i$ such that $g(m_i, m_{-i})\ R_i\ g(m'_i, m_{-i})$ for all $m'_i \in M_i$.

[13] It should be noted that $ME_i(O_i(R); R_i) \neq \emptyset$ by individual maximality.

[14] A social choice function f satisfies *strategy-proofness* if, for all $R \in \mathscr{R}$ and all $i \in N$, there is no $R'_i \in \mathscr{R}_i$ such that $f(R'_i, R_{-i})\ P_i\ f(R)$.

$O_i(\bar{R}) = O_i(\bar{R})$ for all $i \in N$. So, since $LC_i(f(\bar{R});\bar{R}_i) \cap O_i(\bar{R}) \subseteq O_i(\bar{R})$ for all $i \in N$, we have $LC_i(f(\bar{R});\bar{R}_i) \cap O_i(\bar{R}) \subseteq LC_i(f(\bar{R});R_i) \cap O_i(\bar{R})$ for all $i \in N$. This implies $LC_i(f(\bar{R});\bar{R}_i) \cap O_i(\bar{R}) \subseteq LC_i(f(\bar{R});R_i)$ for all $i \in N$, since $LC_i(f(\bar{R});R_i) \cap O_i(\bar{R}) \subseteq LC_i(f(\bar{R});R_i)$ for all $i \in N$. Therefore, restricted monotonicity implies $f(R) = f(\bar{R})$. So, $f(R) = f(\bar{R})$ for any $\bar{R} \in NE^{\Gamma}(R)$. This implies $f(NE^{\Gamma}(R)) = f(R)$. Thus, $f(NE^{\Gamma}(R)) = f(R)$ for all $R \in \mathscr{R}$.

The only if part. By Fact 1, if f is fully-truthfully Nash implementable, then it is fully-truthfully Nash implemented by Γ. So, Lemma 1 implies that f satisfies restricted monotonicity.

Since f is fully-truthfully Nash implemented by Γ, it follows that $R \in NE^{\Gamma}(R)$ for all $R \in \mathscr{R}$. So, for all $R \in \mathscr{R}$ and all $i \in N$, $f(R)\ R_i\ f(R_i', R_{-i})$ for all $R_i' \in \mathscr{R}_i$. Hence, for all $R \in \mathscr{R}$, $f(R) \in ME_i(O_i(R);R_i)$ for all $i \in N$. Thus, f satisfies individual maximality. □

We end this section by briefly discussing the redundancy of individual maximality in characterizing fully-truthfully implementable social choice functions in Nash equilibria. The following is due to Dasgupta et al. (1979).

Proposition 1 (Dasgupta et al. (1979)). *Suppose that $\mathscr{R}$ is rich.*[15] *Then, if a social choice function satisfies monotonicity, then it satisfies strategy-proofness.*

Proposition 1 together with Remark 1 implies that restricted monotonicity implies strategy-proofness if $\mathscr{R}$ is rich. So, restricted monotonicity implies individual maximality if $\mathscr{R}$ is rich. In addition, if A is finite, then individual maximality is automatically satisfied by the completeness and transitivity of preferences regardless of whether or not $\mathscr{R}$ is rich. Thus, we have the following corollary.

Corollary 1. *Suppose that either (i) A is finite or (ii) $\mathscr{R}$ is rich. Then, a social choice function is fully-truthfully implementable in Nash equilibria if and only if it satisfies restricted monotonicity.*

4 The Relationship to Secure Implementation

In this section, we explore the relationship of full-truthful Nash implementation to *secure implementation*[16] (Saijo et al. (2005)), which is identical with double implementation in dominant strategy equilibria and Nash equilibria.

We begin by showing the following lemma, which stems mainly from Proposition 2 (Dasgupta et al. (1979)).

[15] A domain $\mathscr{R}$ is *rich* (Dasgupta et al. (1979)) if, for any $i \in N$, any $R_i, R_i' \in \mathscr{R}_i$, and any $a, b \in A$, if (i) $a\ R_i\ b$ implies $a\ R_i'\ b$ and (ii) $a\ P_i\ b$ implies $a\ P_i'\ b$, then there exists $R_i'' \in \mathscr{R}_i$ such that $LC_i(a;R_i) \subseteq LC_i(a;R_i'')$ and $LC_i(b;R_i') \subseteq LC_i(b;R_i'')$. Examples of rich domains are found in Dasgupta et al. (1979).

[16] A mechanism $\Gamma = (M, g)$ *securely implements* a social choice function f if $g(DSE^{\Gamma}(R)) = g(NE^{\Gamma}(R)) = f(R)$ for any $R \in \mathscr{R}$. A social choice function is *securely implementable* if there exists a mechanism which securely implements it.

Lemma 2. *A social choice function f is securely implemented by the associated direct revelation mechanism if and only if it is fully-truthfully Nash implemented by the associated direct revelation mechanism.*

Proposition 2 (Dasgupta et al. (1979)). *A social choice function is truthfully implemented in Nash equilibria by a direct revelation mechanism if and only if it is truthfully implemented in dominant strategy equilibria by the same direct revelation mechanism.*

Proof of Lemma 2. Let $\Gamma = (\mathscr{R}, f)$ denote the associated direct revelation mechanism.

The if part. Since f is fully-truthfully Nash implemented by Γ, $R \in NE^{\Gamma}(R)$ and $f(NE^{\Gamma}(R)) = f(R)$ for all $R \in \mathscr{R}$. So, since f is truthfully Nash implemented by Γ, Proposition 2 implies that it is truthfully dominant strategy implemented by Γ: $R \in DSE^{\Gamma}(R)$ for all $R \in \mathscr{R}$. This implies $R \in DSE^{\Gamma}(R) \subseteq NE^{\Gamma}(R)$ for all $R \in \mathscr{R}$. Hence, $f(R) \in f(DSE^{\Gamma}(R)) \subseteq f(NE^{\Gamma}(R))$ for all $R \in \mathscr{R}$. Thus, $f(DSE^{\Gamma}(R)) = f(NE^{\Gamma}(R)) = f(R)$ for all $R \in \mathscr{R}$, because $f(NE^{\Gamma}(R)) = f(R)$ for all $R \in \mathscr{R}$.

The only if part. Since f is securely implemented by Γ, $f(DSE^{\Gamma}(R)) = f(NE^{\Gamma}(R)) = f(R)$ for all $R \in \mathscr{R}$. So, the revelation principle for dominant strategy implementation implies $R \in DSE^{\Gamma}(R)$ for all $R \in \mathscr{R}$, implying $R \in DSE^{\Gamma}(R) \subseteq NE^{\Gamma}(R)$ for all $R \in \mathscr{R}$. Thus, f is fully-truthfully Nash implemented by Γ. □

Theorem 2 below follows directly from Fact 1, Lemma 2, Theorem 1, and the results of Saijo et al. (2005) for secure implementation.

Theorem 2. *The following statements are equivalent:*

(i) f is fully-truthfully implementable in Nash equilibria,

(ii) f is securely implementable,

(iii) f satisfies restricted monotonicity and individual maximality,

(iv) f satisfies the rectangular property[17] *(Saijo et al. (2005)) and strategy-proofness.*

Theorem 2 tells us that full-truthful Nash implementation is equivalent to secure implementation, which sheds another light on the structure of secure implementation. Theorem 2 also provides an alternative characterization of securely implementable social choice functions. In contrast to the characterization by Saijo et al. (2005), our characterization has the advantage of using a version of monotonicity, which is familiar to the literature on implementation theory.

[17] A social choice function f satisfies the *rectangular property* if, for all $R, R' \in \mathscr{R}$, if $f(R') \; I_i \; f(R_i, R'_{-i})$ for all $i \in N$, then $f(R') = f(R)$.

5 Ex-Post Implementation

In this section, contrary to the previous sections, let us consider *incomplete information* environments with *interdependent values*. Bergemann and Morris (2007) provided necessary conditions and, separately in general environments and single crossing environments, sufficient conditions for ex-post implementation by the associated direct revelation mechanisms. In this section, we identify a condition necessary and sufficient for ex-post implementation by the associated direct revelation mechanisms in general environments.

Before proceeding, we introduce some notation and definitions that are used in this section. Agent i's *type* is $\theta_i \in \Theta_i$, where Θ_i denotes the set of possible types for her. A *type profile* is a list $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$, where $\Theta := \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$ denotes the *domain*. Given $\theta \in \Theta$, let $R_i(\theta)$ denote a complete and transitive preference relation of agent $i \in N$. An *environment* is a collection (N, A, Θ). Let $LC_i(a;\theta) := \{b \in A \mid a\, R_i(\theta)\, b\}$ be agent i's *lower contour set* of $a \in A$ at $\theta \in \Theta$. Let $ME_i(\bar{A};\theta) := \left\{a \in \bar{A} \,\middle|\, a\, R_i(\theta)\, b \text{ for all } b \in \bar{A}\right\}$ be agent i's set of *maximal elements* in $\bar{A} \subseteq A$ at $\theta \in \Theta$. A *social choice function* is a single-valued function $f\colon \Theta \to A$. Given a mechanism (M, g), let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ denote a *strategy profile* in the mechanism, where $\sigma_i\colon \Theta_i \to M_i$ denotes a *pure strategy* for $i \in N$ in the mechanism. Let Σ^Γ be the set of all strategy profiles in a mechanism Γ. A strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*) \in \Sigma^\Gamma$ is an *ex-post equilibrium* of a mechanism $\Gamma = (M, g)$ if, for all $\theta \in \Theta$, $g(\sigma_i^*(\theta_i), \sigma_{-i}^*(\theta_{-i}))\, R_i(\theta)\, g(m_i, \sigma_{-i}^*(\theta_{-i}))$ for all $i \in N$ and all $m_i \in M_i$. Let $EPE^\Gamma \subseteq \Sigma^\Gamma$ be the set of ex-post equilibria of a mechanism Γ. A mechanism $\Gamma = (M, g)$ *ex-post implements* a social choice function f if $g \circ \sigma^* = f$ for all $\sigma^* \in EPE^\Gamma \neq \emptyset$.

Bergemann and Morris (2007) showed that *ex-post incentive compatibility* is necessary for ex-post implementation.[18] Ex-post incentive compatibility requires that the associated direct revelation mechanism should have the *truthful* ex-post equilibrium for every $\theta \in \Theta$, where each agent adopts the strategy of reporting own true type.

Definition 3 (Ex-Post Incentive Compatibility). A social choice function f satisfies *ex-post incentive compatibility* if, for all $\theta \in \Theta$, all $i \in N$, and all $\theta_i' \in \Theta_i$, $f(\theta)\, R_i(\theta)\, f(\theta_i', \theta_{-i})$.

Proposition 3 (Bergemann and Morris (2007)). *If a social choice function is ex-post implementable, then it satisfies ex-post incentive compatibility.*

We are now ready to state the result regarding a condition both necessary and sufficient for ex-post implementation by the associated direct revelation mechanisms.

[18] Bergemann and Morris (2007) also showed that ex-post incentive compatibility and ex-post monotonicity no veto are sufficient for ex-post implementation when there are three or more agents (see Bergemann and Morris (2007) for details).

Theorem 3. *A social choice function f is ex-post implemented by the associated direct revelation mechanism if and only if it satisfies restricted monotonicity and common maximality.*

Definition 4 (Common Maximality). A social choice function f satisfies *common maximality* if, for all $\theta \in \Theta$, there exists $\bar{\theta} \in \Theta$ such that $f(\bar{\theta}) \in ME_i(O_i(\bar{\theta});\theta)$ for all $i \in N$.

Remark 4. Unlike ex-post incentive compatibility, common maximality does not necessarily require that the associated direct revelation mechanism should have the truthful ex-post equilibrium for each $\theta \in \Theta$. It requires only that there should be an ex-post equilibrium of the mechanism for each $\theta \in \Theta$. So, common maximality is weaker than ex-post incentive compatibility.

Proof of Theorem 3. Let $\Gamma = (\Theta, f)$ denote the associated direct revelation mechanism.

The if part. Let us first show that $EPE^{\Gamma} \neq \emptyset$. Define $\sigma \in \Sigma^{\Gamma}$ as follows: for each $\theta \in \Theta$, $\sigma(\theta) = \bar{\theta}$ where $\bar{\theta}$ is such that $f(\bar{\theta}) \in ME_i(O_i(\bar{\theta});\theta)$ for all $i \in N$. Note that σ is well-defined by common maximality. Then, for all $\theta \in \Theta$, $f(\sigma(\theta))\ R_i(\theta)\ f(\theta'_i, \sigma_{-i}(\theta_{-i}))$ for all $i \in N$ and all $\theta'_i \in \Theta_i$. So, $\sigma \in EPE^{\Gamma}$. Thus, $EPE^{\Gamma} \neq \emptyset$.

Suppose $\sigma^* \in EPE^{\Gamma}$. Then, for all $\theta \in \Theta$, $f(\sigma^*(\theta))\ R_i(\theta)\ f(\theta'_i, \sigma^*_{-i}(\theta_{-i}))$ for all $i \in N$ and all $\theta'_i \in \Theta_i$. This implies $f(\sigma^*(\theta)) \in ME_i(O_i(\sigma^*(\theta));\theta)$ for all $\theta \in \Theta$ and all $i \in N$, implying $LC_i(f(\sigma^*(\theta));\theta) \cap O_i(\sigma^*(\theta)) = O_i(\sigma^*(\theta))$ for all $\theta \in \Theta$ and all $i \in N$. So, since $LC_i(f(\sigma^*(\theta));\sigma^*(\theta)) \cap O_i(\sigma^*(\theta)) \subseteq O_i(\sigma^*(\theta))$ for all $\theta \in \Theta$ and all $i \in N$, we have $LC_i(f(\sigma^*(\theta));\sigma^*(\theta)) \cap O_i(\sigma^*(\theta)) \subseteq LC_i(f(\sigma^*(\theta));\theta) \cap O_i(\sigma^*(\theta))$ for all $\theta \in \Theta$ and all $i \in N$. This implies $LC_i(f(\sigma^*(\theta));\sigma^*(\theta)) \cap O_i(\sigma^*(\theta)) \subseteq LC_i(f(\sigma^*(\theta));\theta)$ for all $\theta \in \Theta$ and all $i \in N$, since $LC_i(f(\sigma^*(\theta));\theta) \cap O_i(\sigma^*(\theta)) \subseteq LC_i(f(\sigma^*(\theta));\theta)$ for all $\theta \in \Theta$ and all $i \in N$. Therefore, restricted monotonicity implies $f(\theta) = f(\sigma^*(\theta))$ for all $\theta \in \Theta$, implying $f = f \circ \sigma^*$. Thus, $f \circ \sigma^* = f$ for all $\sigma^* \in EPE^{\Gamma}$.

The only if part. Since f is ex-post implementable, Proposition 3 implies that it satisfies ex-post incentive compatibility:

$$\text{for all } \theta \in \Theta,\ f(\theta)\ R_i(\theta)\ f(\theta'_i, \theta_{-i}) \text{ for all } i \in N \text{ and all } \theta'_i \in \Theta_i. \tag{1}$$

So, for all $\theta \in \Theta$, $f(\theta) \in ME_i(O_i(\theta);\theta)$ for all $i \in N$, which implies that f satisfies common maximality.

Next, let us check that f satisfies restricted monotonicity. Pick any $\bar{\theta}, \hat{\theta} \in \Theta$ such that $LC_i(f(\bar{\theta});\bar{\theta}) \cap O_i(\bar{\theta}) \subseteq LC_i(f(\bar{\theta});\hat{\theta})$ for all $i \in N$. Let $\sigma \in \Sigma^{\Gamma}$ be such that

$$\sigma(\theta) = \begin{cases} \bar{\theta} & \text{if } \theta = \hat{\theta}, \\ \theta & \text{otherwise.} \end{cases} \tag{2}$$

It follows from (1) that $f(\bar{\theta}) \in ME_i(O_i(\bar{\theta});\bar{\theta})$ for all $i \in N$. This implies $LC_i(f(\bar{\theta});\bar{\theta}) \cap O_i(\bar{\theta}) = O_i(\bar{\theta})$ for all $i \in N$. So, since $LC_i(f(\bar{\theta});\bar{\theta}) \cap O_i(\bar{\theta}) \subseteq LC_i(f(\bar{\theta});\hat{\theta})$ for all

$i \in N$, we have $O_i(\bar{\theta}) \subseteq LC_i(f(\bar{\theta});\hat{\theta})$ for all $i \in N$. This implies $f(\bar{\theta})\ R_i(\hat{\theta})\ f(\theta_i', \bar{\theta}_{-i})$ for all $i \in N$ and all $\theta_i' \in \Theta_i$. Hence,

$$f(\sigma(\hat{\theta}))\ R_i(\hat{\theta})\ f(\theta_i', \sigma_{-i}(\hat{\theta}_{-i})) \text{ for all } i \in N \text{ and all } \theta_i' \in \Theta_i. \quad (3)$$

It also follows from (1) that

$$\text{for all } \theta \in \Theta \setminus \{\hat{\theta}\},\ f(\sigma(\theta))\ R_i(\theta)\ f(\theta_i', \sigma_{-i}(\theta_{-i})) \text{ for all } i \in N \text{ and all } \theta_i' \in \Theta_i. \quad (4)$$

Thus, $\sigma \in EPE^{\Gamma}$ by (3) and (4). So, $f(\sigma(\hat{\theta})) = f(\hat{\theta})$ since f is ex-post implemented by Γ, whereas $f(\sigma(\hat{\theta})) = f(\bar{\theta})$ by (2). Thus, $f(\hat{\theta}) = f(\sigma(\hat{\theta})) = f(\bar{\theta})$. □

Theorem 3 provides a complete characterization of social choice functions that are ex-post implemented by the associated direct revelation mechanisms, whereas Bergemann and Morris (2007) introduced necessary conditions and, in each of general environments and single crossing environments, sufficient conditions for ex-post implementation by the mechanisms. Thus, Theorem 3 closes the gap between the necessary and sufficient conditions of Bergemann and Morris (2007). Moreover, Theorem 3 is true not only in general environments but also in single crossing environments, which is in contrast to the results of Bergemann and Morris (2007).

6 Conclusion

In this paper, we have shown that restricted monotonicity together with individual maximality is both necessary and sufficient for full-truthful Nash implementation. By proving the equivalence of full-truthful Nash implementation and secure implementation, we have also provided an alternative characterization of securely implementable social choice functions. Our characterization sheds new light on the structure of securely implementable social choice functions in terms of monotonicity, a well-known property in implementation theory. Moreover, in incomplete information environments with interdependent values, we have offered a complete characterization of the class of social choice functions ex-post implemented by the associated direct revelation mechanisms. Our characterization closes the gap between Bergemann and Morris (2007)'s necessary and sufficient conditions for ex-post implementation by the associated direct revelation mechanisms.

This paper has considered Nash implementation by a "nice" mechanism, which is in contrast to the one devised by Maskin (1999). It is true that the requirement of truth telling by each agent being a Nash equilibrium of a direct revelation mechanism is appealing. But, as demonstrated in Example 1, the requirement restricts the class of social choice functions Nash implemented by direct revelation mechanisms. Direct revelation mechanisms have been recently received a great deal of attention from both theoretical and practical viewpoints. So, an interesting topic for further research would be to identify a necessary and sufficient condition for Nash implementation by a different direct revelation mechanism in which agents can coordinate their actions.

References

ABDULKADIROĞLU, A. AND T. SÖNMEZ (2003): “School Choice: A Mechanism Design Approach,” *American Economic Review*, 93, 729–747.

BERGEMANN, D. AND S. MORRIS (2007): “Ex Post Implementation,” *Games and Economic Behavior*, forthcoming.

CAMERER, C. F. (2003): *Behavioral Game Theory: Experiments in Strategic Interaction*, Princeton University Press.

CASON, T. N., T. SAIJO, T. SJÖSTRÖM, AND T. YAMATO (2006): “Secure Implementation Experiments: Do Strategy-Proof Mechanisms Really Work?” *Games and Economic Behavior*, 57, 206–235.

DASGUPTA, P., P. J. HAMMOND, AND E. MASKIN (1979): “The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility,” *Review of Economic Studies*, 46, 185–216.

DUTTA, B. AND A. SEN (1991): “A Necessary and Sufficient Condition for Two-Person Nash Implementation,” *Review of Economic Studies*, 58, 121–128.

HURWICZ, L. (1960): “Optimality and Informational Efficiency in Resource Allocation Processes,” in *Mathematical Methods in the Social Sciences*, ed. by K. J. Arrow, S. Karlin, and P. Suppes, Stanford University Press, 27–46.

JACKSON, M. O. (1992): “Implementation in Undominated Strategies: A Look at Bounded Mechanisms,” *Review of Economic Studies*, 59, 757–775.

JACKSON, M. O., T. R. PALFREY, AND S. SRIVASTAVA (1994): “Undominated Nash Implementation in Bounded Mechanisms,” *Games and Economic Behavior*, 6, 474–501.

MASKIN, E. (1999): “Nash Equilibrium and Welfare Optimality,” *Review of Economic Studies*, 66, 23–38.

MOORE, J. AND R. REPULLO (1988): “Subgame Perfect Implementation,” *Econometrica*, 56, 1191–1220.

——— (1990): “Nash Implementation: A Full Characterization,” *Econometrica*, 58, 1083–1099.

PALFREY, T. R. AND S. SRIVASTAVA (1991): “Nash Implementation Using Undominated Strategies,” *Econometrica*, 59, 479–501.

ROTH, A. E. (1984): “The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory,” *Journal of Political Economy*, 92, 991–1016.

ROTH, A. E., T. SÖNMEZ, AND M. U. ÜNVER (2004): "Kidney Exchange," *Quarterly Journal of Economics*, 119, 457–488.

SAIJO, T., T. SJÖSTRÖM, AND T. YAMATO (2005): "Secure Implementation," mimeo., Osaka University.

SCHELLING, T. C. (1960): *The Strategy of Conflict*, Harvard University Press.

SJÖSTRÖM, T. (1991): "On the Necessary and Sufficient Conditions for Nash Implementation," *Social Choice and Welfare*, 8, 333–340.

——— (1994): "Implementation in Undominated Nash Equilibria without Integer Games," *Games and Economic Behavior*, 6, 502–511.

TATAMITANI, Y. (2001): "Implementation by Self-Relevant Mechanisms," *Journal of Mathematical Economics*, 35, 427–444.