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Abstract

In this paper, we consider the problem of allocating indivisible objects with monetary transfers. We study secure implementation (Cason, Saijo, Sjöström, and Yamato, 2006; Saijo, Sjöström, and Yamato, 2006) which signifies double implementation in dominant strategy equilibria and Nash equilibria. We propose a new domain-richness condition—*box-shapedness*—and establish that only constant social choice functions can be securely implemented on a box-shaped domain.

Keywords: Secure implementation, Dominant strategy implementation, Nash implementation, Indivisible goods, Strategy-proofness.

JEL codes: C72, C78, D61, D63, D71.

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1 Introduction

We study the problem of allocating indivisible objects when monetary transfers are possible. We assume that each agent consumes one and only one indivisible object. These objects may represent house locations, jobs, or certain rights.[1] Each *allocation* consists of an assignment of indivisible objects and a vector of monetary transfers. A *social choice function* (or *mechanism*) is a function that associates one allocation with each preference profile.

Since preferences are usually privately known, selfish agents may have an incentive to manipulate the social choice function by misrepresenting these preferences. As a result of such strategic manipulations, the outcome may be far from satisfactory from the social point of view. Thus, it is important for a social choice function to be immune from strategic manipulations. Such social choice functions are termed *strategy-proof* since they insure that no one gains by reference misrepresentation. Strategy-proofness is a primary requirement in the mechanism design. Much of the literature referred to in our model states that strategy-proofness is consistent with certain reasonable requirements.[2]

On the other hand, Saijo, Sjöström, and Yamato (2006) argue theoretical drawbacks of strategy-proof mechanisms. In particular, they point out that a majority of strategy-proof mechanisms produce several Nash equilibrium outcomes thereby making these mechanisms somewhat ineffective. In other words, the existence of "bad" Nash equilibria prevents strategy-proof mechanisms from functioning well. The results of the experiments launched by Cason, Saijo, Sjöström, and Yamato (2006) support this idea. The results show that a strategy-proof mechanism with "bad" Nash equilibria is inferior as compared with one that is free from "bad" Nash equilibria. Based on these facts, Cason, Saijo, Sjöström, and Yamato (2006) and Saijo, Sjöström, and Yamato (2006) developed a new concept, namely, secure implementation. A social choice function is securely implementable if there exists a mechanism that implements it through dominant strategy equilibria, and the set of dominant strategy equilibrium outcomes coincides with the set of Nash equilibrium outcomes.[3]

[1] Basic models of economies with indivisible objects and monetary transfers were provided by Svensson (1983) and Alkan, Demange, and Gale (1991).

[2] Holmström (1979) shows that "Groves schemes" (Groves, 1973) are the only decision-efficient and strategy-proof social choice functions. Miyagawa (2001) provides a characterization of strategy-proof social choice functions satisfying auxiliary axioms such as individual rationality and non-bossiness. Ohseto (2006) characterizes the class of strategy-proof and envy-free social choice functions vis-à-vis homogeneous goods and money.

[3] In other words, secure implementation signifies double implementation in dominant strategy equilibria and Nash equilibria. Fujinaka and Sakai (2006) present a study of Nash implementation

In this case, the secure mechanism does not suffer from any "bad" Nash equilibrium.

Our purpose is to characterize the class of securely implementable social choice functions in our economies. Saijo, Sjöström, and Yamato (2006) provide a characterization that fits a general environment: a social choice function is securely implementable if and only if it satisfies *strategy-proofness* and the *rectangular property*.[4] We provide a new domain-richness condition, *box-shapedness*, and show that a social choice function satisfies *strategy-proofness* and the *rectangular property* in a box-shaped domain if and only if the function is constant. Therefore, we can conclude that in our economies only constant social choice functions can be securely implemented. This implies that all non-trivial strategy-proof mechanisms always have "bad" Nash equilibria and may prove to be ineffective.[5]

This paper is organized as follows: Section 2 defines the basic notions as well as the concept of secure implementation. Section 3 provides the primary result. Section 4 concludes the discussion. In the Appendix, we discuss the independence of the two properties that we survey in this paper.

2 Model

2.1 Basic notion

Let $I \equiv \{1, 2, \ldots, n\}$ be a finite set of *agents*. There are finite types of indivisible objects $T \equiv \{1, 2, \ldots, s\}$ and a divisible good which is called "money." For each type $t \in T$, let $\alpha_t \geq 1$ be the integer number of type-t indivisible objects. We assume that the total number of indivisible objects is at least as great as the number of agents, i.e., $\sum_{t \in T} \alpha_t \geq n$. However, this assumption allows us to deal with the case where the number of indivisible objects to be less than the number of agents. We interpret that agents who is not assigned a "real" object instead receives a "null" object. We assume that each agent consumes precisely one indivisible object and an amount of money.

For each $i \in I$ and each $t \in T$, agent i's *valuation* to a type-t object is given by $v_i(t) \in \mathbb{R}$. Let $\emptyset \neq V_i(t) \subseteq \mathbb{R}$ be a non-empty set of i's valuation to t. Let $v_i \equiv$

in our economies.

[4]Mizukami and Wakayama (2006) provide an alternative characterization of secure implementation.

[5]Bochet and Sakai (2006) study secure implementation in allotment economies with single-peaked preferences. They show that equal division is the unique symmetric and securely implementable social choice function.

$(v_i(t))_{t\in T} \in \mathbb{R}^T$ be a vector of i's valuations over types, and let $V_i \equiv \prod_{t\in T} V_i(t) \subseteq \mathbb{R}^T$ be the set of such vectors. If agent i consumes a type-t object and his monetary transfers is $m_i \in \mathbb{R}$, then his quasi-linear utility is given by

$$u(t, m_i; v_i) \equiv v_i(t) + m_i.$$

Let $v \equiv (v_1, v_2, \ldots, v_n)$ be a profile of valuation vectors of agents. A *domain* is the set of profiles of valuation vectors of agents and is denoted by $V \equiv \prod_{i\in I} V_i$. We often denote $I \setminus \{i\}$ by "$-i$," and $I \setminus \{i, j\}$ by "$-ij$." With this notation, we have

$$\begin{aligned}(v_i', v_{-i}) &\equiv (v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_n);\\ (v_i', v_j', v_{-ij}) &\equiv (v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_{j-1}, v_j', v_{j+1}, \ldots, v_n).\end{aligned}$$

An *assignment vector* is $\sigma \equiv (\sigma_1, \sigma_2, \ldots, \sigma_n) \in T^I$ such that

$$|\{i \in I : \sigma_i = t\}| \leq \alpha_t \text{ for each } t \in T.^6$$

Given $i \in I$, $\sigma_i = t$ means that i receives a type-t object. A *monetary transfer vector* is $m \equiv (m_1, m_2, \ldots, m_n) \in \mathbb{R}^I$. Note that we do not require any feasibility constraint on monetary transfers. If we establish an impossibility result, this strengthens the result. Given $i \in I$, let $m_i \geq 0$ (resp. $m_i < 0$) be the amount of money he is paid (resp. pays). An *allocation* is a pair of an assignment vector and a monetary transfer vector, $x \equiv (\sigma, m)$. Let $(x_i)_{i\in I} \equiv (\sigma_i, m_i)_{i\in I}$. Let X be the set of all allocations.

A *social choice function* on a domain V is a function $f\colon V \to X$ associating with each profile of valuation vectors $v \in V$ an allocation $f(v) = (f_i(v))_{i\in I} \in X$.

2.2 Secure implementation

A social choice function is *securely implementable* if there exists a mechanism that simultaneously implements the social choice function in dominant strategy equilibria and Nash equilibria. In this paper, we would like to characterize the class of secure implementable social choice functions in our economies. Saijo, Sjöström, and Yamato (2006) give a characterization of the class in the general model; a social choice function is securely implementable if and only if it satisfies *strategy-proofness* and the *rectangular property*.

[6] Given a set A, we denote the cardinality of A by $|A|$.

***Strategy-proofness*:** For each $v \in V$, each $i \in I$, and each $v'_i \in V_i$, $u(f_i(v); v_i) \geq u(f_i(v'_i, v_{-i}); v_i)$.

***Rectangular property*:** For each $v, v' \in V$, if $u(f_i(v'_i, v'_{-i}); v_i) = u(f_i(v_i, v'_{-i}); v_i)$ for each $i \in I$, then $f(v') = f(v)$.

3 Theorem

Before establishing our results, we introduce a new domain-richness condition. A domain V is *box-shaped* if for each $i \in I$, each $t \in T$, and each $v_i(t), v'_i(t) \in V_i(t)$ with $v_i(t) \leq v'_i(t)$, $v''_i(t) \in V_i(t)$ for all $v''_i(t) \in [v_i(t), v'_i(t)]$. That is, if a domain V is box-shaped, $V_i(t)$ takes the form of a certain interval for each $i \in N$ and each $t \in T$. In this paper, we restrict our attention to the box-shaped set of valuations.

Remark 1. Given $t \in T$, if $V_i(t) = \{0\}$ for each $i \in I$, i.e., no agent gain any utility from the type-t object, then type-t objects can be interpreted as "null." Our study includes the case where some types are "null" objects, since the singleton is an interval.

The next proposition characterizes the class of social choice functions satisfying *strategy-proofness* and the *rectangular property*. The result suggests that the two property together imply constancy. Therefore, *no* non-constant social choice function can be securely implementable.

Proposition. Suppose that V is a box-shaped domain. Then, a social choice function f on V satisfies *strategy-proofness* and the *rectangular property* if and only if it is constant.

Proof. Since the "if" part is obvious, we only show the "only if " part. Suppose that V is a box-shaped domain, and that a social choice function f satisfies *strategy-proofness* and the *rectangular property*. In what follows, we will prove the following three claims.

Claim 1. For each $\boldsymbol{i \in I}$, each $\boldsymbol{v_{-i} \in V_{-i}}$, and each $\boldsymbol{v_i, v'_i \in V_i}$, letting $\boldsymbol{f_i(v_i, v_{-i}) = (\sigma_i, m_i)}$ and $\boldsymbol{f_i(v'_i, v_{-i}) = (\sigma'_i, m'_i)}$, if $\boldsymbol{\sigma_i = \sigma'_i}$, then $\boldsymbol{m_i = m'_i}$: Suppose, by contradiction, that there exist $i \in N$, $v_{-i} \in V_{-i}$, and $v_i, v'_i \in V_i$ such that $\sigma_i = \sigma'_i$ and $m_i \neq m'_i$, where $f_i(v_i, v_{-i}) = (\sigma_i, m_i)$ and $f_i(v'_i, v_{-i}) = (\sigma'_i, m'_i)$. Without loss of generality, $m_i > m'_i$. Then agent i with the valuation vector v'_i can gain by announcing the false valuation v_i. This contradicts *strategy-proofness*.

Claim 2. For each $i \in I$, and each $v_{-i} \in V_{-i}$, $f_i(v_i, v_{-i}) = f_i(v_i', v_{-i})$ for each $v_i, v_i' \in V_i$: Suppose, by contradiction, that there exist $i \in I$, $v_{-i} \in V_{-i}$, and $v_i', v_i'' \in V_i$ such that $f_i(v_i', v_{-i}) \neq f_i(v_i'', v_{-i})$.

By Claim 1, given the agents except for agent i announce v_{-i}, agent i's consumption bundle depends on the type of the indivisible object that he receives. Let (t, m_i^t) be his consumption bundle when he is assigned to type-t indivisible object. Let

$$O_i(v_{-i}) \equiv \{t \in T : \exists v_i \in V_i \text{ such that } f_i(v_i, v_{-i}) = (t, m_i^t)\},$$

and $f_i(v_i', v_{-i}) = (t', m_i^{t'})$ and $f_i(v_i'', v_{-i}) = (t'', m_i^{t''})$. Since $f_i(v_i', v_{-i}) \neq f_i(v_i'', v_{-i})$, by Claim 1, $t' \neq t''$. *Strategy-proofness* implies that

$$\begin{aligned} v_i'(t') + m_i^{t'} &\geq v_i'(t) + m_i^t \quad \text{for each } t \in O_i(v_{-i}), \\ v_i''(t'') + m_i^{t''} &\geq v_i''(t) + m_i^t \quad \text{for each } t \in O_i(v_{-i}). \end{aligned} \tag{1}$$

Without loss of generality, we assume that $v_i'(t') + m_i^{t'} \geq v_i''(t'') + m_i^{t''}$. Then, (1) implies that $v_i'(t') + m_i^{t'} \geq v_i''(t'') + m_i^{t''} \geq v_i''(t') + m_i^{t'}$, that is, $v_i'(t') \geq v_i''(t')$. Thus, since V is box-shaped, there exists $v_i^* \in V_i$ such that $v_i^*(t') + m_i^{t'} = v_i''(t'') + m_i^{t''}$, and $v_i^*(t) = v_i''(t)$ for each $t \in T \setminus \{t'\}$. Let $f_i(v_i^*, v_{-i}) = (t^*, m_i^{t^*})$. *Strategy-proofness* implies that

$$v_i^*(t^*) + m_i^{t^*} \geq v_i^*(t) + m_i^t \quad \text{for each } t \in O_i(v_{-i}). \tag{2}$$

By (1) and the definition of v_i^*,

$$v_i^*(t'') + m_i^{t''} = v_i''(t'') + m_i^{t''} \geq v_i''(t^*) + m_i^{t^*}. \tag{3}$$

Thus, (2) and (3) imply

$$v_i^*(t^*) + m_i^{t^*} \geq v_i^*(t'') + m_i^{t''} \geq v_i''(t^*) + m_i^{t^*}.$$

Hence, by the definition of v_i^*, we obtain $v_i^*(t') + m_i^{t'} = v_i^*(t'') + m_i^{t''} = v_i^*(t^*) + m_i^{t^*}$.

Case 1. $t^* = t'$: Consider $(v_i^*, v_{-i}), (v_i'', v_{-i}) \in V$. Since

$$u(f_i(v_i'', v_{-i}); v_i^*) = v_i^*(t'') + m_i^{t''} = v_i^*(t^*) + m_i^{t^*} = u(f_i(v_i^*, v_{-i}); v_i^*),$$

and $u(f_j(v_j, v_i'', v_{-ij}); v_j) = u(f_j(v_j, v_i'', v_{-ij}); v_j)$ for each $j \in I \setminus \{i\}$, by the *rect-*

angular property, $f(v_i^*, v_{-i}) = f(v_i'', v_{-i})$. Since $t^* = t'$, this is a contradiction to $t' \neq t''$.

Case 2. $\boldsymbol{t^* \neq t'}$**:** Consider $(v_i^*, v_{-i}), (v_i', v_{-i}) \in V$. Since

$$u(f_i(v_i', v_{-i}); v_i^*) = v_i^*(t') + m_i^{t'} = v_i^*(t^*) + m_i^{t^*} = u(f_i(v_i^*, v_{-i}); v_i^*),$$

and $u(f_j(v_j, v_i', v_{-ij}); v_j) = u(f_j(v_j, v_i', v_{-ij}); v_j)$ for each $j \in I \backslash \{i\}$, by the *rectangular property*, $f(v_i^*, v_{-i}) = f(v_i', v_{-i})$. This is a contradiction to $t^* \neq t'$.

Claim 3. $\boldsymbol{f(v) = f(v')}$ **for each** $\boldsymbol{v, v' \in V}$**:** Let $v, v' \in V$. By Claim 2, $f_i(v_i', v_{-i}') = f_i(v_i, v_{-i}')$ for each $i \in I$. Therefore, $u(f_i(v_i', v_{-i}'); v_i) = u(f_i(v_i, v_{-i}'); v_i)$ for each $i \in I$. By the *rectangular property*, we can conclude $f(v) = f(v')$. □

Remark 2. Our impossibility result relies on the domain to be box-shaped on which social choice functions are required to satisfy *strategy-proofness* and the *rectangular property*. In fact, unless the domain is box-shaped, there may exist a non-constant social choice function satisfying the two properties. Consider the following example: Let $I = \{1, 2\}$ and $T = \{1, 2\}$. For each $i \in I$, let $V_i(2) = \{0\}$. Let $V_1(1) = \mathbb{R} \setminus \{1\}$ and $V_2(1) = \mathbb{R}$. Let f be a social choice function such that for each $v \in V$,

$$f(v) = \begin{cases} ((1,0),(2,0)) & \text{if } v_1(1) > 1 \\ ((2,1),(1,-1)) & \text{otherwise.} \end{cases}$$

Then, the social choice function f satisfies both *strategy-proofness* and the *rectangular property*.

By Saijo, Sjöström, and Yamato (2006) and our Proposition, we can obtain the following theorem.

Theorem. Suppose that V is a box-shaped domain. Then, a social choice function f on V is securely implementable if and only if it is constant.

4 Conclusion

In this study, we applied the notion of secure implementation to the problem of allocating indivisible objects with monetary transfers. We then established that only constant social choice functions can be securely implemented in a box-shaped domain. This negative result suggests that all non-trivial strategy-proof social choice functions do not work well in our environment.

Appendix

We will check the independence of *strategy-proofness* and the *rectangular property*. In what follows, we exhibit a social choice function that does not satisfy either *strategy-proofness* or the *rectangular property*. For convenience, let $f(v) = (f_i(v))_{i \in I} = (\sigma_i(v), m_i(v))_{i \in I}$ for each $v \in V$.

Example 1 (Dropping *strategy-proofness*). Let f be a social choice function such that σ is constant, and for each $v \in V$,

$$\begin{aligned} m_1(v) &= v_1 \\ m_i(v) &= -\frac{v_1}{n-1} \quad \text{for each } i \neq 1. \end{aligned}$$

Then, the social choice function satisfies the *rectangular property*, but it does not satisfy *strategy-proofness*.

Example 2 (Dropping the *rectangular property*). Let f be a social choice function such that σ is constant, and for each $v \in V$,

$$\begin{aligned} m_1(v) &= v_2 \\ m_2(v) &= v_3 \\ &\vdots \\ m_{i-1}(v) &= v_i \\ m_i(v) &= v_{i+1} \\ m_{i+1}(v) &= v_{i+2} \\ &\vdots \\ m_n(v) &= v_1. \end{aligned}$$

Then, the social choice function satisfies *strategy-proofness*, but it does not satisfy the *rectangular property*.

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