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Estimation of the Local Volatility of Discount Bonds Using the Market Prices of Coupon Bond Options

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Abstract. The key issue to price derivatives written on a coupon bond is the volatility structure, such as volatility smiles and skews, of the corresponding discount bonds. This paper proposes a method based on Dupire (1994) to estimate the local volatility of discount bonds when only the prices of coupon bond options are observed in the market. Numerical examples show that our method can construct the local volatility structure consistent with the market data.

Keywords: Volatility estimation, coupon bond options, Dupire's method, local volatility, volatility smile, volatility skew.

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1 Introduction

This paper proposes a method based on Dupire (1994) to estimate the local volatility of discount bonds, when only the prices of coupon bond options are observed in the market. Note that, in order to price derivatives traded in the bond market correctly, we need to model the dynamics of the discount bond prices, in particular the volatility structure, to be consistent with the observed market prices.

It is well known that the volatility smiles and skews are observed not only in the stock options markets but also in the bond options market.[1] Furthermore, in the Japanese bond market, implied volatility curves become very steep with respect to the maturity after the introduction of the so-called "Zero-Interest Rate Policy" by the Bank of Japan.[2] Therefore, it is important to develop models that capture such volatility structures from both theoretical and practical points of view.

The volatility models that deal with the volatility smiles or skews are roughly categorized into three groups. The first group is to use stochastic volatility models. In this approach, the volatility is typically assumed to follow a mean reverting diffusion process. Depending on the parameters of the processes of the underlying asset and the volatility, in particular on the correlation between them, a variety of volatility structure can be generated through this approach. See, e.g., Hull and White (1987) and Heston (1993).

The second group is to add jumps to the underlying asset process. This approach was originally proposed by Merton (1976) and is known as jump-diffusion models. By modeling the jump intensity and the jump size distribution appropriately, jump-diffusion models can generate the volatility smiles. See, e.g., Kou (2003) and references therein for details of jump-diffusion models.

The third group is known as local volatility models. In this approach, the volatility is supposed to be a deterministic function of time to maturity and the price of the underlying asset. This approach was first proposed by Dupire (1994) and becomes popular for stock market practitioners because of its simplicity. Rubinstein (1994) as well as Derman and Kani (1994) provided a binomial tree model that capture the local volatility effect to be consistent with observed market data. Li (2000) extended the binomial tree model and proposed a new algorithm to build the extended tree.

Our method is based on the third approach. As Li (2000) noted, the advantages of this

[1] Rubinstein (1994) pointed out that the effect from the volatility smiles and skews becomes more significant after the October 1987 crash in the US market.

[2] The Bank of Japan quit the Zero-Interest Rate Policy on July 14, 2006. However, the short rate still stays in the lowest range.

approach are: (1) it is a preference-free approach, (2) all contingent claims are priced based on the *single* model consistent with the market data, and (3) no assumption is made on the form of the local volatility function.

In the Japanese Government Bond (JGB) market, only options of coupon bond futures with short maturities are actively traded. There is no market quotation for discount bond options, whereas coupon bond options with longer maturities are rarely traded. Therefore, we want to construct a model that requires only a few parameters to be estimated; but still consistent with the market data. For such a market, the local volatility model can be a useful tool to estimate the volatility structure of discount bonds.

The method we propose in this paper consists of two parts, estimation of the local volatility function of coupon bonds using small samples, and estimation of the local volatility function of discount bonds using the volatility structure of coupon bonds.

As to the first part, the local volatility function of coupon bonds can be determined by Dupire's method (1994) if the prices of coupon bond options with all maturities and all strike prices are observed in the market. In the actual JGB market, however, only a few options are traded. To avoid this unrealistic requirement, we first estimate the implied volatility function from the observed option prices and then, based on Black's formula (1976), calculate the prices of coupon bond options for all maturities and strike prices consistent with the observed implied volatility. The local volatility function of coupon bonds can then be determined.

For the second part, we assume that the volatility functions of coupon bonds and discount bonds differ each other only by their multiplicative factors. This assumption may be justified when the underlying term structure assumes a single-factor model. With the volatility structure of discount bonds at hand, we can calculate the theoretical prices of coupon bond options by Monte Carlo simulation. The multiplicative factors are determined so that the squared errors between the theoretical option prices and the market prices are minimized.

This paper is organized as follows. In the next section, we review Dupire's local volatility model to be applied for the estimation of volatility structure of discount bonds. The forward-neutral method is a key tool for this purpose. Section 3 states the method to convert the volatility function of coupon bonds to that of discount bonds. The performance of the method is verified by a simulation study. We then apply our method for the options in the JGB market in Section 4. The numerical results show that our method works reasonably well even for the actual market. Section 5 concludes the paper. The calculation of volatility function by Dupire's method (1994) is provided in Appendix A for the reader's convenience, while Appendix B calculates the volatility function of coupon bonds when the implied volatility function is given.

2 The Local Volatility Model

Dupire (1994) considered the pricing problem of *stock* options when the risk-free interest rate is constant. That is, let us denote the time t price of a risky asset by $S(t)$, and suppose that $S(t)$ follows the following stochastic differential equation (SDE for short) under the risk-neutral measure Q:

$$\frac{\mathrm{d}S(t)}{S(t)} = r\mathrm{d}t + \sigma \mathrm{d}W_t^Q, \tag{2.1}$$

where r is the risk-free interest rate, σ is the volatility, and W_t^Q denotes the standard Brownian motion under Q.

When both r and σ are constant, the prices of European call options (as well as put options) are given by the Black-Scholes formula (1976). However, it is well known that the implied volatilities calculated from the Black-Scholes formula exhibit the volatility smiles or skews in practice, meaning that the assumption that σ is constant may fail in the actual market. Hence, as Dupire (1993) assumed, it is natural to consider the model in which the volatility is a function of time t and the underlying asset price S. That is, suppose that $S(t)$ follows the SDE

$$\frac{\mathrm{d}S(t)}{S(t)} = r\mathrm{d}t + \sigma(t, S(t))\mathrm{d}W_t^Q, \tag{2.2}$$

rather than (2.1), under the risk-neutral measure Q.

Now consider European put options and suppose that options with all maturities and all strike prices are traded in the market. Denoting the put option price with maturity T and strike price K by $p(T, K)$, Dupire (1994) showed under the differentiability assumption that the volatility function $\sigma(t, S)$ must be given by

$$\sigma^2(T, K) = \frac{2}{K^2 \frac{\partial^2 p(T,K)}{\partial K^2}} \left(\frac{\partial p(T, K)}{\partial T} + rK \frac{\partial p(T, K)}{\partial K} \right) \tag{2.3}$$

in order for the model (2.2) to be consistent with the market prices $p(T, K)$. The derivation of (2.3) in a general setting is given in Appendix A for the reader's convenience.

Of course, not all put options are traded in the actual market. However, equation (2.3) suggests that at least we can obtain an approximated volatility function $\sigma(T, S)$ when a functional form of the put prices with respect to T and K is estimated. This is the reason why Dupire's method is so popular for stock market practitioners.

Next, consider the pricing problem of *bond* options. Then, it is necessary to consider a stochastic interest rate model. A prominent tool in this setting is the forward-neutral method.

Denote the time t price of a non-defaultable (government) discount bond with maturity T by $v(t,T)$. For $\tau \geq T$, the T-forward price of the discount bond $v(t,\tau)$ at time t is given by

$$v_T(t,\tau) := \frac{v(t,\tau)}{v(t,T)}, \qquad t < T \leq \tau. \tag{2.4}$$

It is well known that the T-forward price $v_T(t,\tau)$ is a martingale under the T-forward measure Q^T. That is, for some volatility process $\sigma(t)$, the SDE for the T-forward price $v_T(t,\tau)$ is given by

$$\frac{\mathrm{d}v_T(t,\tau)}{v_T(t,\tau)} = \sigma(t)\mathrm{d}W_t^T, \tag{2.5}$$

where W_t^T denotes the standard Brownian motion under Q^T. Moreover, the put option price with maturity T and strike price K written on the discount bond $v(t,\tau)$ is given by

$$p(T,K) = v(0,T)\mathrm{E}^T[\{K - v(T,\tau)\}^+],$$

where the current time is 0, $\{x\}^+ = \max\{x,0\}$ and E^T is the expectation operator under the T-forward measure. Since $v(T,\tau) = v_T(T,\tau)$, we obtain

$$p^T(T,K) := \frac{p(T,K)}{v(0,T)} = \mathrm{E}^T[\{K - v_T(T,\tau)\}^+], \tag{2.6}$$

where $p^T(T,K)$ denotes the current T-forward price of the put option. See, e.g., Kijima (2002) for details of the forward-neutral method.

In order to obtain the local volatility function of discount bonds, we follow the idea of Dupire (1994). That is, consider the SDE (2.5) with a deterministic volatility function $\sigma(t)$ in time t. When the Vasicek model (1977) is assumed for the risk-free interest rate $r(t)$, the volatility function $\sigma(t)$ is given by[3]

$$\sigma(t) = \sigma_T(t,\tau) := \frac{\sigma}{a}\left(\mathrm{e}^{-a(\tau-t)} - \mathrm{e}^{-a(T-t)}\right), \qquad t < T \leq \tau. \tag{2.7}$$

In this setting, the current T-forward prices of European options (call and put options) are given by Black's formula (1976) with constant volatility ν where

$$\nu^2 = \int_0^T \sigma_T^2(s,\tau)\mathrm{d}s.$$

However, again, the implied volatilities calculated from Black's formula exhibit the volatility smiles or skews in practice.

[3] We take the Vasicek model as the basis of the local volatility model, since it seems the simplest model in the stochastic interest rate setting. Of course, it is possible to assume other volatility functions as the basis. Note that the Hull–White model (1990) produces the same volatility function, since the mean-reverting level does not contribute to derivative prices. Also, recall that the Hull–White model is a special case of the HJM model (1992) with constant volatility structure. See Inui and Kijima (1998) for details.

Suppose that the volatility is not only a function of time t but also a function of the underlying asset $v_T(t,\tau)$. It is assumed that the T-forward price follows the SDE

$$\frac{\mathrm{d}v_T(t,\tau)}{v_T(t,\tau)} = \sigma(t, v_T(t,\tau))\mathrm{d}W_t^T, \qquad t \le T, \tag{2.8}$$

under the T-forward measure Q^T. The current T-forward price of put option with maturity T and strike price K is given by (2.6). Hence, following the idea of Dupire (1994) presented in Appendix A, we obtain the local volatility function of the T-forward price $v_T(t,\tau)$ as

$$\sigma^2(T,K) = \frac{2\frac{\partial p^T(T,K)}{\partial K}}{K^2 \frac{\partial^2 p^T(T,K)}{\partial K^2}} \tag{2.9}$$

in order for the model (2.8) to be consistent with the market prices $p^T(T,K)$. However, this formula is valid only when discount bond options with all maturities and strike prices are actively traded in the market.

3 Main Results

In this section, we consider the case that only coupon bond options are traded in the market. For this purpose, we first obtain the volatility function of coupon bonds, assuming that the prices of coupon bond options with all maturities and all strike prices are observed in the market. The volatility function is then converted to that of discount bonds under some assumptions on the relationship between them. For notational simplicity, we denote in what follows the T-forward price $v_T(t,T_i)$ by $v_i(t)$, where $T < T_1 < T_2 < \cdots < T_N$ for some $N \ge 1$.

Suppose, rather than (2.8), that the T-forward price $v_i(t)$ follows the SDE

$$\frac{\mathrm{d}v_i(t)}{v_i(t)} = \sigma_T(t,T_i)\eta_i(t, v_i(t))\mathrm{d}W^T(t) \tag{3.1}$$

under the T-forward measure Q^T, where $\sigma_T(t,\tau)$ is given by (2.7). Note that the volatility structure for the T-forward price $v_T(t,\tau)$ may depend on the term structure of the maturity τ.[4] In order to take this effect into account, we leave $\sigma_T(t,\tau)$ in the SDE (3.1) and determine the volatility function $\eta_i(t,v)$ from the market data; see the assumption (3.7).

[4] Of course, the dependence on τ appears implicit in the formula (2.9).

3.1 Volatility function of coupon bonds

Suppose that a coupon bond pays constant cash flows C_i at T_i, $i = 1, \cdots, N$. The time t price of the coupon bond is then given by

$$\Theta(t, T_N) = \sum_{i=1}^{N} C_i v(t, T_i), \qquad t < T.$$

The T-forward price of the coupon bond at time t is expressed as

$$\Theta(t) := \frac{\Theta(t, T_N)}{v(t, T)} = \sum_{i=1}^{N} C_i v_i(t). \tag{3.2}$$

Note that practitioners use Black's formula (1976) for the pricing of coupon bond options. This means that they assume the SDE (2.5) with a deterministic volatility function $\sigma(t)$ for the T-forward price $\Theta(t)$. Hence, it is natural to assume the following SDE for $\Theta(t)$:

$$\frac{\mathrm{d}\Theta(t)}{\Theta(t)} = \sigma_B \xi(t, \Theta(t)) \mathrm{d}W^T(t), \qquad t \leq T, \tag{3.3}$$

where σ_B is a constant and $\xi(t, \Theta)$ is a deterministic function that characterizes the local volatility of the T-forward price of coupon bonds. We leave the constant σ_B in (3.3) to match the implied volatility function (4.1) defined later.

Consider plain options (call or put options) written on the coupon bond. We denote the put option price with maturity T and strike price K by $p(T, K)$. Let $q(T, K) = p(T, K)/v(0, T)$ denote the current T-forward price of the put option. Since

$$q(T, K) = \mathrm{E}^T[\{K - \Theta(T)\}^+] \tag{3.4}$$

according to the forward-neutral method, it is readily seen from Appendix A that $q(T, K)$ satisfies the partial differential equation (PDE for short)

$$\frac{\partial q(T, K)}{\partial T} = \frac{1}{2} K^2 \sigma_B^2 \xi^2(T, K) \frac{\partial^2 q(T, K)}{\partial K^2}. \tag{3.5}$$

It follows that

$$\xi^2(T, K) = \frac{2 \frac{\partial q(T,K)}{\partial T}}{\sigma_B^2 K^2 \frac{\partial^2 q(T,K)}{\partial K^2}}. \tag{3.6}$$

Appendix B provides the calculation of the volatility function $\xi(T, K)$ when an implied volatility function is given.

Consider next call options written on the coupon bond. The price of the call option with maturity T and strike price K is denoted by $c(T, K)$. Let $c^T(T, K) = c(T, K)/v(0, T)$. Since

$$c^T(T, K) = \mathrm{E}^T[\{\Theta(T) - K\}^+] = q(T, K) + \Theta(0) - K$$

due to the put-call parity, the T-forward price $c^T(T, K)$ also satisfies the PDE (3.5). Hence, in either cases, we can derive the functional form of $\xi(T, K)$ from the observed prices of coupon bond (put and call) options using the equation (3.6).

3.2 Volatility function of discount bonds

To link the volatility function $\xi(t, \Theta)$ of the coupon bond to the volatility functions $\eta_i(t, v)$ of the discount bonds, we assume that they differ each other only by the multiplicative factors. That is, suppose that

$$\eta_i(t, x) = \alpha_i \xi(t, x), \qquad i = 1, \ldots, N, \tag{3.7}$$

where α_i are some constants to be determined. This assumption may be plausible since discount bond prices are perfectly correlated in the single-factor term structure model.[5] When the coupon size is small compared to the principal, the dominant term of the coupon bond price $\Theta(t, T_N)$ is the discount bond price $v(t, T_N)$.

We determine the unknown factors α_i as follows. First, note that the functional form of $\xi(t, \Theta)$ is estimated by (3.6). Hence, assuming some values for α_i, we can generate the sample paths of $v_i(t)$ using (3.1) and (3.7). Using these sample values, we can determine sample values of $\Theta(T)$ from (3.2). Hence, theoretical option values can be obtained, based on (3.4), by the simple Monte Carlo method. The unknown parameters α_i are determined so that the squared errors between the theoretical values and the observed option prices are minimized.

More specifically, let N_{sim} be the number of samples in the Monte Carlo simulation. Given the parameter set $\alpha = \{\alpha_i\}$, let $\{v_i^k(T;\alpha)\}_{k=1}^{N_{sim}}$ be the set of the T-forward prices of the discount bond with maturity T_i. Each value $v_i^k(T;\alpha)$ is generated by simulating the sample path $v_i^k(t)$ according to (3.1). Namely, we apply the Euler approximation for (3.1):

$$v_i(t_{n+1}) = v_i(t_n) \exp\left(-\frac{1}{2}\sigma_T^2(t_n, T_i)\alpha_i^2 \xi^2(t_n, v_i(t_n))\Delta t_n + \sigma_T(t_n, T_i)\alpha_i \xi(t_n, v_i(t_n))\sqrt{\Delta t_n}\epsilon_n\right),$$

where $\Delta t_n := t_{n+1} - t_n$, and ϵ_n are independent, identically distributed random variables that follow the standard normal distribution. Using the samples, the T-forward price of the coupon bond with maturity T_N is calculated as

$$\Theta^k(T;\alpha) = \sum_{i=1}^{N} C_i v_i^k(T;\alpha)$$

for the given parameter set $\alpha = \{\alpha_i\}$.

We compute $\{\alpha_i\}$ as

$$\{\alpha_i\}_{i=1}^N = \arg\min_{\{\alpha_i\}} \sum_{j=1}^{M_{op}} \left(q_{obs}(T_j, K_j) - q_{sim}(T_j, K_j)\right)^2, \tag{3.8}$$

[5] Empirical studies suggest the use of multi-factor term structure models (see, e.g., Rebonato (1996)). However, because we chose the local volatility model to explain the volatility smiles and skews observed in the bond options market, we decided to start with the single-factor Vasicek model. Development of multi-factor local volatility models will be our future research.

where

$$q_{sim}(T_j, K_j) = \frac{1}{N_{sim}} \sum_{k=1}^{N_{sim}} \{K_j - \Theta^k(T_j; \alpha)\}^+$$

are the theoretical option values obtained by simulation, and where $q_{obs}(T_j, K_j)$ are the option prices observed in the market, respectively.

3.3 Numerical experiments

In order to verify the applicability of our method, we perform the following simulation study. Namely, we assume that the current yield curve is flat at 2%. The underlying asset is the 10-year bond with annual coupons of 2%. The maturity of options are all 0.5 year ($T = 0.5$). We set $\sigma_B = 0.2$ in (3.3) and $a = \sigma = 0.01$ in (2.7). Furthermore, we assume that the volatility function for the coupon bonds is given by

$$\xi(K, T) = \exp\left(1 - \frac{K}{100}\right). \tag{3.9}$$

The volatility functions $\eta_i(t, x)$ for discount bonds are determined by (3.7).

In this setting, we first generate the T-forward price $\Theta(T)$ by the Monte Carlo simulation using (3.3) to obtain the theoretical prices of put options using (3.4). Note that this is possible since we assume the local volatility function (3.9) for the coupon bonds. In the actual calculation, we compute the put option values with strike prices 80, 81, 82, . . ., 100.

Next, we apply our method to compute the multiplicative factors $\{\alpha_i\}$ using (3.8). Table 1 lists the estimated multiplicative factors $\{\alpha_i\}$. Note that the factors are quite different but, in general, α_i is smaller for short maturity discount bonds than that for longer ones. This means that the investors are more sensitive to risks of discount bonds with shorter maturities.

(Table 1 is inserted here)

Table 2 shows the put option prices written on the coupon bond. The column "Sim" indicates that the prices are obtained by the Monte Carlo simulation using (3.4). The number of simulation runs is ??????, and we suppose that these option prices are true. On the other hand, the column "FKN" indicates that the prices are calculated by our method. The absolute differences between the theoretical prices and those obtained by our method are listed in the column "Diff". In this numerical experiments, the absolute errors are less than 0.06 and we believe that our method is quite useful to estimate the local volatility of discount bonds as far as the volatility function of coupon bonds is estimated correctly.

(Table 2 is inserted here)

4 Application to the JGB Market

In this section, we apply our method to estimate the local volatility functions of discount bonds for the actual market, namely for the JGB market. As explained in the introductory section, the JBG market has a number of special features that make it difficult to determine the accurate volatility model.[6] In the authors' best knowledge, this is the first paper that analyses the local volatility structure of the JGB market.

4.1 Market data

In the JGB market, options written on the 10-year bond futures are actively traded and the volatility skews are observed. We provide the market quotations of the put options with times to maturity 0.104 year and 0.274 year on the 10-year bond futures on the date May 23, 2006 in Table 3. For convenience, we express the options with time to maturity 0.104 year as "option 1" and the options with time to maturity 0.274 year as "option 2". In either case, a significant volatility skew is observed.

(Table 3 is inserted here)

The cheapest deliverable bond of the underlying 10-year bond futures in Table 3 is the JGB 10-year bond series #253 (JGB # 253 for short). Thus, the options written on the 10-year bond futures are considered as the options written on the JGB #253 by using the conversion factor. We provide the details of the JGB # 253 in Table 4.

(Table 4 is inserted here)

[6] The JGB yield curve exhibits the so-called S shape. Recently, many attempts have been made to construct term structure models that can capture the special shape of the yield curve. See Kabanov, Kijima and Rinaz (2005) and references therein for details.

4.2 Implied volatilities of the coupon bond options

In order to estimate the volatility function $\xi(t,\Theta)$ of the coupon bond, option prices for all strikes and all maturities are required. To avoid this unrealistic requirement, we first estimate the implied volatility function using the observed option prices and then, based on Black's formula (1976), calculate the prices of coupon bond options for all maturities and strike prices consistent with the observed implied volatility. The volatility function of coupon bonds can then be determined.

Suppose that the functional form of the implied volatility is given by

$$\sigma_{mkt}(T,K) = \sigma_B \exp\left(\sum_{n=1}^{M} a_n \left(\frac{K}{K_{ATM}(T)} - 1\right)^n\right), \tag{4.1}$$

where $K_{ATM}(T)$ is the *at-the-money* option strike with maturity T. We estimate the parameters σ_B, a_n by the least square fitting to the observed implied volatilities. Because we have only a few options traded in the JGB market, we take $M = 2$ to avoid the overfitting problem. That is, the number of parameters to be estimated is 3 for 12 data in option 1 and 8 data in option 2. The estimated parameters are listed in Table 5, while the fitting results are shown in Table 6. The volatility function of the coupon bond calculated from the the implied volatility model (4.1) is given in Appendix B.

(Tables 5 and 6 are inserted here)

4.3 Volatility function of discount bonds

Once we obtain the volatility function $\xi(t,\Theta)$ of the coupon bond, we can estimate the local volatility of discount bonds using the procedure presented in the previous section.

For the JGB #253, the number of coupon payments is 15 ($N = 15$). We use $a = 0.01508$ and $\sigma = 0.007$ for $\sigma_T(t,\tau)$ in (2.7). These values are obtained by fitting the swaption volatilities using the Hull–White model (1990). Moreover, we use the implied volatility parameters (a_1, a_2, σ_B) of option 1 for $t < 0.104$ and those of option 2 for $t > 0.104$.

The estimated multiplicative factors $\{\alpha_i\}$ are presented in Table 7. It is observed that, as in the previous case (see Table 2) where the yield curve is flat, the factor α_i is decreasing in i except $i = 1, 2$. In particular, the factor α_1 is very small, meaning that the discount bond with short maturity ($T_1 = 0.33$) has a very low volatility. Recall that the factor α_i describes the magnitude of the volatility of the discount bond with maturity T_i. This result is consistent with the actual JGB market where the short rate stays near zero under the "Zero-Interest Rate Policy" by the Bank of Japan.

(Table 7 is inserted here)

On the other hand, Table 8 presents the option prices calculated by the Monte Carlo simulation using the estimated volatility functions of discount bonds. It is observed that the absolute differences between the observed prices and the estimated prices are less than 0.9. When the relative differences are used, the fitness becomes better for option 2 (with a longer maturity), while it becomes poor for the out-of-the-money option 1. This is because we used the absolute error in option prices in the fitting procedure (3.7).

(Table 8 is inserted here)

5 Conclusion

In this paper, we proposed a methodology to estimate the local volatility of discount bonds, when only the prices of coupon bond options are observed in the market. Note that, in order to price derivatives traded in the bond market correctly, we need to model the dynamics of the discount bond prices, in particular the volatility structure, to be consistent with the observed market prices. Recent empirical studies reveal that the effect from the volatility smiles and skews becomes more significant in the bond options market than ever. It is therefore important to develop such models that capture the volatility structure from both theoretical and practical points of view.

Our method consists of two parts, estimation of the volatility function of coupon bonds using small samples, and estimation of the volatility function of discount bonds using the volatility structure of coupon bonds.

As to the first part, given the prices of coupon bond options, the volatility function of coupon bonds can be determined by Dupire's method (1994). For this purpose, we first estimate the implied volatility from the observed option prices and then, based on Black's formula (1976), calculate the prices of coupon bond options for all maturities and strikes consistent with the observed implied volatility.

For the second part, we assume the relationship (3.7) between the volatility functions of coupon bonds and discount bonds. This assumption may be justified when the underlying term structure assumes a single-factor model. However, it is well known that all bond price dynamics of different maturities are perfectly correlated. Thus, many empirical studies suggest the use of multi-factor term structure models in order to make the model more flexible. Development of multi-factor local volatility models will be our future research.

A Duprire's Local Volatility Model

As in Dupire (1993), suppose that $S(t)$ follows the following SDE under the risk-neutral measure Q:

$$\frac{\mathrm{d}S(t)}{S(t)} = (r-\delta)\mathrm{d}t + \sigma(t, S(t))\mathrm{d}W_t^Q,$$

where δ is an instantaneous dividend rate.

Let $\phi(T, S)$ be the density function of $S(T)$. From the *forward* Kolmogorov equation, we have

$$\frac{\partial \phi}{\partial T} + \frac{\partial}{\partial S}\{(r-\delta)S\phi(T,S)\} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\{\sigma^2(T,S)S^2\phi(T,S)\} = 0.$$

Twice integrating the above equation with respect to S yields

$$\frac{1}{2}\sigma^2(T,S)S^2\phi(T,S) = \frac{\partial}{\partial T}\int_0^S \int_0^v \phi(T,u)\mathrm{d}u\mathrm{d}v + (r-\delta)\int_0^S v\phi(T,v)\mathrm{d}v. \tag{A.1}$$

On the other hand, consider a European put option with maturity T and strike price K, and denote its time t price by $p(T, K)$. Then, it is readily seen that $p(T, K)$ can be expressed as

$$p(T,K) = \mathrm{e}^{-r(T-t)}\int_0^K (K-S)\phi(T,S)\mathrm{d}S \tag{A.2}$$

$$= \mathrm{e}^{-r(T-t)}\left(K\int_0^K \phi(T,S)\mathrm{d}S - \int_0^K S\phi(T,S)\mathrm{d}S\right). \tag{A.3}$$

By differentiating (A.2) with respect to K, we get

$$\frac{\partial p(T,K)}{\partial K} = \mathrm{e}^{-r(T-t)}\int_0^K \phi(T,u)\mathrm{d}u. \tag{A.4}$$

Therefore, from (A.3), we have an expression for $p(T, K)$ as

$$p(T,K) = K\frac{\partial p(T,K)}{\partial K} - \mathrm{e}^{-r(T-t)}\int_0^K S\phi(T,S)\mathrm{d}S,$$

or equivalently,

$$\int_0^K v\phi(T,v)\mathrm{d}v = \mathrm{e}^{r(T-t)}\left[K\frac{\partial p(T,K)}{\partial K} - p(T,K)\right]. \tag{A.5}$$

Once again, differentiating (A.4) with respect to K yields

$$\frac{\partial^2 p(T,K)}{\partial K^2} = \mathrm{e}^{-r(T-t)}\phi(T,K). \tag{A.6}$$

Equation (A.6) indicates that the price of the European put option satisfies

$$\int_0^K \int_0^v \phi(T,u)\mathrm{d}u\mathrm{d}v = \int_0^K \int_0^v \mathrm{e}^{r(T-t)}\frac{\partial^2 p(T,u)}{\partial u^2}\mathrm{d}u\mathrm{d}v = \mathrm{e}^{r(T-t)}p(T,K), \tag{A.7}$$

since $\dfrac{\partial p(T,u)}{\partial u} = p(T,u) = 0$ at $u = 0$.

Finally, setting $S = K$ and substituting (A.5)–(A.7) into (A.1), we have

$$\begin{aligned} & \frac{1}{2}\sigma^2(T,K)K^2 e^{r(T-t)} \frac{\partial^2 p(T,K)}{\partial K^2} \\ = \quad & e^{r(T-t)} \left[\frac{\partial p(T,K)}{\partial T} + rp(T,K) + (r-\delta) \left\{ K \frac{\partial p(T,K)}{\partial K} - p(T,K) \right\} \right]. \end{aligned} \tag{A.8}$$

Solving (A.8) with respect to the volatility function, we get

$$\sigma^2(T,K) = \frac{2\left[\dfrac{\partial p(T,K)}{\partial T} + \delta p(T,K) + (r-\delta)K \dfrac{\partial p(T,K)}{\partial K} \right]}{K^2 \dfrac{\partial^2 p(T,K)}{\partial K^2}}. \tag{A.9}$$

B Volatility Function of Coupon Bonds

In this appendix, we derive the volatility function of the coupon bond $\xi(T, \Phi)$ when we assume the implied volatility function (4.1). We only demonstrate the calculation for the case of put options, since the calculation for call options is similar.

Under the implied volatility model (4.1), it follows from (3.4) that the option price is expressed as

$$q(K,T) = KN(\epsilon_K) - \Theta(0)N(\epsilon_K - \sigma_{mkt}(K,T)\sqrt{T}),$$

where

$$\epsilon_K = \frac{\log K - \log \Theta(0)}{\sigma_{mkt}(K,T)\sqrt{T}} + \frac{1}{2}\sigma_{mkt}(K,T)\sqrt{T}.$$

Here, $N(x)$ denotes the cumulative distribution function of the standard normal distribution.

Using the above expression, the derivatives in the right-hand-side of (3.6) are calculated as

$$\begin{aligned} \frac{\partial q(K,T)}{\partial T} &= KN'(\epsilon_K)\frac{\partial \epsilon_K}{\partial T} - \frac{\partial \Theta(0)}{\partial T} N(\epsilon_K - \sigma_{mkt}(K,T)\sqrt{T}) \\ &- \Theta(0)N'(\epsilon_K - \sigma_{mkt}(K,T)\sqrt{T}) \\ &\quad \times \left(\frac{\partial \epsilon_K}{\partial T} - \frac{\partial \sigma_{mkt}(K,T)}{\partial T}\sqrt{T} - \frac{\sigma_{mkt}(K,T)}{2\sqrt{T}} \right) \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} \frac{\partial^2 q(K,T)}{\partial K^2} &= 2N'(\epsilon_K)\frac{\partial \epsilon_K}{\partial K} + KN''(\epsilon_K)\left(\frac{\partial \epsilon_K}{\partial K}\right)^2 + KN'(\epsilon_K)\frac{\partial^2 \epsilon_K}{\partial K^2} \\ &\quad -\Theta(0)N''(\epsilon_K - \sigma_{mkt}(K,T)\sqrt{T})\left(\frac{\partial \epsilon_K}{\partial K} - \frac{\partial \sigma_{mkt}(K,T)}{\partial K}\sqrt{T} \right)^2 \\ &\quad -\Theta(0)N'(\epsilon_K - \sigma_{mkt}(K,T)\sqrt{T})\left(\frac{\partial^2 \epsilon_K}{\partial K^2} - \frac{\partial^2 \sigma_{mkt}(K,T)}{\partial K^2}\sqrt{T} \right). \end{aligned} \tag{B.2}$$

Here, the first and second derivatives of $N(x)$ are given by

$$N'(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2}, \qquad N''(x) = \frac{-x}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2}$$

respectively.

The partial derivatives in (B.1) and (B.2) are expressed as

$$\begin{aligned} \frac{\partial \epsilon_K}{\partial T} &= -\frac{\frac{\partial \Theta(0)}{\partial T}}{\sigma_{mkt}(K,T)\sqrt{T}\Theta(0)} \\ &\quad -\frac{\log K - \log \Theta(0)}{\sigma_{mkt}^2(K,T)T} \left(\frac{\sigma_{mkt}(K,T)}{\partial T}\sqrt{T} + \frac{\sigma_{mkt}(K,T)}{2\sqrt{T}} \right) \\ &\quad + \frac{1}{2}\frac{\partial \sigma_{mkt}(K,T)}{\partial T}\sqrt{T} + \frac{\sigma_{mkt}(K,T)}{4\sqrt{T}}, \end{aligned} \tag{B.3}$$

$$\frac{\partial \epsilon_K}{\partial K} = \frac{1}{\sigma_{mkt}(K,T)K\sqrt{T}} - \left[\frac{\log K - \log \Theta(0)}{\sigma_{mkt}^2(K,T)\sqrt{T}} - \frac{\sqrt{T}}{2} \right] \frac{\partial \sigma_{mkt}(K,T)}{\partial K}, \tag{B.4}$$

$$\begin{aligned} \frac{\partial^2 \epsilon_K}{\partial K^2} &= \frac{-1}{\sigma_{mkt}^2(K,T)K^2\sqrt{T}} \left[\sigma_{mkt}(K,T) + 2K\frac{\partial \sigma_{mkt}(K,T)}{\partial K} \right. \\ &\quad \left. + K^2 \log\left(\frac{K}{\Theta(0)}\right) \frac{\partial^2 \sigma_{mkt}(K,T)}{\partial K^2} \right] \\ &\quad + 2\frac{\log K - \log \Theta(0)}{\sigma_{mkt}^3(K,T)\sqrt{T}} \left(\frac{\partial \sigma_{mkt}(K,T)}{\partial K} \right)^2 + \frac{1}{2}\frac{\partial^2 \sigma_{mkt}(K,T)}{\partial K^2}\sqrt{T} \end{aligned} \tag{B.5}$$

and

$$\frac{\partial \Theta(0)}{\partial T} = R(T)\Theta(0),$$

where $R(T) = -\dfrac{\log P(0,T)}{T}$ is the spot rate.

Finally, from (4.1), the derivatives appeared in (B.3)–(B.5) are calculated as

$$\begin{aligned} \frac{\partial \sigma_{mkt}(K,T)}{\partial T} &= -\sigma_{mkt}(K,T)\frac{K}{K_{ATM}^2(T)}\frac{\partial K_{ATM}(T)}{\partial T}\sum_{n=1}^{M} a_n n \left(\frac{K}{K_{ATM}(T)} - 1 \right)^{n-1}, \\ \frac{\partial \sigma_{mkt}(K,T)}{\partial K} &= \frac{\sigma_{mkt}(K,T)}{K_{ATM}(T)} \sum_{n=1}^{M} a_n n \left(\frac{K}{K_{ATM}(T)} - 1 \right)^{n-1}, \\ \frac{\partial^2 \sigma_{mkt}(K,T)}{\partial K^2} &= \frac{\sigma_{mkt}(K,T)}{K_{ATM}^2(T)} \left[\sum_{n=2}^{M} a_n n(n-1) \left(\frac{K}{K_{ATM}(T)} - 1 \right)^{n-2} \right. \\ &\quad \left. + \left(\sum_{n=1}^{M} a_n n \left(\frac{K}{K_{ATM}(T)} - 1 \right)^{n-1} \right)^2 \right], \end{aligned}$$

and

$$K_{ATM}(T) = \frac{\Theta(0,T_N)}{v(0,T)} = \Theta(0).$$

Thus, we can express $\xi(T,\Phi)$ by substituting (B.3)–(B.5) into (3.6).

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Table 1: Estimated multiplicative factors α_i. The factor α_i is in general smaller for discount bonds with shorter maturities than that for longer maturities.

α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
41.469	27.154	19.326	15.431	11.510	13.673	12.017	17.152	12.571	2.171

Table 2: Put option prices written on the coupon bond. The column "Sim" indicates the theoretical prices obtained by simulation, while the column "FKN" shows the prices calculated by our method. The absolute errors are listed in the column "Diff".

Strike	Sim	FKN	Diff	Strike	Sim	FKN	Diff
80	0.41	0.37	0.05	91	2.02	1.97	0.05
81	0.49	0.44	0.05	92	2.28	2.23	0.04
82	0.57	0.52	0.05	93	2.56	2.52	0.04
83	0.67	0.62	0.06	94	2.86	2.83	0.03
84	0.78	0.72	0.06	95	3.19	3.17	0.02
85	0.91	0.85	0.06	96	3.54	3.54	0.01
86	1.05	0.99	0.06	97	3.92	3.93	(0.01)
87	1.20	1.14	0.06	98	4.33	4.35	(0.02)
88	1.38	1.32	0.06	99	4.77	4.80	(0.03)
89	1.57	1.51	0.06	100	5.23	5.27	(0.04)
90	1.78	1.73	0.05				

Table 3: Market quotations of the put options written on the 10-year bond futures on the date May 23, 2006. The implied volatilities are presented in the column IV. Note the significant volatility skews.

option 1		option 2	
Time to Maturity: 0.104 year		Time to Maturity: 0.274 year	
Strike	IV	Strike	IV
127.5	6.45 %	127	5.13 %
128	6.22 %	128	4.92 %
128.5	6.04 %	129	4.89 %
129	5.23 %	130	4.80 %
129.5	5.05 %	131	4.62 %
130	4.67 %	132	4.41 %
130.5	4.74 %	133	3.98 %
131	4.58 %	134	3.83 %
131.5	4.13 %		
132	3.86 %		
132.5	3.56 %		
133	3.39 %		

Table 4: Details of JGB # 253 on the date May 23, 2006. This JGB is the cheapest deliverable bond, and so we can think that the bond options traded in the market are written on this bond.

Maturity date	September 20, 2013
Time to Maturity	7.33 year
Coupon rate	1.6 %
Conversion Factor	0.751486
Market price	99.459

Table 5: Estimated parameters of the implied volatility function.

	option 1	option 2
a_1	-17.19	-6.05
a_2	-25.25	-9.06
σ_B	0.036	0.042

Table 6: Fitting results of the estimated volatilities. The column "IV" means the observed implied volatility, while the column "Model" represents the volatility calculated from (4.1) using the parameter values listed in Table 5.

option 1			option 2		
Strike	IV	Model	Strike	IV	Model
127.5	6.45 %	6.54 %	127	5.13 %	5.27 %
128	6.22 %	6.17 %	128	4.92 %	5.06 %
128.5	6.04 %	5.82 %	129	4.89 %	4.85 %
129	5.23 %	5.48 %	130	4.80 %	4.65 %
129.5	5.05 %	5.16 %	131	4.62 %	4.45 %
130	4.67 %	4.85 %	132	4.41 %	4.25 %
130.5	4.74 %	4.56 %	133	3.98 %	4.06 %
131	4.58 %	4.28 %	134	3.83 %	3.88 %
131.5	4.13 %	4.02 %			
132	3.86 %	3.77 %			
132.5	3.56 %	3.53 %			
133	3.39 %	3.31 %			

Table 7: Estimated multiplicative factors α_i. The factor α_i is decreasing in i except $i = 1, 2$

i	T_i	α_i	i	T_i	α_i
1	0.33	0.100	9	4.33	1.352
2	0.83	0.336	10	4.83	1.226
3	1.33	2.511	11	5.33	1.135
4	1.83	2.459	12	5.83	1.052
5	2.33	2.256	13	6.33	0.975
6	2.83	1.950	14	6.83	0.915
7	3.33	1.689	15	7.33	0.852
8	3.83	1.496			

Table 8: Option prices calculated by the Monte Carlo simulation using the estimated volatility functions of discount bonds. It is observed that the absolute differences between the observed prices and the estimated prices are sufficiently small.

option 1	Time to Maturity: 0.104		option 2	Time to Maturity: 0.274	
Strike	Observation	Estimation	Strike	Observation	Estimation
127.5	0.0318	0.0402	127	0.0793	0.0780
128	0.0375	0.0479	128	0.1276	0.1215
128.5	0.0455	0.0644	129	0.2039	0.1894
129	0.0569	0.0843	130	0.3219	0.2915
129.5	0.0733	0.1066	131	0.4995	0.4485
130	0.0969	0.1366	132	0.7569	0.6882
130.5	0.1312	0.1795	133	1.1124	1.0299
131	0.1823	0.2374	134	1.5766	1.4953
131.5	0.2564	0.3182			
132	0.3622	0.4249			
132.5	0.5113	0.5796			
133	0.7131	0.7774			