Research Paper Series

No. 15

Pricing of Ratchet Equity-Indexed Annuities under Stochastic Interest Rates

Masaaki Kijima, Tony Wong

2006 年 11 月

Department of Business Administration, Graduate School of Sciences, Tokyo Metropolitan University.

1-1 Minami-Ohsawa, Hachiohji, Tokyo 192-0397, Japan

Pricing of Ratchet Equity-Indexed Annuities under Stochastic Interest Rates [★]

Masaaki Kijima [a], Tony Wong [b,*,1]

(This Version: 13 November 2006)

[a] *Graduate School of Social Sciences, Tokyo Metropolitan University, and*
Daiwa Securities Group Chair, Graduate School of Economics, Kyoto University
[b] *Graduate School of Economics, Kyoto University*

Abstract

We consider the valuation of simple and compound Ratchet equity-indexed annuities (EIAs) in the presence of stochastic interest rates. We assume that the equity index follows a geometric Brownian motion and the short rate follows the extended Vasicek model. Under a given forward measure, we obtain an explicit multivariate normal characterization for multiple log-returns on the equity index. Using such a characterization, closed-form price formulas are derived for both simple and compound Ratchet EIAs. An efficient Monte Carlo simulation scheme is also established to overcome the computational difficulties resulted from the evaluation of high-dimensional multivariate normal cumulative distribution functions (CDFs) embedded in the price formulas as well as the consideration of additional complex contract features. Finally, numerical results are provided to illustrate the computational efficiency of our simulation scheme and the effects of various model and contract parameters on pricing.

Key words: Ratchet EIA, extended Vasicek model, forward measure, forward valuation method, multivariate normal CDF.

1. Introduction

EIAs are hybrid annuity products that allow investors to participate in some proportion of returns on an equity index while entitling them to some minimum return guarantee. There exists various contract designs underlying each type of EIAs; e.g., the Point-to-Point design, the Ratchet design, and the Water Mark design. Among the existing designs, Ratchet EIAs with annual reset are the most popular ones comprising about 70% of the EIAs sold in marketplaces according to Morrion (2001). EIAs usually have a maturity ranging from one to ten years.

Pricing of EIAs has been studied by several authors. The analysis, however, is mostly restricted to the standard Black-Scholes model; e.g., see Gerber and Shiu (2003), Hardy (2003), Lee (2003), and Tiong (2000). In order to take into account volatility smiles, Jaimungal (2004) assumes that the underlying index follows a Variance-Gamma model and derives closed-form solutions for the Point-to-Point EIA and the compound Ratchet EIA. Lin and Tan (2003) argues that the effects of stochastic interest rates are crucial in EIA pricing since most EIAs have a long maturity. Consequently, they assume that the short rate follows the Vasicek model and obtain prices for various EIAs using the risk-minimizing scheme of Föllmer and Sondermann (1986). They resort to simulation to compute the value of the EIAs.

In this paper, we consider the pricing of simple and compound Ratchet EIAs when the short rate follows the extended Vasicek model. We adopt the ordinary arbitrage-free pricing principle for the purpose of pricing. This valuation methodology can be problematic in the current setting since EIA markets are typically incomplete due to the non-traded mortality risks embedded in EIA contracts. However, the arbitrage-free pricing principle still serves as a natural benchmark valuation method. To the least extent, it still holds that if EIAs prices evolve in a way as suggested by the arbitrage-free pricing principle, then the model remains arbitrage-free. One may also argue that emerging markets for mortality derivatives facilitate the use of the arbitrage-free pricing method since market prices of mortality risks may be extracted from trades in such markets. In addition, the arbitrage-free pricing principle can be more easily justified when mortality risks are assumed to be deterministic (this is the case if mortality risks are treated by the conventional actuarial present value principle). Following Kijima and Muromachi (2001), we show that under a given forward measure multiple log-returns on the underlying index can be characterized by a zero-mean multivariate normal vector whose variance-covariance matrix can be explicitly computed. Using such a characterization, we not only obtain closed-form solutions for simple and compound Ratchet EIAs, but also provide an efficient simulation scheme for evaluating Ratchet EIAs with additional complex contract features such as a cap, arithmetic index averaging, and a global minimum contract value. It is worth mentioning that unlike the simulation scheme of Lin and Tan (2003) our simulation scheme does not require discretization of the sample paths of the equity index price process and the short rate process. Thus, this may allow us to reduce the pricing errors as well as the computational times.

The remainder of this paper is organized as follows. Section 2 presents the financial model that character-izes the underlying market. In Section 3, we first review the forward valuation method for derivative pricing. Then, we provide a multivariate normal characterization for multiple log-returns on the underlying index. The variance-covariance matrix associated with the multivariate normal vector is also supplied in explicit form. In Section 4, we derive closed-form price formulas for simple and compound Ratchet EIAs. Section 5 discusses some computational problems associated with the price formulas derived in Section 4 and suggests an *exact* simulation method to handle these problems. Section 6 provides modified price formulas taking into account mortality risk. In Section 7, numerical results are provided to test the computational efficiency of our simulation scheme against the conventional simulation scheme used in the EIA pricing literature. Additional numerical results are also given to illustrate the effects of various model and contract parameters on the pricing of Ratchet EIAs. Finally, we conclude the paper in Section 8.

2. The Financial Model

We assume that the economy consists of two traded assets, namely a risky equity index $S(t)$ and a bank account $B(t)$ that satisfy the following stochastic system:

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma_1 dz_1(t) + \sigma_2 dz_2(t), \tag{1}$$

$$\frac{dB(t)}{B(t)} = r(t)dt, \tag{2}$$

where $r(t)$ denotes the instantaneous short rate. Here, $z_1(t)$ and $z_2(t)$ are two independent Wiener processes. In addition, we assume that the short rate satisfies the extended Vasicek model:

$$dr(t) = \left\{ \kappa(t) \left[f(0,t) - r(t) \right] + \frac{\partial f(0,t)}{\partial t} + \phi(t) \right\} dt + \gamma dz_1(t), \tag{3}$$

where $f(0,t)$ denotes the initial instantaneous forward curve that is differentiable with respect to t. The function $\kappa(t)$ is assumed to be a deterministic function of time and

$$\phi(t) = \gamma^2 \int_0^t e^{-2 \int_s^t \kappa(u)du} ds. \tag{4}$$

Note that the volatilities of the equity index and the short rate are $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$ and γ, respectively. The stochastic differential equations (SDEs) above are assumed to be defined on some probability space $\left(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{Q} \right)$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by the processes $\{z_1(t)\}_{t \geq 0}$ and $\{z_2(t)\}_{t \geq 0}$. For valuation purposes, we assume that $\mathbb{Q}$ is the *risk-neutral measure*.

Under the model assumptions above, we have

$$d \ln S(t) = \left(r(t) - \frac{\sigma^2}{2} \right) dt + \sigma_1 dz_1(t) + \sigma_2 dz_2(t), \tag{5}$$

$$\rho \equiv Corr \left(d \ln S(t), dr(t) \right) = \frac{\sigma_1}{\sigma}. \tag{6}$$

When σ_1 is negative, the log-price of the equity index and the short rate are negatively correlated.

Let $P(t,T)$ denote the time t price of one unit of risk-free zero-coupon bond maturing at time T, where $0 \leq t \leq T$. Then, we have the following bond price expressions:

$$P(0,T) = \exp \left\{ - \int_0^T f(0,u) du \right\}, \tag{7}$$

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp \left\{ -\frac{1}{2} \beta^2(t,T) \phi(t) + \beta(t,T) \left[f(0,t) - r(t) \right] \right\}, \tag{8}$$

where

$$\beta(t,T) = \int_t^T e^{- \int_t^s \kappa(u)du} ds. \tag{9}$$

For more details on bond pricing under the extended Vasicek model; e.g., see Brigo and Mercurio (2001).

It is easy to see that for all $t > 0$ the short rate $r(t)$ is Gaussian under the extended Vasicek model. Although this implies that the short rate can become negative with a positive probability, this probability is often negligible for many practical applications. The extended Vasicek model offers two advantages in derivative pricing. First, it is consistent with the current term structure of interest rates. Second, it is analytically tractable. As we shall see later, closed-form solutions can be obtained for Ratchet EIAs under this model.

3. The Forward Valuation Method

The *forward valuation method* is a special case of the *change-of-numeraire method* of German et al. (1995), which is a very useful approach for pricing derivatives under stochastic interest rates. This valuation method works as follows.

Let Θ be the payoff of a European contingent claim written on the equity index that is maturing at time T. Using the usual risk-neutral valuation principle, the time t price of this contract is given by

$$V(t) = \mathbb{E}_t \left[\frac{B(t)}{B(T)} \Theta \right], \tag{10}$$

where $E_t[\cdot]$ denotes the expectation taken under the risk-neutral measure $\mathbb{Q}$ and conditional on $\mathcal{F}_t$. Define a *forward measure* $\mathbb{Q}^T$ using the following Radon-Nikodym derivative:

$$L^T(t) \equiv \frac{d\mathbb{Q}^T}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} = \frac{P(t,T)}{B(t)P(0,T)}. \tag{11}$$

We then obtain

$$
\begin{aligned}
V(t) &= \mathbb{E}_t\left[\frac{B(t)}{B(T)}\Theta\right] \\
&= \mathbb{E}_t\left[L^T(T)\frac{B(t)}{B(T)L^T(T)}\Theta\right] \\
&= \mathbb{E}_t\left[L^T(T)\right]\mathbb{E}_t^T\left[\frac{B(t)}{B(T)L^T(T)}\Theta\right] \\
&= L^T(t)\frac{B(t)}{P(0,T)}\mathbb{E}_t^T\left[\Theta\right] \\
&= P(t,T)\mathbb{E}_t^T\left[\Theta\right],
\end{aligned}
\tag{12}
$$

where $E_t^T[\cdot]$ denotes the expectation taken under the forward measure $\mathbb{Q}^T$ and conditional on $\mathcal{F}_t$. Note that the third equality in (12) follows from the Abstract Bayes' Formula; see Björk (2004). The forward valuation method can be a powerful pricing tool in practice. It is particularly useful when the distribution of the random component embedded in the payoff function can be identified under $\mathbb{Q}^T$. We shall see that this is indeed the case for Ratchet EIAs.

The pricing methodology adopted in this paper follows the lines of Kijima and Muromachi (2001) where the focus is given to the pricing of equity swaps under a model that essentially coincides with model (1) - (3). Their approach is based on the forward valuation formula (12) and hinges on an analytical characterization of the dynamics of the equity return $\frac{S(T)}{S(t)}$ under the forward measure $\mathbb{Q}^T$, where $0 \le t < T$. We give a brief summary of their approach below.

Define

$$a(s,t) \equiv -\int_s^t \gamma e^{-\int_s^u \kappa(\tau)d\tau} du, \quad 0 \le s \le t, \tag{13}$$

$$\psi_S^2(s,t) \equiv \int_0^s \left[\sigma_1 - a(u,t)\right]^2 du + \sigma_2^2 s, \quad 0 \le s \le t, \tag{14}$$

$$\psi_V^2(s,t,T) \equiv \int_0^s \left[a(u,T) - a(u,t)\right]^2 du, \quad 0 \le s < t \le T. \tag{15}$$

Kijima and Muromachi (2001) show that the forward dynamics of the forward equity price $S_T(t) \equiv \frac{S(t)}{P(t,T)}$ is given by

$$S_T(t) = S_T(0)\exp\left\{-\frac{\psi_S^2(t,T)}{2} + \int_0^t \left[\sigma_1 - a(u,t)\right]dz_1^T(u) + \sigma_2 z_2^T(t)\right\}, \tag{16}$$

where $z_1^T(t)$ and $z_2^T(t)$ are two independent $\mathbb{Q}^T$-Wiener processes on $[0,T]$ which are related to the processes $z_1(t)$ and $z_2(t)$ through the following transformations

$$z_1^T(t) = z_1(t) - \int_0^t a(s,T)ds, \tag{17}$$

$$z_2^T(t) = z_2(t). \tag{18}$$

Using (8) and (16), and with some algebraic manipulation, the authors obtain the following expression for $\frac{S(T)}{S(t)}$:

4

$$\frac{S(T)}{S(t)} = \frac{P(0,t)}{P(0,T)} \exp\left\{ -\frac{\psi_S^2(T,T)}{2} + \frac{\psi_S^2(t,T)}{2} - \frac{\psi_V^2(t,t,T)}{2} + \int_0^t [a(u,t) - a(u,T)] \, dz_1^T(u) \right. \tag{19}$$

$$\left. + \int_t^T [\sigma_1 - a(u,T)] \, dz_1^T(u) + \sigma_2 \left[z_2^T(T) - z_2^T(t) \right] \right\}.$$

Under the forward valuation approach, the expression above plays a crucial role in pricing payoffs involving the return $\frac{S(T)}{S(t)}$. More specifically, it allows one to identify the log-return $\ln \frac{S(T)}{S(t)}$ as a normal random variable and compute its mean and variance explicitly. The theorem below provides a slight extension of expression (19).

Theorem 3.1 *Assume $0 \le s \le t \le T$. Under model (1) - (3), we have*

$$\frac{S(t)}{S(s)} = C(s,t) e^{W(s,t)}, \tag{20}$$

where

$$C(s,t) = \frac{P(0,s)}{P(0,t)} \exp\left\{ -\frac{\psi_S^2(t,T)}{2} + \frac{\psi_S^2(s,T)}{2} + \frac{\psi_V^2(t,t,T)}{2} - \frac{\psi_V^2(s,s,T)}{2} \right\}, \tag{21}$$

$$W(s,t) = \int_0^s [a(u,s) - a(u,t)] \, dz_1^T(u) + \int_s^t [\sigma_1 - a(u,t)] \, dz_1^T(u) + \sigma_2 \left[z_2^T(t) - z_2^T(s) \right], \tag{22}$$

and the processes $z_1^T(\cdot)$ and $z_2^T(\cdot)$ are given by (17) and (18), respectively.

PROOF. Use the fact $\frac{S(t)}{S(s)} = \frac{\frac{S(T)}{S(s)}}{\frac{S(T)}{S(t)}}$ and apply expression (19). ∎

To price Ratchet EIAs, we need a characterization of the joint distribution of multiple log-returns on the underlying index, which is given in the corollary below.

Corollary 3.2 *Assume $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_{N-1} \le t_N \le T$. Under model (1) - (3), $(W(0,t_1), W(t_1,t_2), \cdots, W(t_{N-1},t_N))$, where $W(\cdot,\cdot)$ is given by (22), is a multivariate normal vector with mean zero and variance-covariance matrix $\Xi = (v_{j,i})$, where*

$$v_{j,j} = \psi_S^2(t_j,t_j) - \psi_S^2(t_{j-1},t_j) + \psi_V^2(t_{j-1},t_{j-1},t_j), \tag{23}$$

for $j = 1,2,...,N$, and

$$v_{j,i} = \int_0^{t_{j-1}} [a(s,t_{j-1}) - a(s,t_j)] [a(s,t_{i-1}) - a(s,t_i)] \, ds \tag{24}$$

$$+ \int_{t_{j-1}}^{t_j} [\sigma_1 - a(s,t_j)] [a(s,t_{i-1}) - a(s,t_i)] \, ds,$$

for $N \ge i > j \ge 1$.

PROOF. From expression (22), it is obvious that each $W(t_{j-1},t_j)$, $1 \le j \le N$, is normally distributed. The results then follow readily from direct calculation. ∎

Although pricing of Ratchet EIAs can be as well done under the risk-neutral measure, we believe that the forward valuation formulation presented in this section is algebraically simpler. This simplification hinges on the fact that under the forward formulation the stochastic discount factor embedded in the corresponding risk-neutral valuation formula becomes irrelevant as it can be factored out from the forward expectation operator. As such, one only needs to work with the joint distribution of multiple log-returns one the index

$S(t)$ under the forward measure, for which a simple multivariate normal representation is available, as given in Corollary 3.2. Moreover, in contrast to the case dealing with the risk-neutral measure, the short rate $r(t)$ does not appear in the dynamics of $S(t)$ under the forward measure which further reduces the algebraic complexity involved in pricing.

4. Valuation of Ratchet EIAs

We consider Ratchet EIAs with *annual reset* meaning that the returns to be credited are reset annually based on the realized annual returns on the equity index, a participation rate, and a guaranteed minimum annual return. Note that the results obtained in this section can be easily applied to Ratchet EIAs with arbitrary reset frequency. To reduce the volatility of credited returns, a common variant is to use some average of the index levels when calculating annual returns on the index. Both index averaging and imposing a cap on returns can be considered as means to reduce the cost of an EIA. The existing EIA pricing literature considers mainly *arithmetic index averaging*. It is, however, difficult to obtain closed-form solutions when arithmetic index averaging is used. In this section, we consider an alternative index averaging method, *geometric index averaging*. The case of arithmetic index averaging is discussed in Section 5.1.4.

The idea of geometric index averaging is not new in the finance literature as it is analogous to the averaging method used for a geometric Asian option [1]; e.g., see Nielsen and Sandmann (1996), Cheung and Wong (2004). To our best knowledge, however, we are the first to introduce geometric index averaging to the EIA pricing literature. With geometric index averaging, we not only achieve the purpose of reducing the volatility of the credited returns on an EIA, but also preserve the analytical tractability of the pricing model.

Throughout this section, we shall adhere to the following notations:

$$R_j \equiv \frac{S(j)}{S(j-1)}, \tag{25}$$

$$R_j^{(m)} \equiv \left[\prod_{k=0}^{m-1} \frac{S\left(j - \frac{k}{m}\right)}{S(j-1)} \right]^{\frac{1}{m}}, \tag{26}$$

for $j = 1, 2, \dots$. Hence, R_j denotes the index return over the jth year without index averaging while $R_j^{(m)}$ denotes the index return over the jth year with geometric index averaging sampled at an interval of $\frac{1}{m}$.

4.1. *The Simple Ratchet EIA (SR-EIA)*

4.1.1. *The Price Formula: Without Index Averaging*

An N-year SR-EIA without index averaging pays at maturity N the following amount:

$$\Lambda_{SR} = 1 + \sum_{j=1}^{N} \max\left[F, \alpha\left(R_j - 1\right)\right], \tag{27}$$

where F is the guaranteed minimum annual return and $\alpha \geq 0$ is the participation rate.

Let $V_{SR}(N, F, \alpha)$ denote the time 0 price of the N-year SR-EIA. Using the forward valuation method described in the previous section, we have

---

[1] Under the extended Vasicek model, a simple closed-form price formula can be derived for the geometric Asian option following the pricing methodology presented in this section. On the other hand, the Monte Carlo simulation scheme established in Section 5 can serve as an efficient tool for pricing an arithmetic Asian option under the extended Vasicek model.

$$V_{SR}(N, F, \alpha) = P(0, N) \left[1 + \mathbb{E}_0^N \left[\sum_{j=1}^N \max \left[F, \alpha \left(\frac{S(j)}{S(j-1)} - 1 \right) \right] \right] \right] \tag{28}$$

$$= P(0, N) \left[1 + NF + \alpha \sum_{j=1}^N \mathbb{E}_0^N \left[\max \left(\frac{S(j)}{S(j-1)} - \frac{F + \alpha}{\alpha}, 0 \right) \right] \right].$$

The result below follows easily from Theorem 3.1 and is obtained independently by Muromachi (2002).

Theorem 4.1 *For $K > 0$ and $j = 1, 2, ..., N$, we have*

$$\Delta(j, N, K) \equiv \mathbb{E}_0^N \left[\max \left(\frac{S(j)}{S(j-1)} - K, 0 \right) \right] = C(j-1, j) e^{\frac{\sigma_j^2}{2}} \Phi(d_1(j, N, K)) - K \Phi(d_2(j, N, K)), \tag{29}$$

where $\Phi(\cdot)$ denotes the standard normal CDF, $C(j-1, j)$ is given by (21), and

$$d_1(j, N, K) = \frac{\ln C(j-1, j) - \ln K}{\sigma_j} + \sigma_j, \tag{30}$$

$$d_2(j, N, K) = d_1(j, N, K) - \sigma_j, \tag{31}$$

$$\sigma_j = \sqrt{\psi_S^2(j, j) - \psi_S^2(j-1, j) + \psi_V^2(j-1, j-1, j)}. \tag{32}$$

PROOF. From (20) - (22), we have $\frac{S(j)}{S(j-1)} = C(j-1, j) e^{W(j-1, j)}$, where $C(j-1, j)$ is a constant given by (21) and $W(j-1, j)$ is a normal random variable with mean 0 and variance σ_j^2. The result then follows by evaluating the integral

$$\mathbb{E}_0^N \left[\max \left(\frac{S(j)}{S(j-1)} - K, 0 \right) \right] = \int_{\frac{1}{\sigma_j} \ln \frac{K}{C(j-1,j)}}^{\infty} [C(j-1, j) e^{\sigma_j z} - K] \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz. \tag{33}$$

∎

Proposition 4.2 *The time 0 price of an N-year SR-EIA without index averaging is given by*

$$V_{SR}(N, F, \alpha) = P(0, N) \left[1 + NF + \alpha \sum_{j=1}^N \Delta \left(j, N, \frac{F + \alpha}{\alpha} \right) \right], \tag{34}$$

where $\Delta(\cdot)$ is given by (29).

4.1.2. *The Price Formula: With Geometric Index Averaging*

An N-year SR-EIA with the $\frac{1}{m}^{th}$ geometric index averaging has the following payoff at time N:

$$\Lambda_{SR}^{(m)} = 1 + \sum_{j=1}^N \max \left[F, \alpha \left(R_j^{(m)} - 1 \right) \right]. \tag{35}$$

Note that the case $m = 1$ reduces to the SR-EIA without index averaging. Define

$$C_j^{(m)} \equiv \left[\prod_{k=0}^{m-1} C \left(j - 1, j - \frac{k}{m} \right) \right]^{\frac{1}{m}}, \tag{36}$$

$$W_j^{(m)} \equiv \frac{1}{m} \sum_{k=0}^{m-1} W \left(j - 1, j - \frac{k}{m} \right), \tag{37}$$

for $j = 1, 2, ..., N$. It is easy to verify that $R_j^{(m)} = C_j^{(m)} e^{W_j^{(m)}}$.

Let $V_{SR}^{(m)}(N, F, \alpha)$ denote the time 0 price of the N-year SR-EIA defined above. Then, we have

$$V_{SR}^{(m)}(N, F, \alpha) = P(0, N) \left[1 + NF + \alpha \sum_{j=1}^{N} \mathbb{E}_0^N \left[\max \left(C_j^{(m)} e^{W_j^{(m)}} - \frac{F + \alpha}{\alpha}, 0 \right) \right] \right]. \tag{38}$$

It is easy to show that $W_j^{(m)}$ is a normal random variable with mean 0 and variance $\sigma_j^{(m)^2}$, given by

$$\sigma_j^{(m)^2} = \frac{1}{m^2} \left[\sum_{k=0}^{m-1} \left[\psi_S^2 \left(j - \frac{k}{m}, j - \frac{k}{m} \right) - \psi_S^2 \left(j - 1, j - \frac{k}{m} \right) + \psi_V^2 \left(j - 1, j - 1, j - \frac{k}{m} \right) \right] \tag{39}$$

$$+ 2 \sum_{0 \le y < i \le m-1} \left[\int_0^{j-1} \left[a(s, j-1) - a \left(s, j - \frac{y}{m} \right) \right] \left[a(s, j-1) - a \left(s, j - \frac{i}{m} \right) \right] ds \right.$$

$$\left. + \int_{j-1}^{j - \frac{i}{m}} \left[\sigma_1 - a \left(s, j - \frac{i}{m} \right) \right] \left[\sigma_1 - a \left(s, j - \frac{y}{m} \right) \right] ds + \sigma_2^2 \left(1 - \frac{i}{m} \right) \right] \right].$$

Following the same idea in the proof of Theorem 4.1, we obtain the results below.

Theorem 4.3 *For $K > 0$ and $j = 1, 2, ..., N$, we have*

$$\Delta^{(m)}(j, N, K) \equiv \mathbb{E}_0^N \left[\max \left(C_j^{(m)} e^{W_j^{(m)}} - K, 0 \right) \right] \tag{40}$$

$$= C_j^{(m)} e^{\frac{\sigma_j^{(m)^2}}{2}} \Phi \left(d_1^{(m)}(j, N, K) \right) - K \Phi \left(d_2^{(m)}(j, N, K) \right),$$

where

$$d_1^{(m)}(j, N, K) = \frac{\ln C_j^{(m)} - \ln K}{\sigma_j^{(m)}} + \sigma_j^{(m)}, \tag{41}$$

$$d_2^{(m)}(j, N, K) = d_1^{(m)}(j, N, K) - \sigma_j^{(m)}. \tag{42}$$

Proposition 4.4 *The time 0 price of an N-year SR-EIA with the $\frac{1}{m}^{th}$ geometric index averaging is given by*

$$V_{SR}^{(m)}(N, F, \alpha) = P(0, N) \left[1 + NF + \alpha \sum_{j=1}^{N} \Delta^{(m)} \left(j, N, \frac{F + \alpha}{\alpha} \right) \right], \tag{43}$$

where $\Delta^{(m)}(\cdot)$ is given by (40).

4.2. The Compound Ratchet EIA (CR-EIA)

4.2.1. The Price Formula: Without Index Averaging

An N-year CR-EIA without index averaging pays at maturity N the following amount:

$$\Lambda_{CR} = \prod_{j=1}^{N} \max \left[1 + F, 1 + \alpha (R_j - 1) \right]. \tag{44}$$

Let $V_{CR}(N, F, \alpha)$ denote the time 0 price of the CR-EIA. With some simple algebra, we obtain

8

$$V_{CR}(N, F, \alpha) = \alpha^N P(0, N) \mathbb{E}_0^N \left[\prod_{j=1}^N (M + \Upsilon_j) \right], \tag{45}$$

where

$$M \equiv \frac{1 - \alpha}{\alpha}, \tag{46}$$

$$K \equiv \frac{F + \alpha}{\alpha}, \tag{47}$$

$$\Upsilon_j \equiv \max \left[K, C(j-1, j)e^{W(j-1, j)} \right]. \tag{48}$$

By expanding the product operator in (45) and using the linearity of expectation, we have

$$\mathbb{E}_0^N \left[\prod_{j=1}^N (M + \Upsilon_j) \right] = M^N + \sum_{k=1}^N \sum_{\{j_1, \ldots, j_k\} \in A_k} M^{N-k} \mathbb{E}_0^N \left[\Upsilon_{j_1} \Upsilon_{j_2} \cdots \Upsilon_{j_k} \right], \tag{49}$$

where A_k, $1 \le k \le N$, is the collection of all k-subsets of $\{1, 2, \ldots, N\}$. For each $\{j_1, \ldots, j_k\} \in A_k$, define

$$\Gamma_{N,k}(j_1, j_2, \ldots, j_k) = \mathbb{E}_0^N \left[\Upsilon_{j_1} \Upsilon_{j_2} \cdots \Upsilon_{j_k} \right]. \tag{50}$$

It is possible to derive an analytical expression for the quantity $\Gamma_{N,k}(j_1, j_2, \ldots, j_k)$ for each $\{j_1, \ldots, j_k\} \in A_k$, $k = 1, 2, \ldots, N$. We shall demonstrate this by deriving an explicit expression for $\Gamma_{N,N}(1, 2, \ldots, N)$.

From Corollary 3.2, we know that $W \equiv (W(0, 1), W(1, 2), \ldots, W(N-1, N))$ is a multivariate normal vector with mean zero and variance-covariance matrix Σ that can be computed using (23) and (24). In general, the joint probability density function (PDF) of a N-dimensional multivariate normal vector with mean μ and variance-covariance matrix G is given by

$$f(w; \mu, G) = \frac{1}{(2\pi)^{\frac{N}{2}} |det(G)|^{\frac{1}{2}}} \exp \left\{ -\frac{(w - \mu)G^{-1}(w - \mu)'}{2} \right\}, \ \forall w \in \mathbb{R}^N, \tag{51}$$

where $det(G)$ denotes the determinant of the matrix G, G^{-1} denotes the matrix inverse of G, and $(w - \mu)'$ denotes the transpose of the *row vector* $w - \mu$. Therefore, the joint PDF of W is given by $f(w; 0, \Sigma)$.

Now define the following sets in $\mathbb{R}$:

$$E(j, 0) \equiv \left(-\infty, \ln \frac{K}{C(j-1, j)} \right], \tag{52}$$

$$E(j, 1) \equiv \left(\ln \frac{K}{C(j-1, j)}, \infty \right), \tag{53}$$

for $j = 1, 2, \ldots, N$, and the following sets in $\mathbb{R}^N$:

$$H_N(e) \equiv E(1, e_1) \times E(2, e_2) \times \ldots \times E(N, e_N), \tag{54}$$

for each $e = (e_1, e_2, \ldots, e_N) \in \{0, 1\}^N$. It is clear that $\left\{ H_N(e) \mid e \in \{0, 1\}^N \right\}$ forms a partition of $\mathbb{R}^N$. Moreover, we have

$$\Gamma_{N,N}(1, 2, \ldots, N) = \sum_{e \in \{0,1\}^N} K^{N-(e_1+e_2+\ldots+e_N)} \left(\prod_{j=1}^N C^{e_j}(j-1, j) \right) \tag{55}$$

$$\times \int_{H_N(e)} \exp\{ew'\} f(w; 0, \Sigma) \, dw.$$

It remains to compute the integral $\int_{H_N(e)} \exp\{ew'\} f(w; 0, \Sigma) \, dw$ for each $e \in \{0, 1\}^N$.

9

Let $u(e) = e\Sigma$. Then, we have

$$\int_{H_N(e)} \exp\{ew\prime\} f(w; 0, \Sigma)\, dw \tag{56}$$

$$= \exp\left\{\frac{u(e)\Sigma^{-1}u(e)'}{2}\right\} \int_{H_N(e)} \frac{1}{(2\pi)^{\frac{N}{2}}|det(\Sigma)|^{\frac{1}{2}}} \exp\left\{-\frac{(w - u(e))\Sigma^{-1}(w - u(e))'}{2}\right\} dw$$

$$= \exp\left\{\frac{u(e)\Sigma^{-1}u(e)'}{2}\right\} q(e),$$

where

$$q(e) \equiv \int_{H_N(e)} f(w; u(e), \Sigma)\, dw. \tag{57}$$

Therefore, $q(e)$ is equal to $Prob[W(e) \in H_N(e)]$, where $W(e) \equiv (W_1(e), W_2(e), ..., W_N(e))$ is a multivariate normal vector with mean $u(e)$ and variance-covariance matrix Σ. It is worth mentioning that (56) can also be obtained using the change of measure method.

It is well known that the joint CDF of a correlated multivariate normal vector can only be expressed up to an integral form. There exists computer packages that provide numerical implementations for evaluating multivariate normal CDFs. Most of these implementations, however, cannot evaluate the probability of a correlated multivariate normal distribution over arbitrary sets. In our case, if the set $H_N(e)$ is not in the form $E(1, 0) \times E(2, 0) \times \ldots \times E(N, 0)$ (i.e., e is not the zero vector), we may still compute the probability $q(e)$ by combining various CDF evaluations recursively. However, this method can become very inefficient when N is large. We resolve this problem by making use of the specific forms of the sets $H_N(e)$.

For each non-zero enumeration $e \in \{0, 1\}^N$, let $J(e) \equiv \{j \mid e_j = 1, \ 1 \leq j \leq N\}$ and $\bar{J}(e) \equiv \{j \mid e_j = 0, \ 1 \leq j \leq N\}$. We have

$$q(e) = Prob\left[\left(\cap_{j \in J(e)} \{W_j(e) \in E(j, 1)\}\right) \cap \left(\cap_{j \in \bar{J}(e)} \{W_j(e) \in E(j, 0)\}\right)\right] \tag{58}$$

$$= Prob\left[\left(\cap_{j \in J(e)} \{-W_j(e) \in E_-(j, 0)\}\right) \cap \left(\cap_{j \in \bar{J}(e)} \{W_j(e) \in E(j, 0)\}\right)\right],$$

where

$$E_-(j, 0) \equiv \left(-\infty, -\ln\frac{K}{C(j-1, j)}\right]. \tag{59}$$

Now let $U(e)$ be a N-by-N diagonal matrix with diagonal entries $U_{j,j}(e) = (-1)^{e_j}$, where $(-1)^0 \equiv 1$. Then, it is clear that the probability $q(e)$ can be seen as a CDF evaluation of a multivariate normal distribution with mean $u(e)U(e)$ and variance-covariance matrix $U(e)\Sigma U(e)$. Therefore, for any e, the probability $q(e)$ can always be evaluated as the CDF of some adjusted multivariate normal distribution. In other words, the computation of the quantities $\Gamma_{N,k}(j_1, j_2, ..., j_k)$ boils down to the evaluation of various multivariate normal CDFs. We summarize the results above in the following theorem.

Theorem 4.5 *The time 0 price of an N-year CR-EIA without index averaging is given by*

$$V_{CR}(N, F, \alpha) = \alpha^N P(0, N)\left[M^N + \sum_{k=1}^{N} \sum_{\{j_1, j_2, ..., j_k\} \in A_k} M^{N-k}\Gamma_{N,k}(j_1, j_2, ..., j_k)\right], \tag{60}$$

where A_k is the collection of all k-subsets of $\{1, 2, ..., N\}$.

4.2.2. *The Price Formula: With Geometric Index Averaging*

Consider now an N-year CR-EIA with the $\frac{1}{m}^{th}$ *geometric index averaging* that pays at maturity N the following amount:

$$\Lambda_{CR}^{(m)} = \prod_{j=1}^{N} \max\left[1 + F, 1 + \alpha\left(R_j^{(m)} - 1\right)\right]. \tag{61}$$

Let C_j^m and $W_j^{(m)}$ be defined as in (36) and (37), respectively. With some simple algebra, it is easy to show that $W^{(m)} = \left(W_1^{(m)}, W_2^{(m)}, ..., W_N^{(m)}\right)$ is a multivariate normal vector with mean zero and variance-covariance matrix

$$\Sigma^{(m)} = \begin{pmatrix} \sigma_1^{(m)^2} & v_{1,2}^{(m)} & v_{1,3}^{(m)} & \cdots & \cdots & v_{1,N}^{(m)} \\ & \sigma_2^{(m)^2} & v_{2,3}^{(m)} & \cdots & \cdots & v_{2,N}^{(m)} \\ & & \vdots & \vdots & \vdots & \vdots \\ & & & \sigma_{N-2}^{(m)^2} & v_{N-2,N-1}^{(m)} & v_{N-2,N}^{(m)} \\ & & & & \sigma_{N-1}^{(m)^2} & v_{N-1,N}^{(m)} \\ & & & & & \sigma_N^{(m)^2} \end{pmatrix}, \tag{62}$$

where $\sigma_j^{(m)^2}$ is given by (39), and

$$v_{j,i}^{(m)} = \frac{1}{m^2} \sum_{k=0}^{m-1} \sum_{y=0}^{m-1} \left[\int_0^{j-1} \left[a(s, j-1) - a\left(s, j - \frac{k}{m}\right)\right] \left[a(s, i-1) - a\left(s, i - \frac{y}{m}\right)\right] ds \right. \tag{63}$$
$$\left. + \int_{j-1}^{j-\frac{k}{m}} \left[\sigma_1 - a\left(s, j - \frac{k}{m}\right)\right] \left[a(s, i-1) - a\left(s, i - \frac{y}{m}\right)\right] ds \right],$$

for $N \geq i > j \geq 1$.

Now define the quantity $\Gamma_{N,k}^{(m)}(j_1, j_2, ..., j_k)$ in the same manner as $\Gamma_{N,k}(j_1, j_2, ..., j_k)$, *but with $C(j-1, j)$ and Σ replaced by $C_j^{(m)}$ and by $\Sigma^{(m)}$*, respectively. Following the same ideas as before, we can obtain a similar expression for $\Gamma_{N,k}^{(m)}(j_1, j_2, ..., j_k)$. The computation of the quantities $\Gamma_{N,k}^{(m)}(j_1, j_2, ..., j_k)$ also boils down to the evaluation of various multivariate normal CDFs. We summarize the results in the following theorem.

Theorem 4.6 *The time 0 price of an N-year CR-EIA with the $\frac{1}{m}^{th}$ geometric index averaging is given by*

$$V_{CR}^{(m)}(N, F, \alpha) = \alpha^N P(0, N) \left[M^N + \sum_{k=1}^{N} \sum_{\{j_1, j_2, ..., j_k\} \in A_k} M^{N-k} \Gamma_{N,k}^{(m)}(j_1, j_2, ..., j_k) \right], \tag{64}$$

where A_k is the collection of all k-subsets of $\{1, 2, ..., N\}$.

5. Valuation by Monte Carlo Simulation

In this section, we discuss a few computational problems associated with the price formulas derived for the SR-EIA and the CR-EIA in the previous section. The first two problems apply to the CR-EIA only while the last two problems apply to both the SR-EIA and the CR-EIA. Using the multivariate normal characterization derived in Corollary 3.2, we then establish an efficient Monte Carlo simulation scheme to resolve these computational problems in valuation.

5.1. Some Computational Problems

5.1.1. Numerical Evaluation of High-Dimensional Multivariate Normal CDFs

The price formulas (60) and (64) require numerical evaluation of the multivariate normal CDFs embedded in the price formulas. This can be computationally problematic when the dimensions of these multivariate normal CDFs become high, as numerical integration of high-dimensional integrals can be very time-consuming. For a CR-EIA with N reset points, the number of evaluations of k-*dimensional* multivariate normal CDFs, where $1 \leq k \leq N$, is given by

$$L(N,k) = \sum_{j=k}^{N} \binom{N}{j} \binom{j}{k}. \tag{65}$$

Hence, the total number of evaluations of multivariate normal CDFs is

$$L(N) = \sum_{k=1}^{N} \sum_{j=k}^{N} \binom{N}{j} \binom{j}{k} = 3^N - 2^N. \tag{66}$$

When N is large, the computational efficiency of the price formulas can deteriorate substantially as the numbers $L(N,k)$ are increasing in N.

5.1.2. Application of Caps

Another related computational problem associated with the CR-EIA arises when the number of reset points N is large and when a cap C is applied to the contract. Imposing a cap can be regarded as an alternative way to reduce the cost of an EIA as opposed to the use of index averaging. When a cap is in effect, the CR-EIA payoff takes the following form:

$$\Lambda_{CR} = \prod_{j=1}^{N} \min \left[\max \left[1 + F, 1 + \alpha \left(R_j - 1 \right) \right], 1 + C \right]. \tag{67}$$

To see why this poses computational difficulty in valuation, simply observe that the sum in (55) is now taken over all possible enumerations in the form $e \in \{0, 1, 2\}^N$ due to the addition of a cap. This can dramatically increase the numbers $L(N,k)$ discussed above.

Remark 5.1 *The price formula for a capped SR-EIA can be easily obtained using the results derived in Section 4.1. To demonstrate this, we consider an N-year SR-EIA with the $\frac{1}{m}^{th}$ geometric index averaging and a cap C. In this case, the payoff can be written as*

$$\Lambda_{SR}^{(m)} = 1 + \sum_{j=1}^{N} \min \left[\max \left[F, \alpha \left(R_j^{(m)} - 1 \right) \right], C \right] \tag{68}$$

$$= 1 + NC + \sum_{j=1}^{N} \max \left[F, \alpha \left(R_j^{(m)} - 1 \right) \right] - \sum_{j=1}^{N} \max \left[C, \alpha \left(R_j^{(m)} - 1 \right) \right].$$

Following the derivation of (43), we obtain the following price formula:

$$V_{SR}^{(m)} \left(N, F, C, \alpha \right) = P(0, N) \left\{ 1 + NF + \alpha \sum_{j=1}^{N} \left[\Delta^{(m)} \left(j, N, \frac{F + \alpha}{\alpha} \right) - \Delta^{(m)} \left(j, N, \frac{C + \alpha}{\alpha} \right) \right] \right\}. \tag{69}$$

Note that the price formula above is computationally feasible since it involves univariate normal CDFs only. Therefore, the value of the SR-EIA can be computed analytically even in the presence of a cap.

5.1.3. *Application of a Minimum Contract Value (MCV)*

For EIAs that are not registered as securities in the US, the Non-Forfeiture regulations require that the payoff received by the investor at withdrawal or at maturity must be greater than some MCV, which equals a certain percent β of the initial premium compounded annually at some guaranteed effective annual rate g. For a single premium EIA contract, β and g are typically 90% and 3%, respectively. Some EIA issuers have recently launched registered EIA products. Since registered products are considered as securities, the issuers do not have to comply the MCV requirement and therefore have the flexibility to adjust the contract payoff at withdrawal according to market conditions. This essentially means that part of the investment risk has been shifted to the investor.

Let's now look at the difficulty in valuation resulted from an MCV. For simplicity, we ignore early withdrawal and index averaging here. With an MCV, the payoffs of the SR-EIA and the CR-EIA are given as

$$\Lambda_{SR} = \max \left[\beta(1+g)^N, 1 + \sum_{j=1}^{N} \max \left[F, \alpha \left(R_j - 1 \right) \right] \right], \tag{70}$$

$$\Lambda_{CR} = \max \left[\beta(1+g)^N, \prod_{j=1}^{n} \max \left[1 + F, 1 + \alpha \left(R_j - 1 \right) \right] \right], \tag{71}$$

respectively. Since there are no simple closed-form expressions for the distributions of $\sum_{j=1}^{N} \max \left[F, \alpha \left(R_j - 1 \right) \right]$ and $\prod_{j=1}^{n} \max \left[1 + F, 1 + \alpha \left(R_j - 1 \right) \right]$, the additional max operator in the payoffs above poses difficulty in obtaining analytical solutions for the SR-EIA and the CR-EIA.

5.1.4. *The Case of Arithmetic Index Averaging*

In practice, the method of arithmetic index averaging is often applied to reduce the cost of an EIA. As a simple example, with arithmetic index averaging the CR-EIA payoff in (61) becomes

$$\Lambda_{CR}^{(m)} = \prod_{j=1}^{N} \max \left[1 + F, 1 + \alpha \left(\bar{R}_j^{(m)} - 1 \right) \right], \tag{72}$$

where

$$\bar{R}_j^{(m)} \equiv \frac{1}{m} \sum_{k=0}^{m-1} \frac{S \left(j - \frac{k}{m} \right)}{S(j-1)}. \tag{73}$$

Unlike the case of geometric index averaging, there exists no closed-form solution for this CR-EIA even in the absence of a cap and a MCV. This is because the averaged index return $\bar{R}_j^{(m)}$ is an arithmetic average of m log-normal random variables under the forward measure whose PDF is not known in simple analytical form. If arithmetic index averaging is used, a price computed based on the assumption of geometric index averaging only serves as a lower bound for the true price and therefore is an approximation for the true price. Fortunately, the multivariate normal representation derived in Corollary 3.2 also allows us to deal with arithmetic index averaging in an efficient manner through Monte Carlo simulation. However, simulating Ratchet payoffs with arithmetic index averaging is expected to be more time-consuming than simulating their geometric counterparts (see Remark 5.3 in the next subsection).

In addition to Monte Carlo simulation, there exist several other approximation methods for pricing payoffs involving an arithmetic sum of log-normal random variables. Levy (1992) suggests that a single log-normal random variable with its first two moments matching those of the true variable can be used as an approximating variable to obtain a reasonable price approximation. Some theoretical justifications of this approach are provided by Dufresne (2004). A related approach involving the use of Edgeworth expansion of the true density function is also supplied by Turnbull and Wakeman (1991). Recently, the concept of comonotonicity

of random variables has been successfully applied to various problems involving arithmetic sums of log-normal random variables. Using this approach, Dhaene et al. (2002) is able to derive accurate bounds for an arithmetic Asian option. Applications of this approach in approximating risk measures of sums of log-normal random variables can be found in Vanduffel et al. (2005) and Vanduffel et al. (2006). The former authors have demonstrated that the so-called comonotonic maximal variance lower bound approximation outperforms the moment matching log-normal approximation and the reciprocal Gamma approximation of Milevsky and Posner (1998) for a wide range of model parameters. Although not addressed in this paper, we believe that the approximation methods mentioned above merit further investigation related to the pricing of Ratchet EIAs .

Remark 5.2 *In spite of the computational problems discussed above, we shall point out that the price formulas derived in Section 4 can still be very useful in practice. For the CR-EIA, the price formulas can yield very accurate results in less than a few seconds provided that the contract consists of a small number of reset points. For the SR-EIA, the number of reset points has an negligible effect on the computational efficiency of the price formulas since they involve univariate normal CDFs only.*

5.2. *The Use of Monte Carlo Simulation*

In this subsection, we present an exact Monte Carlo simulation scheme which can overcome the computational problems discussed in the previous subsection. To simplify the illustration, we shall focus on the CR-EIA only and assume that there is no index averaging.

Let's again consider the following expected value appeared in the price expression (45) for the CR-EIA:

$$\mathbb{E}_0^N \left[\prod_{j=1}^N \left[M + \max\left(K, C(j-1, j) e^{W(j-1, j)} \right) \right] \right]. \tag{74}$$

Since the mean and the variance-covariance matrix of the multivariate normal vector $W = (W(0, 1), W(1, 2), ..., W(N-1, N))$ can be explicitly computed, the expected value above can be evaluated using Monte Carlo simulation. Unlike Lin and Tan (2003), path discretization of $S(t)$ and $r(t)$ is avoided here since we only need to simulate the multivariate normal vectors W. Consequently, a reduction in pricing errors and computational times may be achieved.

The simulation of correlated multivariate normal vectors has been well studied in the literature. One common approach is by matrix transformation of uncorrelated multivariate normal vectors. We now describe this approach briefly in the context of our pricing problem. Recall that the variance-covariance matrix of W is denoted by Σ. Suppose that Σ admits the following decomposition

$$\Sigma = HH', \tag{75}$$

where H is some non-singular square matrix. Such a matrix exists if Σ is positive definite. When H is a lower triangle matrix, the decomposition is known as *Cholesky's Decomposion*. Given an *uncorrelated multivariate standard normal vector* $Z = (Z_1, Z_2, ..., Z_N)$, it is easy to see that

$$W =^d HZ', \tag{76}$$

where ' $=^d$ ' means *equal in distribution*. Therefore, an n-point sample of W can be obtained as follows:
  (a) Simulate a sample of Nn independent univariate standard normal numbers $\{Z^{(k)}; \ k = 1, ..., nN\}$,
  (b) Set $Y^{(i)} \equiv (Z^{(N(i-1)+1)}, Z^{(N \cdot (i-1)+2)}, ..., Z^{(N(i-1)+N-1)}, Z^{(Ni)})'$, $i = 1, 2, ..., n$,
  (c) Set $\left(W_1^{(i)}, W_2^{(i)}, ..., W_N^{(i)} \right) \equiv HY^{(i)}$, $i = 1, 2, ..., n$.
The Monte Carlo approximation for the expected value (74) is then given by

$$\frac{1}{n} \sum_{i=1}^n \left[\prod_{j=1}^N \left[M + \max\left(K, C(j-1, j) e^{W_j^{(i)}} \right) \right] \right]. \tag{77}$$

Remark 5.3 *The Monte Carlo simulation scheme discussed above can be applied to CR-EIAs with either geometric index averaging or arithmetic index averaging. With the $\frac{1}{m}^{th}$ geometric index averaging, the multivariate normal vectors to be simulated are of dimension N. On the other hand, it is clear from (72) and (73) that the multivariate normal vectors of to be simulated are of dimension mN when the $\frac{1}{m}^{th}$ arithmetic index averaging is used. Therefore, geometric index averaging has an additional advantage that it can reduce the dimension of the simulation problem.*

Remark 5.4 *It is worth mentioning that the Monte Carlo simulation scheme above can also be used to evaluate EIAs with Water-Mark designs whose payoffs depend on the running maximum or minimum of the index returns over the contract term. In particular, if the index is monitored in a discrete fashion, the problem again reduces to the simulation of multivariate normal vectors.*

6. Incorporation of Mortality Risk

In this section, we modify the price formulas derived in Section 4 to reflect mortality risk. For illustration, we focus on an N-year CR-EIA without index averaging.

Suppose that an investor of age x enters a CR-EIA contract that is maturing in N years. Let τ_x and K_x denote the residual future lifetime and the curtate future lifetime of the investor, respectively. Note that K_x represents the remaining number of complete years the investor lives, i.e. $K_x = [\tau_x]$, where $[\tau_x]$ denotes the greatest integer that is smaller than or equal to τ_x. We assume that τ_x is a random variable measurable with respect to $(\Omega, \mathcal{F}, \mathbb{Q})$. The benefit of the CR-EIA is paid at the end of the year of the investor's death if it happens prior to time N, and is paid at time N if the investor survives at time N. Should death happen prior to time N, the benefit payable is equal to the account value accumulated up to the end of the year of death. In effect, this means that the CR-EIA has a random maturity $N_x = \min(K_x + 1, N)$ and the benefit payable is given by

$$\Lambda_{CR}(x) = \prod_{j=1}^{N_x} \max[1 + F, 1 + \alpha(R_j - 1)]. \tag{78}$$

Alternatively, we can regard the benefit of the CR-EIA as a series of payments D_t at $t = 1, 2, ..., N$, where

$$D_t = I_{\{t-1 < \tau_x \leq t\}} \prod_{j=1}^{t} \max[1 + F, 1 + \alpha(R_j - 1)]. \tag{79}$$

Let $V_{CR}(N, F, \alpha; x)$ denote the time 0 price of the CR-EIA taking into account the mortality risk of this investor. As in Cairns et al. (2006), we make the assumption that mortality risk and financial risk are independent, i.e. τ_x is independent of the processes $S(t)$ and $r(t)$. Then, we have

$$V_{CR}(N, F, \alpha; x) = \sum_{t=1}^{N} \mathbb{E}_0 \left[\frac{1}{B(t)} I_{\{t-1 < \tau_x \leq t\}} \prod_{j=1}^{t} \max[1 + F, 1 + \alpha(R_j - 1)] \right] \tag{80}$$

$$= \sum_{t=1}^{N} \mathbb{E}_0 \left[I_{\{t-1 < \tau_x \leq t\}} \right] \mathbb{E}_0 \left[\frac{1}{B(t)} \prod_{j=1}^{t} \max[1 + F, 1 + \alpha(R_j - 1)] \right]$$

$$= \sum_{t=0}^{N} {}_{t-1|}q_x V_{CR}(t, F, \alpha),$$

where $V_{CR}(t, F, \alpha)$, $t = 1, 2, ..., N$, is given in Theorem 4.5, and

$${}_{t-1|}q_x \equiv Prob[t - 1 < \tau_x \leq t]. \tag{81}$$

Similarly, we can obtain the following mortality-adjusted price formulas for the other Ratchet EIAs discussed in Section 4:

$$V_{SR}(N, F, \alpha; x) = \sum_{t=0}^{N} {}_{t-1|}q_x V_{SR}(t, F, \alpha),$$ (82)

$$V_{SR}^{(m)}(N, F, \alpha; x) = \sum_{t=0}^{N} {}_{t-1|}q_x V_{SR}^{(m)}(t, F, \alpha),$$ (83)

$$V_{CR}^{(m)}(N, F, \alpha; x) = \sum_{t=0}^{N} {}_{t-1|}q_x V_{CR}^{(m)}(t, F, \alpha).$$ (84)

It is clear that the mortality adjustments are embedded in the price formulas above only through the probabilities $\left\{ {}_{t-1|}q_x \mid t = 1, 2, ..., N \right\}$. These price formulas hold regardless of the choice of the mortality model. There are various ways to specify a mortality model. One traditional approach used by actuaries is the actuarial present value principle (APVP). The APVP relies on the assumption that the future lifetimes of homogeneous individuals are independently identically distributed. This implies that mortality risk is diversifiable. Based on this assumption, one can then fit a deterministic parametric model to the historical mortality data and use it to project the probabilities $\left\{ {}_{t-1|}q_x \mid t = 1, 2, ..., N \right\}$. This method essentially treats group mortality in a deterministic fashion. However, many studies suggest that mortality risk is systematic in the long run and is therefore non-diversifiable; e.g., see Currie et al. (2004). For EIAs with a long maturity, it is crucial to incorporate systematic mortality risk into pricing. A number of recent studies have proposed new models to capture the systematic nature of mortality risk. Most of these models focus on the specification of the dynamics of the force of mortality based on existing interest rate or credit risk frameworks. We refer the interested readers to Dahl (2004), Luciano and Vigna (2005), and Schrager (2006) for more details on such models.

7. Numerical Results

In this section, we carry out a detailed numerical analysis using the results obtained in Sections 4 and 5. Under the assumption that mortality risk and financial risk are independent, the inclusion of mortality risk in the numerical experiments does not provide any additional insight. We therefore ignore mortality risk in our analysis. We first test the computational efficiency of the Monte Carlo simulation scheme established in Section 5 numerically against the conventional Monte Carlo simulation scheme that is widely used in the EIA pricing literature. For simplicity, we shall refer to the two simulation schemes as MCS1 scheme and MCS2 scheme, respectively. We then analyze the effects of various model and contract parameters (σ, ρ, m, β, and g) on the pricing of the SR-EIA and the CR-EIA.

Following the existing literature, the pricing analysis for the SR-EIA and the CR-EIA focuses on the *break-even participation rate* (BPR), which is defined to be the participation rate at which the initial premium or price of an EIA equals its notional principal ($1 in our case). As imposing a cap allows an EIA issuer to raise the participation rate for marketing purposes, we also provide some results on the *break-even cap rate* (BCR), which is defined in a similar manner as the BPR. Since arithmetic index averaging has already been widely considered in the literature, we choose to focus on geometric index averaging in our analysis. Whenever a cap or an MCV is imposed, we combine the Monte Carlo simulation scheme with the bisection method to obtain the estimates of the BPRs and the BCRs. This procedure works as follows. For each set of model parameter values, we simulate a sample (of size 100,000) of the multivariate normal vector required to evaluate the payoff of the EIA. Using the simulated sample, the corresponding BPR or BCR can then be computed using the bisection method. We repeat these two steps 10 times to obtain an estimate of the standard errors of the estimated BPRs and BCRs.

All numerical results are summarized in Tables 1 - 12 in the Appendix. Except for Tables 1 and 7 where the underlying EIA is assumed to have a maturity $N = 3$, we assume $N = 7$ for all remaining numerical

experiments. The following parameter values are assumed throughout all experiments: $F = 0\%$, $f(0, t) = 0.04 + 0.0045t - 0.00015t^2$, and $\kappa = 0.05$. Note that the forward curve $f(0, t)$ is upward-sloping for $t \leq 15$.

7.1. *Computational Efficiency of the MCS1 and MCS2 Schemes*

The MSC2 scheme has been widely used in the EIA pricing literature, which requires discretization of the sample paths of $S(t)$ and $r(t)$. On the other hand, the MSC1 scheme suggested in this paper is exact and requires only the simulation of multivariate normal vectors. For practical purposes, it is interesting to compare the computational efficiency of the two simulation schemes. For illustration, we focus on a three-year CR-EIA without a cap, a MCV, and index averaging. The comparison is carried out as follows.

We first compute the prices of the EIA using the analytical price formulas derived in Section 4. These prices provide benchmarks for the comparison between the two simulation schemes. For each simulation scheme, we then record the estimated prices, the estimated standard errors of the price estimates, the percentage errors in price relative to the benchmark prices, and the computational times used. For the MCS1 scheme, we simulate 10 samples of size 100,000 to produce 10 price estimates for each scenario. The final price estimate and its associated standard error are computed as the average and the sample standard deviation of the 10 price estimates, respectively. For the MCS2 scheme, we also simulate 10 samples each consisting of 100,000 sample paths of $S(t)$ and $r(t)$. The final price estimate and its associated standard errors are then computed in the same manner as in the previous case. Since path discretization is required for the MCS2 scheme, we consider three step sizes for path discretization: 1, $\frac{1}{12}$, and $\frac{1}{252}$.

The numerical results for the comparison are summarized in Table 1. As expected, for the MCS2 scheme a smaller step size gives smaller percentage errors relative to the benchmark prices. However, the differences do not appear to be significant for the step sizes under consideration. On the other hand, it is clear that the prices produced by the MSC1 scheme are at least as accurate as those produced by the MCS2 scheme. In addition, the MSC1 scheme dominates the MCS2 scheme significantly in computational time. Computational time can be an important issue in practice. Our results indicate that performing a pricing or sensitivity analysis for the CR-EIA can be a very time-consuming task when the MCS2 scheme is adopted while the MCS1 scheme can speed up the analysis considerably. Moreover, most EIAs have a maturity longer than the three-year CR-EIA assumed in our experiment. The MCS2 scheme is expected to perform even more slowly when an EIA with a longer maturity is encountered.

7.2. *The SR-EIA*

Numerical results for the SR-EIA are provided in Tables 2 - 6 in the Appendix. The underlying EIA is assumed to have a maturity of $N = 7$. The results in Table 2 are based on the analytical price formulas derived in Section 4.1 while the results in Tables 3 - 6 are obtained using the MCS1 scheme. We summarize our findings below.

Result 1: Table 2 gives the analytical BPRs obtained using the price formulas (34) and (43). A few observations can be drawn. First, the BPRs under monthly geometric index averaging are consistently higher than those under no index averaging (the BPRs are almost double in some cases). This is to be expected as index averaging helps lower the volatility of the credit returns and thus reduces the cost of the EIA. Second, when $\gamma = 0$, the correlation efficient ρ is irrelevant in determining the BPR as the the short rate is deterministic in this case. When $\gamma \neq 0$, the BPR appears to be increasing in ρ in general. As ρ increases, the index return and the short rate are more likely to move in the same direction. This implies that the discounting effect on higher index returns are more pronounced for a larger ρ. As a result, a higher BPR is required to compensate the more pronounced discounting effect on the right tail of the index returns. Third, when $\rho \geq 0$, the BPR increases monotonically in the short rate volatility γ. In contrast, when $\rho < 0$, the BPR decreases initially as γ increases. As γ increases further, the BPR increases. This phenomenon can be attributed to the interaction between the effects of γ and ρ on the value of the SR-EIA. The net effect depends on the particular set of model and contract parameters being considered. Finally, it can be seen

that the BPR always declines as the index volatility σ increases. This is consistent with the intuition that a higher index volatility should give a higher gain potential when the return is floored, which is analogous to the relation between the value of a call option and the volatility parameter.

Result 2: In Table 3, we consider the impact of imposing an MCV on the BPR. An immediate observation is that the BPRs are lower when an MCV is in effect. This is to be expected as the MCV provides extra downside protection and thus increases the value of the EIA. Moreover, when g is fixed, the smaller β is, the larger the BPR is. This is simply because a smaller β provides a smaller downside protection and therefore requires a higher BPR to bring up the value of the EIA.

Result 3: In Table 4, we record the BPRs under various caps, when the short rate volatility γ is fixed at 4%. Unlike the case without a cap, increasing the index volatility can sometimes result in a higher BPR depending on the values of the other parameters. As an illustration, observe that when $\rho = 0.3$, $cap = 16\%$, $\beta = 100\%$, and $g = 3\%$, the BPR increases from 0.9581 to 0.9888 when σ increases from 20% to 30%. This reversed effect on the BPR arises because for certain sets of parameters the additional upside potential resulted from an increase in the index volatility may not be fully realized due to the limitation on gain resulted from the cap. Table 5 gives some additional results on the BPRs when fixing the cap at 16%.

Result 4: In Table 6, we fix the participation at 100% and assess the cost of the SR-EIA using the BCR. Clearly, the value of an EIA is monotonically increasing in both the participation rate and the cap rate. The results indicate that the behavior of the BCR relative to the other model and contract parameters agree with that of the BPR qualitatively. In particular, the reversed effect observed in the BPR when increasing the index volatility σ is also observed in the BCR. Comparing to the BPR, the BCR appears to be less sensitive to σ in general. For example, when $\rho = 0.3$, $\gamma = 8\%$, $\beta = 100\%$, $g = 3\%$, and $m = 1$ (see Table 3 and Table 6), increasing σ from 20% to 30% results in a percentage drop of 22.50% in the BPR, but a percentage drop of only 2.80% in the BCR.

7.3. *The CR-EIA*

Numerical results for the CR-EIA are given in Tables 7 - 12 in the Appendix. In Table 7, we compare the BPRs obtained using the analytical price formulas (60) and (64) with those obtained using the MCS1 scheme for a three-year CR-EIA. We assume a three-year maturity since the analytical price formulas are computationally better suited for CR-EIAs with a short maturity. It is clear that the BPRs obtained under the two approaches agree with each other closely. On the other hand, the results given in Tables 8 - 12 are obtained using the MCS1 scheme for a seven-year CR-EIA. The results agree qualitatively with those obtained for the SR-EIA previously. We now make a few additional comments.

Result 5: The BPRs and the BCRs obtained for the CR-EIA are consistently lower than those obtained for the SR-EIA. This is simply because under the CR-EIA each periodic return is re-invested at the remaining reset returns while it is not the case for the SR-EIA. This phenomenon is analogous to the relation between simple interests and compound interests. In other words, the CR-EIA has a higher value than its SR-EIA counterpart when holding the contract and model parameters fixed, and therefore requires a lower BPR or BCR.

Result 6: The effect of the short rate volatility γ on the BPR is mixed in general. The net effect seems to depend on the interaction between γ and the other parameters. Under certain sets of parameters, the BPR can be quite sensitive to the short rate volatility for both the SR-EIA and the CR-EIA, especially when a cap is in effect (see Table 5 and Table 11). This reassures the importance of introducing randomness to the short rate when pricing long-term EIAs.

8. Conclusion

In this paper, we make an attempt to incorporate stochastic interest rates into the valuation of Ratchet EIAs by assuming that the short rate follows the extended Vasicek model. This is an important issue in the valuation of EIAs as these contracts usually have a long maturity for which the effects of stochastic interest rates become crucial. Analytical price formulas are derived for simple and compound Ratchet EIAs using the forward valuation approach. We also show how to modify the derived price formulas to reflect mortality risk. For the SR-EIA, the number of rest points or the presence of a cap has negligible effects on the computational feasibility of the price formulas, while the price formulas for the CR-EIA are computationally better suited for contracts with a small number of reset points. To overcome the computational difficulties in evaluating Ratchet EIAs with a large number of reset points, a cap, an MCV, or arithmetic index averaging, we establish an easy and efficient Monte Carlo simulation scheme for pricing which is based on the simulation of multivariate normal vectors. Finally, detailed numerical results are provided to illustrate the computational efficiency of our simulation scheme and the impacts of various model and contract parameters on pricing. These numerical results provide further insights on the role of stochastic interest rates in evaluating EIA contracts.

There still remains some important issues in the valuation of EIAs that need to be addressed in future research. First, it may be interesting to consider other types of stochastic interest rate models to overcome the known problems of the extend Vasicek model. In particular, one undesired property of the extend Vasicek model is that it permits negative interest rates, which can be problematic if the term of EIA is long enough giving rise to a significant likelihood of negative rates. Moreover, the short rate is not observable in the market. Therefore, a proxy of the spot short rate is needed as an input of the pricing. One possible solution to these problems is to consider the LIBOR Market model; e.g., see Brace et al. (1997), where the interest rates being modelled are both positive and observable. Second, the hedging of EIA contracts remains a challenging yet important task. In recent years, insurers have become increasingly aware of the role of hedging in managing EIA portfolios. Hedging of EIAs is a challenging task particularly because these products usually have maturities that are substantially longer than those of exchange-traded options. On the other hand, dynamic hedging with risk-free bonds and the underlying is too costly to carry out. Finally, the existing EIA pricing literature has largely ignored early surrender risk. This should be taken into account for future studies as early surrender risk has always been an important concern for insurers.

References

[1] Bowers L., Gerber H., Hickman J., Jones D., and Nesbitt C., 1997, Actuarial Mathematics, 2nd ed., *Society of Actuaries.*

[2] Björk T., 2004, Arbitrage Theory in Continuous Time, 2nd ed., *Oxford University Press.*

[3] Bodmayr M., "Equity-Indexed Annuities in the United States.", *Münchener Rückversicherungs-Gesellschaft.*

[4] Brace G., Gatarek D., and Musiela M., 1997, "The Market Model of Interest Rate Dynamics.", *Mathematical Finance*, 7: 127-155.

[5] Brigo D. and Mercurio F., 2001, Interest Rate Models: Theory and Practice, 1st ed., *Springer.*

[6] Cairns A., Blake D., and Dowd K., 2006, "Pricing Death: Frameworks for The Valuation and Securitization of Mortality Risk.", *Astin Bulletin*, 36: 79-120.

[7] Cheung Y.L. and Wong H.Y., 2004, "Geometric Asian Options: Valuation and Calibration with Stochastic Volatility.", *Quantitative Finance*, 4(3): 301-314.

[8] Currie I.D., Durban M., and Eilers P.H.C., 2004, "Smoothing and Forecasting Mortality Rates.", *Statistical Modelling*, 4:279-298.

[9] Dahl M., 2004, "Stochastic Mortality in Life Insurance: Market Reserves and Mortality-Linked Insurance Contracts.", *Insurance: Mathematics and Economics*, 35(1): 113-136.

[10] Denuit M., Dhaene J., Goovaerts M.J., Kaas R., and Vyncke D., 2002, "The Concept of Comonotonicity in Actuarial Science and Finance: Theorey.", *Insurance: Mathematics and Economics*, 31(2): 3-33.

[11] Denuit M., Dhaene J., Goovaerts M.J., Kaas R., and Vyncke D., 2002, "The Concept of Comonotonicity in Actuarial Science and Finance: Applications.", *Insurance: Mathematics and Economics*, 31(2): 133-161.

[12] Dhaene J., Hoedemakers T., and Vanduffel S., 2005, "Comparing Approximations for Risk Measures of Sums of Non-Independent Lognormal Random Variables.", *North American Actuarial Journal*, 9(4): 71 - 82.

[13] Chen X., Dhaene J., Goovaerts M.J., Henrard L., Kaas R., and Vanduffel S., 2006, "A Note on Optimal Lower Bound Approximations for Risk Measures of Sums of Lognormals.", *Preprint*, Katholic University Leuven.

[14] Dufresne D., 2004, "The Lognormal Approximation in Financial and Other Computations.", *Advances in Applied Probability*, 36: 747-773.

[15] Föllmer H. and Sondermann D., 1986, "Hedging of Non- Redundant Contingent Claims.", *Contributions to Mathematical Economics: 20523*, Edited by W. Hildenbrand and A. Mas-Colell. Amsterdam: North-Holland.

[16] Gerber H. and Shiu E., 2003, "Pricing Lookback Options and Dynamic Guarantees.", *North American Actuarial Journal*, 7(1): 48-67.

[17] German H., Karoui N., and Rochet J., 1995, "Change of Numeraire, Change of Probability Measure, and Option Pricing.", *Journal of Applied Probability*, 32: 443-458.

[18] Hardy M., 2001, "A Regime Switching Model of Long Term Stock Returns.", *North American Actuarial Journal*, 5: 41-53.

[19] Hardy M., 2003, "Ratchet Equity Indexed Annuities.", The 14th Annual International AFIR Colloquium, *The International Actuarial Association*.

[20] Jaimungal S., 2004, "Pricing and Hedging Equity Indexed Annuities with Variance Gamma Deviates.", preprint, `http://www.utstat.utoronto.ca/sjaimung/papers/eiaVG.pdf`.

[21] Kijima M. and Muromachi Y., 2001, "Pricing Equity Swaps in A Stochastic Interest Rate Economy.", *Journal of Derivatives*, 8: 19-35.

[22] Lee H., 2003, "Pricing Equity-Indexed Annuities with Path-Dependent Options.", *Insurance, Mathematics, and Economics*, 33(3): 677-690.

[23] Levy E., 1992, 'The Valuation of Average Rate Currency Options.", *Journal of International Money and Finance*, 11: 474491.

[24] Lin X.D. and Tan K.S., 2003, "Valuation of Equity-Indexed Annuities under Stochastic Interest Rates.", *North American Actuarial Journal*, 6: 72-91.

[25] Luciano E. and Vigna E., 2005, "Non-Mean-Reverting Affine Processes for Stochastic Mortality.", The Proceedings of the 15th International AFIR Colloquium, *The International Actuarial Association*.

[26] Marrion J., 2001, "Advantage March, 2001 Equity Index Report.", online article, `http://www.indexannuity.org`.

[27] Muromachi Y., 2002, "Evaluation of Equity-Indexed Annuities in a Stochastic Interest Rate Economy.", *Preprint*.

[28] Nielsen J.A. and Sandmann K., 1996, "The Pricing of Asian Options under Stochastic Interest Rates.", *Applied Mathematical Finance*, 3:209-236.

[29] Schrager F., 2006, "Affine Stochastic Mortality.", *Insurance: Mathematics and Economics*, 38(1): 81-97.

[30] Tiong S., 2000, "Valuing Equity-Indexed Annuities.", *North American Actuarial Journal*, 4: 149-170.

[31] Turnbull S.M. and Wakeman L. M., 1991, "Quick Algorithm for Pricing European Average Options.", *Journal of Financial and Quantitative Analysis*, 26(3): 377389.

APPENDIX: Tables

MCS1 Scheme			
ρ	-0.3	0	0.3
Price	1.0496	1.0522	1.0542
Estimated Standard Error	0.0003	0.0004	0.0004
Percentage Error	-0.0038%	0.0076%	-0.0322%
Computational Time (in second)	1.8030	1.7020	1.8630

MCS2 Scheme				
Step Size*	ρ	-0.3	0	0.3
1	Price	1.0515	1.0499	1.0482
	Estimated Standard Error	0.0006	0.0003	0.0008
	Percentage Error	0.1766%	-0.2157%	-0.6001%
	Computational Time (in second)	72.9450	72.9950	74.6570
$\frac{1}{12}$	Price	1.0502	1.0520	1.0537
	Estimated Standard Error	0.0005	0.0006	0.0005
	Percentage Error	0.0534%	-0.0114%	-0.0797%
	Computational Time (in second)	90.1600	90.2700	90.1490
$\frac{1}{252}$	Price	1.0493	1.0521	1.0544
	Estimated Standard Error	0.0006	0.0005	0.0005
	Percentage Error	-0.0324%	-0.0019%	-0.0133%
	Computational Time (in second)	420.9210	419.2020	419.5140

Table 1
The prices and other statistics for a three-year CR-EIA without index averaging obtained using the two simulation schems, MCS1 and MCS2. The other model and contract parameters are: $\kappa = 0.05$, $\gamma = 0.04$, $\sigma = 20\%$, $\alpha = 0.6$, $F = 0$, no cap, and no MCV. (*The step size used for path discretization)

		No Index Averaging: $m = 1$			Monthly Index Averaging: $m = 12$		
		ρ			ρ		
σ	γ	-0.3	0	0.3	-0.3	0	0.3
20%	0	0.5760	0.5760	0.5760	1.0189	1.0189	1.0189
	4%	0.5729	0.5826	0.5921	1.0163	1.0367	1.0569
	8%	0.5835	0.6024	0.6205	1.0489	1.0899	1.1301
30%	0	0.4255	0.4255	0.4255	0.7576	0.7576	0.7576
	4%	0.4216	0.4325	0.4432	0.7513	0.7719	0.7926
	8%	0.4314	0.4532	0.4748	0.7728	0.8151	0.8579

Table 2
Analytical BPRs for a seven-year SR-EIA based on the analytical price formulas: no MCV, no cap.

Index Averaging	σ	γ	MCV: $\beta = 100\%$, $g = 3\%$ ρ			MCV: $\beta = 90\%$, $g = 3\%$ ρ		
			-0.3	0	0.3	-0.3	0	0.3
$m = 1$	20%	0	0.5557 (0.0007)	0.5552 (0.0009)	0.5555 (0.0005)	0.5739 (0.0011)	0.5730 (0.0011)	0.5729 (0.0007)
		4%	0.5503 (0.0004)	0.5508 (0.0008)	0.5524 (0.0012)	0.5697 (0.0012)	0.5775 (0.0007)	0.5833 (0.0010)
		8%	0.5405 (0.0011)	0.5431 (0.0013)	0.5463 (0.0013)	0.5752 (0.0010)	0.5879 (0.0015)	0.6004 (0.0011)
	30%	0	0.4048 (0.0004)	0.4050 (0.0006)	0.4049 (0.0008)	0.4224 (0.0006)	0.4217 (0.0006)	0.4220 (0.0005)
		4%	0.4015 (0.0009)	0.4073 (0.0008)	0.4121 (0.0008)	0.4184 (0.0006)	0.4278 (0.0009)	0.4370 (0.0009)
		8%	0.4042 (0.0008)	0.4144 (0.0007)	0.4234 (0.0005)	0.4260 (0.0010)	0.4444 (0.0011)	0.4616 (0.0011)
$m = 12$	20%	0	0.9831 (0.0017)	0.9831 (0.0014)	0.9835 (0.0010)	1.0141 (0.0016)	1.0137 (0.0018)	1.0143 (0.0015)
		4%	0.9782 (0.0013)	0.9859 (0.0021)	0.9932 (0.0017)	1.0110 (0.0012)	1.0292 (0.0022)	1.0442 (0.0022)
		8%	0.9825 (0.0019)	0.9980 (0.0025)	1.0136 (0.0016)	1.0368 (0.0020)	1.0693 (0.0018)	1.0991 (0.0023)
	30%	0	0.7230 (0.0013)	0.7234 (0.0015)	0.7232 (0.0012)	0.7517 (0.0012)	0.7523 (0.0020)	0.7521 (0.0015)
		4%	0.7173 (0.0013)	0.7300 (0.0011)	0.7420 (0.0010)	0.7457 (0.0015)	0.7642 (0.0021)	0.7824 (0.0016)
		8%	0.7286 (0.0015)	0.7530 (0.0013)	0.7786 (0.0026)	0.7653 (0.0009)	0.8014 (0.0024)	0.8377 (0.0022)

Table 3
BPRs for a seven-year SR-EIA based on the MCS1 scheme: no cap. The numbers in the parentheses are the estimated standard errors associated with the BPR estimates. The same applies to what follows.

σ	cap	MCV: $\beta = 100\%$, $g = 3\%$ ρ			MCV: $\beta = 90\%$, $g = 3\%$ ρ		
		-0.3	0	0.3	-0.3	0	0.3
20%	20%	0.6669 (0.0015)	0.6831 (0.0017)	0.6988 (0.0033)	0.6982 (0.0016)	0.7276 (0.0023)	0.7589 (0.0028)
	18%	0.7350 (0.0022)	0.7567 (0.0012)	0.7824 (0.0029)	0.7727 (0.0028)	0.8145 (0.0026)	0.8619 (0.0018)
	16%	0.8690 (0.0042)	0.9113 (0.0043)	0.9581 (0.0057)	0.9172 (0.0031)	0.9904 (0.0036)	1.0773 (0.0077)
30%	20%	0.5518 (0.0018)	0.5778 (0.0010)	0.6045 (0.0019)	0.5835 (0.0018)	0.6203 (0.0024)	0.6624 (0.0017)
	18%	0.6371 (0.0020)	0.6742 (0.0023)	0.7171 (0.0034)	0.6772 (0.0016)	0.7376 (0.0034)	0.8027 (0.0028)
	16%	0.8218 (0.0055)	0.8961 (0.0048)	0.9888 (0.0067)	0.8915 (0.0045)	1.0084 (0.0076)	1.1609 (0.0152)

Table 4
BPRs for a seven-year SR-EIA based on the MCS1 scheme: no index averaging, $\gamma = 4\%$.

σ	γ	MCV: $\beta = 100\%$, $g = 3\%$ ρ			MCV: $\beta = 90\%$, $g = 3\%$ ρ		
		-0.3	0	0.3	-0.3	0	0.3
20%	0	0.6662 (0.0016)	0.6675 (0.002)	0.6663 (0.0012)	0.6943 (0.0017)	0.6944 (0.0014)	0.6947 (0.0011)
	4%	0.6669 (0.0015)	0.6831 (0.0017)	0.6988 (0.0033)	0.6982 (0.0016)	0.7276 (0.0023)	0.7589 (0.0028)
	8%	0.7070 (0.0024)	0.7475 (0.0041)	0.7939 (0.0043)	0.7772 (0.0029)	0.8547 (0.0044)	0.9462 (0.0026)
30%	0	0.5570 (0.0017)	0.5572 (0.0012)	0.5571 (0.002)	0.5905 (0.0013)	0.5910 (0.0017)	0.5893 (0.0015)
	4%	0.5518 (0.0018)	0.5778 (0.0010)	0.6045 (0.0019)	0.5835 (0.0018)	0.6203 (0.0024)	0.6624 (0.0017)
	8%	0.5892 (0.0019)	0.6488 (0.0029)	0.7199 (0.0039)	0.6382 (0.0027)	0.7340 (0.0046)	0.8526 (0.0075)

Table 5
BPRs for a seven-year SR-EIA based on the MCS1 scheme: no index averaging, 20% cap

| | | MCV: $\beta = 100\%$, $g = 3\%$ | | | MCV: $\beta = 90\%$, $g = 3\%$ | | |
| | | | ρ | | | ρ | |
σ	γ	-0.3	0	0.3	-0.3	0	0.3
20%	0	19.29% (0.0004)	19.28% (0.0003)	19.27% (0.0004)	19.53% (0.0003)	19.53% (0.0006)	19.52% (0.0005)
	4%	19.48% (0.0003)	20.13% (0.0004)	20.78% (0.0006)	19.76% (0.0006)	20.62% (0.0005)	21.52% (0.0005)
	8%	21.34% (0.0008)	22.65% (0.0006)	23.90% (0.0009)	22.17% (0.0006)	23.95% (0.0007)	25.86% (0.0007)
30%	0	19.41% (0.0004)	19.42% (0.0003)	19.43% (0.0004)	19.68% (0.0002)	19.70% (0.0003)	19.68% (0.0005)
	4%	19.32% (0.0003)	19.98% (0.0005)	20.71% (0.0004)	19.57% (0.0002)	20.37% (0.0005)	21.20% (0.0005)
	8%	20.39% (0.0004)	21.80% (0.0004)	23.23% (0.0007)	20.81% (0.0003)	22.54% (0.0005)	24.37% (0.0005)

Table 6
BCRs for a seven-year SR-EIA based on the MCS1 scheme: no index averaging, full index participation, i.e. $\alpha = 100\%$.

| | | | No Index Averaging: $m = 1$ | | | Monthly Index Averaging: $m = 12$ | | |
| | | | | ρ | | | ρ | |
	σ	γ	-0.3	0	0.3	-0.3	0	0.3
Analytical	20%	0	0.4411	0.4411	0.4411	0.7765	0.7765	0.7765
		4%	0.4410	0.4359	0.4311	0.7756	0.7721	0.7689
		8%	0.4307	0.4218	0.4139	0.7663	0.7603	0.7549
	30%	0	0.3219	0.3219	0.3219	0.5710	0.5710	0.5710
		4%	0.3218	0.3205	0.3192	0.5693	0.5704	0.5716
		8%	0.3189	0.3165	0.3142	0.5666	0.5690	0.5714
MCS1	20%	0	0.4418 (0.0015)	0.4412 (0.0016)	0.4413 (0.0009)	0.7772 (0.0021)	0.7771 (0.0022)	0.7772 (0.0020)
		4%	0.4404 (0.0012)	0.4356 (0.0018)	0.4313 (0.0011)	0.7761 (0.0021)	0.7729 (0.0018)	0.7692 (0.0014)
		8%	0.4306 (0.0010)	0.4221 (0.0012)	0.4139 (0.0007)	0.7662 (0.0034)	0.7603 (0.0021)	0.7541 (0.0032)
	30%	0	0.3215 (0.0010)	0.3215 (0.0010)	0.3218 (0.0012)	0.5707 (0.0017)	0.5718 (0.0015)	0.5710 (0.0017)
		4%	0.3214 (0.0007)	0.3205 (0.0011)	0.3198 (0.0009)	0.5691 (0.0017)	0.5709 (0.0019)	0.5718 (0.0015)
		8%	0.3189 (0.0009)	0.3166 (0.0014)	0.3144 (0.0006)	0.5669 (0.0009)	0.5686 (0.0015)	0.5719 (0.0015)

Table 7
BPRs for a three-year CR-EIA based on the the analytical price formulas and the MCS1 scheme: no MCV, no cap.

| | | No Index Averaging: $m = 1$ | | | Monthly Index Averaging: $m = 12$ | | |
| | | | ρ | | | ρ | |
σ	γ	-0.3	0	0.3	-0.3	0	0.3
20%	0	0.4883 (0.0008)	0.4884 (0.0010)	0.4884 (0.0008)	0.8640 (0.0013)	0.8639 (0.0015)	0.8636 (0.0012)
	4%	0.4840 (0.0007)	0.4864 (0.0012)	0.4885 (0.0010)	0.8600 (0.0017)	0.8687 (0.0020)	0.8786 (0.0016)
	8%	0.4771 (0.0009)	0.4834 (0.0013)	0.4879 (0.0014)	0.8687 (0.0017)	0.8869 (0.0027)	0.9033 (0.0020)
30%	0	0.3604 (0.0007)	0.3612 (0.0006)	0.3606 (0.0007)	0.6421 (0.0010)	0.6414 (0.0012)	0.6419 (0.0008)
	4%	0.3578 (0.0006)	0.3637 (0.0007)	0.3692 (0.0010)	0.6379 (0.0012)	0.6507 (0.0014)	0.6639 (0.0012)
	8%	0.3610 (0.0006)	0.3723 (0.0009)	0.3821 (0.0008)	0.6502 (0.0015)	0.6757 (0.0006)	0.7001 (0.0025)

Table 8
BPRs for a seven-year CR-EIA based on the MCS1 scheme: no MCV, no cap.

			MCV: $\beta = 100\%$, $g = 3\%$			MCV: $\beta = 90\%$, $g = 3\%$		
Index				ρ			ρ	
Averaging	σ	γ	-0.3	0	0.3	-0.3	0	0.3
$m = 1$	20%	0	0.4689 (0.0008)	0.4689 (0.0008)	0.4693 (0.0003)	0.4858 (0.0009)	0.4856 (0.0006)	0.4860 (0.0008)
		4%	0.4632 (0.0011)	0.4590 (0.0007)	0.4553 (0.0008)	0.4809 (0.0008)	0.4823 (0.0008)	0.4823 (0.0014)
		8%	0.4418 (0.0007)	0.4362 (0.0015)	0.4324 (0.0008)	0.4716 (0.0011)	0.4727 (0.0014)	0.4731 (0.0018)
	30%	0	0.3423 (0.0005)	0.3422 (0.0007)	0.3422 (0.0008)	0.3572 (0.0008)	0.3576 (0.0008)	0.3574 (0.0008)
		4%	0.3400 (0.0004)	0.3416 (0.0007)	0.3425 (0.0008)	0.3552 (0.0004)	0.3594 (0.0007)	0.3638 (0.0006)
		8%	0.3374 (0.0009)	0.3395 (0.0010)	0.3426 (0.0013)	0.3564 (0.0007)	0.3651 (0.0011)	0.3729 (0.0008)
$m = 12$	20%	0	0.8306 (0.0013)	0.8306 (0.0007)	0.8301 (0.0014)	0.8598 (0.0015)	0.8583 (0.0015)	0.8588 (0.0018)
		4%	0.8253 (0.0011)	0.8236 (0.0019)	0.8235 (0.0017)	0.8549 (0.0019)	0.8617 (0.0016)	0.8683 (0.0018)
		8%	0.8122 (0.0027)	0.8110 (0.0022)	0.8125 (0.0016)	0.8579 (0.0014)	0.8707 (0.0023)	0.8821 (0.0017)
	30%	0	0.6114 (0.0010)	0.6118 (0.0010)	0.6109 (0.0011)	0.6367 (0.0012)	0.6376 (0.0012)	0.6373 (0.0011)
		4%	0.6075 (0.0010)	0.6138 (0.0010)	0.6203 (0.0018)	0.6333 (0.0010)	0.6444 (0.0014)	0.6554 (0.0015)
		8%	0.6112 (0.0010)	0.6242 (0.0010)	0.6355 (0.0018)	0.6432 (0.0012)	0.6640 (0.0025)	0.6868 (0.0019)

Table 9
BPRs for a seven-year CR-EIA based on the MCS1 scheme: no cap.

		MCV: $\beta = 100\%$, $g = 3\%$			MCV: $\beta = 90\%$, $g = 3\%$		
			ρ			ρ	
σ	cap	-0.3	0	0.3	-0.3	0	0.3
20%	20%	0.5110 (0.0010)	0.5112 (0.0013)	0.5119 (0.0009)	0.5352 (0.0009)	0.5437 (0.0023)	0.5514 (0.0017)
	18%	0.5389 (0.0015)	0.5420 (0.0013)	0.5427 (0.0018)	0.5652 (0.0016)	0.5779 (0.0019)	0.5893 (0.0016)
	16%	0.5874 (0.0014)	0.5941 (0.0020)	0.6001 (0.0017)	0.6201 (0.0017)	0.6401 (0.0019)	0.6589 (0.0021)
30%	20%	0.4052 (0.0009)	0.4141 (0.0011)	0.4223 (0.0011)	0.4282 (0.0007)	0.4438 (0.0008)	0.4584 (0.0019)
	18%	0.4390 (0.0017)	0.4501 (0.0013)	0.4622 (0.0016)	0.4653 (0.0015)	0.4857 (0.0023)	0.5075 (0.0016)
	16%	0.4991 (0.0012)	0.5176 (0.0014)	0.5386 (0.0025)	0.5341 (0.0015)	0.5659 (0.0026)	0.5989 (0.0022)

Table 10
BPRs rates for a seven-year CR-EIA based on the MCS1 scheme: no index averaging, $\gamma = 4\%$.

		MCV: $\beta = 100\%$, $g = 3\%$			MCV: $\beta = 90\%$, $g = 3\%$		
			ρ			ρ	
σ	γ	-0.3	0	0.3	-0.3	0	0.3
20%	0	0.5154 (0.0007)	0.5147 (0.0014)	0.5146 (0.0007)	0.5366 (0.0006)	0.5369 (0.0013)	0.5363 (0.0017)
	4%	0.5110 (0.0010)	0.5112 (0.0013)	0.5119 (0.0009)	0.5352 (0.0009)	0.5437 (0.0023)	0.5514 (0.0017)
	8%	0.5097 (0.0016)	0.5140 (0.0018)	0.5214 (0.0018)	0.5540 (0.0022)	0.5747 (0.0017)	0.5980 (0.0023)
30%	0	0.4082 (0.0011)	0.4088 (0.0013)	0.4087 (0.0007)	0.4325 (0.0013)	0.4318 (0.0009)	0.4316 (0.0013)
	4%	0.4052 (0.0009)	0.4141 (0.0011)	0.4223 (0.0011)	0.4282 (0.0007)	0.4438 (0.0008)	0.4584 (0.0019)
	8%	0.4152 (0.0012)	0.4340 (0.0013)	0.4543 (0.0020)	0.4484 (0.0017)	0.4806 (0.0019)	0.5191 (0.0027)

Table 11
BPRs for a seven-year CR-EIA based on the MCS1 scheme: no index averaging, 20% cap

		MCV: $\beta = 100\%$, $g = 3\%$			MCV: $\beta = 90\%$, $g = 3\%$		
			ρ			ρ	
σ	γ	-0.3	0	0.3	-0.3	0	0.3
20%	0	11.84% (0.0002)	11.84% (0.0002)	11.83% (0.0002)	12.11% (0.0002)	12.11% (0.0002)	12.11% (0.0003)
	4%	11.91% (0.0002)	12.12% (0.0002)	12.33% (0.0002)	12.22% (0.0001)	12.57% (0.0003)	12.88% (0.0002)
	8%	12.58% (0.0002)	12.97% (0.0004)	13.36% (0.0003)	13.21% (0.0002)	13.85% (0.0004)	14.48% (0.0004)
30%	0	12.34% (0.0002)	12.34% (0.0001)	12.33% (0.0002)	12.66% (0.0001)	12.66% (0.0002)	12.66% (0.0001)
	4%	12.31% (0.0002)	12.57% (0.0002)	12.85% (0.0002)	12.61% (0.0002)	13.01% (0.0002)	13.38% (0.0003)
	8%	12.77% (0.0002)	13.31% (0.0003)	13.87% (0.0004)	13.21% (0.0002)	14.00% (0.0003)	14.79% (0.0003)

Table 12

BCRs for a seven-year CR-EIA based on the MCS1 scheme: no index averaging, full index participation, i.e. $\alpha = 100\%$.