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Masaaki KIJIMA and Yukio MUROMACHI

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An Extension of the Wang Transform Derived from Bühlmann's Economic Premium Principle for Insurance Risk *

Masaaki Kijima[a] and Yukio Muromachi[b]

[a] *Graduate School of Social Sciences, Tokyo Metropolitan University, and Daiwa Securities Chair, Graduate School of Economics, Kyoto University*

[b] *Financial Research Group, NLI Research Institute*

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Abstract. It is well known that the Wang transform (2002) for the pricing of financial and insurance risks is derived from Bühlmann's economic premium principle (1980). The transform is extended to the multivariate setting by Kijima (2006). This paper further extend the results to derive a class of probability transforms that are consistent with Bühlmann's pricing formula. The class of transforms is extended to the multivariate setting by using a Gaussian copula, while the multiperiod extension is also possible within the equilibrium pricing framework.

Keywords: Bühlmann's equilibrium price, Wang transform, Gaussian copula, non-central t distribution

*Send all correspondence to Yukio Muromachi, Financial Research Group, NLI Research Institute, 4-1-7 Kudankita, Chiyoda-ku, Tokyo 102-0073, Japan. TEL:+81-3-3512-1850, FAX:+81-3-5211-1082, Email: muromati@nli-research.co.jp.

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1 Introduction

In the actuarial literature, there have been developed many probability transforms for pricing financial and insurance risks. Such methods include the variance loading, the standard deviation loading, and the Esscher transform. Recently, Wang [14] proposed a universal pricing method based on the following transformation from $F(x)$ to $F^*(x)$:

$$F^*(x) = \Phi[\Phi^{-1}(F(x)) + \theta], \tag{1.1}$$

where Φ denotes the standard normal cumulative distribution function (CDF for short) and θ is a constant. The transform is now called the *Wang transform* and produces a risk-adjusted CDF $F^*(x)$. The mean value evaluated under $F^*(x)$ will define a risk-adjusted "fair value" of risk X with CDF $F(x)$ at some future time, which can be discounted to time zero using the risk-free interest rate. The parameter θ is considered to be a risk premium.

The Wang transform not only possesses various desirable properties as a pricing method, but also has a sound economic interpretation. For example, the Wang transform (as well as the Esscher transform) is the only distortion function, among the family of distortions, that can recover CAPM (the capital asset pricing model) for underlying assets and the Black-Scholes formula for options. See Wang [15] and Kijima [6] for details.

Among them, the most striking result on the transform (1.1) is that it is consistent with Bühlmann's economic premium principle. More precisely, Bühlmann [1] considered risk exchanges among a set of agents. Each agent is characterized by his/her exponential utility function $u_j(x) = -\mathrm{e}^{-\lambda_j x}$, $j = 1, 2, \ldots, n$, and faces a risk of potential loss X_j. In a pure risk exchange model, Bühlmann [1] derived the equilibrium pricing formula

$$\pi(X) = E[\eta X], \qquad \eta = \frac{\mathrm{e}^{-\lambda Z}}{E[\mathrm{e}^{-\lambda Z}]}, \tag{1.2}$$

where $Z = \sum_{j=1}^n X_j$ is the aggregate risk and λ is given by

$$\lambda^{-1} = \sum_{j=1}^{n} \lambda_j^{-1}, \qquad \lambda_j > 0.$$

The parameter λ is thought of the *risk aversion index* of the representative agent in the market. Wang [16] showed that the transform (1.1) can be derived from the equilibrium pricing formula (1.2) under some assumptions on the aggregate risk. The result is extended to the multivariate setting by Kijima [6].

In this paper, we derive a class of transforms that are consistent with Bühlmann's economic premium principle (1.2), thereby extending the results of Wang [15]. Namely, based on the idea of Kijima [6], we obtain the transform

$$F^*(x) = E[\Phi(G^{-1}(F(x))Y + \theta)], \tag{1.3}$$

where Y is *any* positive random variable and the expectation is taken with respect to Y. Here, $G(x)$ denotes the CDF of random variable U/Y and U represents a standard normal random variable, independent of Y. In particular, when $Y = 1$ almost surely, we have $G(x) = \Phi(x)$, so that the transform (1.3) is reduced to the Wang transform (1.1). The transform (1.3) can be extended to the multivariate setting by using a Gaussian copula, in order to preserve the linearity for the pricing functional.[1] A multiperiod extension is also possible within the equilibrium pricing framework.

It is often said that a drawback of the Wang transform (1.1) is the normal distributions, that never match the fat-tailness observed in the actual markets. In fact, some empirical studies suggest to use t distributions, whose CDF is denoted by $T_\nu(x)$, with $\nu = 3$ or 4 degree of freedom for return distributions of financial and insurance assets (see, e.g., Platen and Stahl [13]). Hence, it is natural to consider the case that $Y = \sqrt{\chi^2_\nu/\nu}$, where χ^2_ν denotes a chi-square random variable with ν degree of freedom. As Kijima and Muromachi [9] observed, this case leads to the two-parameter transformation

$$F^*(x) = P_{\nu:-\lambda}[T_\nu^{-1}(F(x))], \tag{1.4}$$

where $P_{\nu:\delta}$ denotes the CDF of non-central t distribution with ν degree of freedom and non-centrality parameter δ. However, contrary to our intuition, the risk-adjusted distribution (1.4) derived from t distributions is not fatter in tail parts than the original Wang transform (1.1) that is derived from normal distributions.

The present paper is organized as follows. In the next section, we review the result of Wang [16] and consider an alternative (and simple) derivation of the Wang transform (1.1). Using the idea presented in Section 2, the general transform (1.3) is derived in Section 3. Of interest is the comparison of tail distributions derived by the general transform. It is shown in Section 4 that the original Wang transform (1.1) produces the fattest tail distribution among the class of transforms (1.3) that are derived from Bühlmann's economic premium principle (1.2). While the result is further extended to a multivariate setting in Section 5, Section 6 states a multiperiod extension of our result within the equilibrium pricing framework. Section 7 concludes the paper.

[1] The pricing functional π is said to be linear if $\pi(aX + bY) = a\pi(X) + b\pi(Y)$ for all risks X, Y and constants a, b. If it is not linear, arbitrage opportunities are not precluded. See, e.g., Harrison and Kreps [2] and Kijima [5] for details.

2 A pure risk exchange economy

Bühlmann [1] considered a single-period economy for risk exchanges among a set of agents $j = 1, 2, \ldots, n$. Each agent is characterized by an exponential utility function $u_j(x) = -\mathrm{e}^{-\lambda_j x}$, $x \geq 0$, and initial wealth w_j. Suppose that agent j faces a risk of potential loss X_j and is willing to buy/sell a risk exchange Y_j. If agent j is an insurance company, the risk exchange Y_j is thought of the sum of all insurance policies sold by j. While the original risk X_j belongs to agent j, the risk exchange Y_j can be freely bought/sold by the agents in the market. Denoting the price of Y_j by $\pi(Y_j)$, the equilibrium price for this risk exchange economy is characterized by:

1. For any j, $E[u_j(w_j - X_j + Y_j - \pi(Y_j))]$ is maximized with respect to Y_j, and
2. $\sum_{j=1}^n Y_j = 0$ for all possible states.

In this setting, Bühlmann [1] showed that the equilibrium price $\pi(Y)$ for the risk exchange is given by (1.2).

2.1 The Wang transform

Wang [16] showed that the transform (1.1) can be derived from Bühlmann's economic premium principle (1.2) under the following assumptions on the aggregate risk $Z = \sum_{j=1}^n X_j$:

1. There are so many individual risks X_j in the market that the aggregate risk Z can be approximated by a normal random variable, and
2. The correlation coefficient between $Z_0 = (Z - \mu_Z)/\sigma_Z$ and $U = \Phi^{-1}[F(X)]$ is ρ,[2] where $\mu_Z = E[Z]$ and $\sigma_Z^2 = V[Z]$.

Here, $F(x)$ denotes the CDF of risk X of interest. For the sake of simplicity, it is assumed that $F(x)$ is strictly increasing in x. The inverse function of $F(x)$ is denoted by $F^{-1}(x)$.

Recall that, if the random vector (Z_0, U) follows a bivariate normal distribution with correlation coefficient ρ, there exists a normal random variable ξ, independent of (Z_0, U), such that $Z_0 = \rho U + \xi$. Hence, from (1.2), it follows that

$$\pi(X) = \frac{E[X\mathrm{e}^{-\theta U}]}{E[\mathrm{e}^{-\theta U}]}, \qquad \theta = \lambda \sigma_Z \rho.$$

[2] For any random variable X and its CDF $F(x)$, the random variable $\Phi^{-1}[F(X)]$ follows a standard normal distribution. Hence, the second assumption can be stated that the random vector $(F(X), \Phi(Z_0))$ follows a bivariate Gaussian copula with correlation coefficient ρ.

Since $U = \Phi^{-1}[F(X)]$ follows a standard normal distribution, we obtain

$$\pi(X) = \mathrm{e}^{-\theta^2/2} E[X\mathrm{e}^{-\theta U}] = \mathrm{e}^{-\theta^2/2} \int_R x\mathrm{e}^{-\theta\Phi^{-1}[F(x)]} \mathrm{d}F(x), \tag{2.1}$$

where R stands for the real line.

We intend to write the pricing functional $\pi(X)$ in terms of a transformed CDF $F^*(x)$ such that

$$\pi(X) = \int_R x \mathrm{d}F^*(x) = E^*[X],$$

where E^* stands for the expectation operator associated with the CDF $F^*(x)$. It follows from (2.1) that

$$F^*(x) = \mathrm{e}^{-\theta^2/2} \int_{-\infty}^{x} \mathrm{e}^{-\theta\Phi^{-1}[F(y)]} \mathrm{d}F(y). \tag{2.2}$$

After some algebraic manipulation, Wang [16] showed that the transform (2.2) is given by (1.1), i.e.

$$F^*(x) = \Phi[\Phi^{-1}(F(x)) + \theta], \qquad \theta = \lambda\sigma_Z\rho.$$

Hence, the mean value evaluated under the transformed CDF $F^*(x)$ will define a risk-adjusted fair value of risk X.

2.2 Alternative derivation of the Wang transform

For any x, let $I_x(y) = 1$ if $y \le x$ and $I_x(y) = 0$ otherwise. Then, using the function $I_x(y)$, (2.2) can be written as

$$\begin{aligned} F^*(x) &= \mathrm{e}^{-\theta^2/2} E\left[I_x(X)\mathrm{e}^{-\theta\Phi^{-1}[F(X)]}\right] \\ &= \mathrm{e}^{-\theta^2/2} E\left[I_x(F^{-1}(\Phi(U)))\mathrm{e}^{-\theta U}\right], \end{aligned} \tag{2.3}$$

where the second equality follows since $U = \Phi^{-1}[F(X)]$ by assumption.

The following result is well known and useful for our purposes. See, e.g., Kijima and Muromachi [8] for the proof.

Lemma 2.1 *For any bivariate normal random vector (X, Y) and any function $h(x)$, we have*

$$E[h(X)\mathrm{e}^{-Y}] = E[\mathrm{e}^{-Y}]E[h(X - Cov[X, Y])]$$

for which the expectations exist.

Applying the lemma to (2.3), we obtain

$$F^*(x) = E\left[I_x(F^{-1}(\Phi(U - \theta)))\right],$$

since $E[\mathrm{e}^{-\theta U}] = \mathrm{e}^{\theta^2/2}$ and $Cov[U, \theta U] = \theta$. It follows from the definition of $I_x(y)$ that

$$\begin{aligned} F^*(x) &= P\{F^{-1}(\Phi(U-\theta)) \le x\} \\ &= P\{U - \theta \le \Phi^{-1}(F(x))\} \\ &= \Phi[\Phi^{-1}(F(x)) + \theta], \qquad \theta = \lambda\sigma_Z\rho, \end{aligned}$$

which is the Wang transform (1.1). Here, the third equality holds, since U follows a standard normal distribution. In the next section, following the above idea, we obtain a more general transformation.

3 An extension of the Wang transform

Given a CDF $F(x)$ of risk X, suppose that there exist random variables U and $Y > 0$, independent of each other, such that U follows a standard normal distribution and

$$F(X) = G(U/Y), \qquad Y > 0,$$

where $G(x)$ denotes the CDF of random variable U/Y. For example, when $Y = 1$, we have $G(x) = \Phi(x)$ as for the univariate Wang transform. When $Y = \sqrt{\chi^2_\nu/\nu}$ as assumed by Kijima and Muromachi [9], where χ^2_ν denotes a chi-square random variable with ν degree of freedom, the random variable U/Y follows a t distribution with ν degree of freedom. However, at this point, we do not specify the CDF $G(x)$. We only assume that $G(x)$ is strictly increasing in x for the sake of simplicity. Note that, for any random variable X and its CDF $F(x)$, $F(X)$ follows a uniform distribution on the interval $[0, 1]$.

In summary, our assumption on the aggregate risk can be stated as

$$Z_0 = \rho U + \xi, \qquad U = Y\alpha(X), \tag{3.1}$$

where $\alpha(x) \equiv G^{-1}(F(x))$ and U and ξ are normally distributed random variables, independent of each other.

3.1 The main result

As for (2.1), we have

$$\begin{aligned} \pi(X) &= \mathrm{e}^{-\theta^2/2} E\left[E\left[X\mathrm{e}^{-\theta Y\alpha(X)}\middle|Y\right]\right] \\ &= \mathrm{e}^{-\theta^2/2} E\left[\int_R x\mathrm{e}^{-\theta Y\alpha(x)}\mathrm{d}F(x)\right]. \end{aligned}$$

It follows that (cf. (2.2))

$$F^*(x) = \mathrm{e}^{-\theta^2/2} E\left[\int_{-\infty}^{x} \mathrm{e}^{-\theta Y \alpha(z)} \mathrm{d}F(z)\right],$$

where the expectation is taken with respect to Y. Using the function $I_x(y)$, where $I_x(y) = 1$ if $y \le x$ and $I_x(y) = 0$ otherwise, we obtain

$$\begin{aligned} F^*(x) &= \mathrm{e}^{-\theta^2/2} E\left[E\left[I_x(X)\mathrm{e}^{-\theta Y \alpha(X)} \middle| Y\right]\right] \\ &= \mathrm{e}^{-\theta^2/2} E\left[E\left[I_x(\alpha^{-1}(U/Y))\mathrm{e}^{-\theta U} \middle| Y\right]\right]. \end{aligned}$$

Since Y and U are independent of each other by assumption, we obtain from Lemma 2.1 that

$$E\left[I_x(\alpha^{-1}(U/Y))\mathrm{e}^{-\theta U} \middle| Y\right] = \mathrm{e}^{\theta^2/2} E\left[I_x(\alpha^{-1}((U-\theta)/Y)) \middle| Y\right].$$

It follows from the definition of $I_x(y)$ that

$$\begin{aligned} F^*(x) &= E\left[E\left[I_x(\alpha^{-1}((U-\theta)/Y)) \middle| Y\right]\right] \\ &= E\left[P\left\{\alpha^{-1}((U-\theta)/Y)) \le x \middle| Y\right\}\right] \\ &= E\left[P\left\{U \le \alpha(x)Y + \theta \middle| Y\right\}\right]. \end{aligned}$$

Therefore, we finally obtain the transformation (1.3), i.e.

$$F^*(x) = E[\Phi(\alpha(x)Y + \theta)], \qquad \theta = \lambda\sigma_Z\rho, \quad \alpha(x) = G^{-1}(F(x)), \tag{3.2}$$

since U follows the standard normal distribution.

For the CDF $F^*(x)$ given by (3.2), we denote the random variable associated with $F^*(x)$ by X^*. Recall that the risk under consideration is the random variable X with CDF $F(x)$. If these random variables represent the amount of a profit/loss, we take $\theta > 0$. The other case can be treated similarly. In the next theorem, we call X greater than X^* in the sense of *first order stochastic dominance*, denoted by $X \ge_{\mathrm{FSD}} X^*$, if $F(x) < F^*(x)$ for all x.[3]

Theorem 3.1 *If $\theta > 0$, then $X \ge_{\mathrm{FSD}} X^*$. That is, the investor evaluates the loss worse than the actual loss when pricing the risk X.*

Proof. Note that, by definition, we have

$$F(x) = G(\alpha(x)) = P\{U/Y \le \alpha(x)\} = E[\Phi(\alpha(x)Y)]. \tag{3.3}$$

Since $\theta > 0$ and $\Phi(x)$ is increasing in x, it follows from (3.2) and (3.3) that

$$F^*(x) = E[\Phi(\alpha(x)Y + \theta)] > E[\Phi(\alpha(x)Y)] = F(x), \qquad x \in R,$$

whence the result.

[3]See Kijima and Ohnishi [10] for the application of stochastic dominance relations to finance.

Remark 3.1 When the aggregated risk Z and $U = \Phi^{-1}[F(X)]$ are uncorrelated, i.e. $\rho = 0$ in (3.1), so that $\theta = 0$ in (3.2), we have $F^*(x) = F(x)$. That is, no distortion takes place for the uncorrelated case, as expected.

3.2 Some special cases

Special cases of the transform (3.2) we consider are the following. Other cases can be treated similarly, as far as the CDF $G(x)$ of the random variable U/Y is obtained.

First, assume $Y = 1$. Then, we have $G(x) = \Phi(x)$ so that we obtain the Wang transform (1.1). More generally, for a positive discrete random variable Y with

$$p_i = P\{Y = y_i\}, \qquad y_i > 0, \quad i = 1, 2, \ldots, n,$$

we have

$$G(x) = E[\Phi(xY)] = \sum_{i=1}^{n} \Phi(xy_i)p_i$$

so that we obtain

$$F^*(x) = \sum_{i=1}^{n} \Phi(\alpha(x)y_i + \theta)p_i, \qquad \theta = \lambda\sigma_Z\rho,$$

where $\alpha(x) = G^{-1}(F(x))$.

Next, suppose that $Y > 0$ is a continuous random variable with probability density function (PDF for short) $g(x)$. In order to obtain the PDF of U/Y, we need to consider the joint PDF of $(U/Y, Y)$. Denoting its joint PDF by $h(x, y)$, we have

$$h(x, y) = \phi(xy)g(y)y, \qquad y > 0, \quad x \in R,$$

where $\phi(u)$ denotes the PDF of the standard normal distribution. Hence, the CDF of U/Y is obtained as

$$G(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} h(u, v)\mathrm{d}v\mathrm{d}u.$$

The risk-adjusted CDF $F^*(x)$ is then calculated by (3.2).

In particular, when $Y = \sqrt{\chi^2_\nu/\nu}$, it is well known that the random variable U/Y follows a t distribution with ν degree of freedom, whose CDF is denoted by $T_\nu(x)$, i.e. $G(x) = T_\nu(x)$. On the other hand, the transform (3.2) is written as

$$F^*(x) = E[\Phi(\alpha(x)Y + \theta)] = P\left\{\frac{U - \theta}{Y} \le \alpha(x)\right\}.$$

Recall that the random variable $(U + \delta)/Y$ follows a non-central t distribution with ν degree of freedom and non-centrality parameter δ whose PDF is given by (see, e.g., Johnson and

Kotz [4])

$$p_{\nu:\delta}(t) = \left(\frac{1}{2}\right)^{\frac{\nu-1}{2}} \frac{\mathrm{e}^{-\frac{1}{2}\delta^2}}{\sqrt{\pi\nu}\Gamma(\frac{\nu}{2})} \int_0^\infty x^\nu \exp\left\{-\frac{1}{2}\left[\left(1+\frac{t^2}{\nu}\right)x^2 - 2\frac{t\delta}{\sqrt{\nu}}x\right]\right\} \mathrm{d}x.$$

Denoting the CDF of the non-central t distribution by $P_{\nu:\delta}(x)$, we arrive at the two-parameter transformation (1.4), i.e.

$$F^*(x) = P_{\nu:-\theta}[T_\nu^{-1}(F(x))], \qquad \theta = \lambda\sigma_Z\rho. \tag{3.4}$$

Note that, when $F(x) = T_\nu(x)$, the transform (3.4) distorts the t distribution with ν degree of freedom to a non-central t distribution with the same degree of freedom and non-centrality parameter $-\theta$, where θ represents the risk premium. A more elementary derivation of (3.4) is given by Kijima and Muromachi [9].

Remark 3.2 Since χ^2_ν/ν converges to unity as $\nu \to \infty$, the non-central t distribution converges to a shifted normal distribution with shift parameter δ. It follows that the new transformation (3.4) converges to the Wang transform (1.1) as $\nu \to \infty$.

4 Comparison of tail distributions

Return distributions of financial and insurance assets are often claimed that they are fat-tailed. In fact, some empirical studies suggest to use t distributions with 3 or 4 degree of freedom for asset returns (see, e.g., Platen and Stahl (2003)). Of interest is therefore how fat-tailed the transformed CDF $F^*(x)$ is. In this section, we compare the tail distributions of the new transform (3.4) with the Wang transform (1.1).

For a given CDF $F(x)$, let $\beta(x) = \Phi^{-1}(F(x))$ in the Wang transform (1.1). Also, let

$$K_x(\theta) \equiv \Phi(\beta(x)+\theta) - E[\Phi(\alpha(x)Y+\theta)], \qquad \theta \geq 0.$$

Note that, from (3.3), we obtain

$$K_x(0) = \Phi(\beta(x)) - E[\Phi(\alpha(x)Y)] = F(x) - F(x) = 0. \tag{4.1}$$

The next theorem shows that $K_x(\theta) \geq 0$ for all x such that $F(x) < 1/2$. Hence, the Wang transform (1.1) produces the fattest tail distributions among the class of probability transforms given by (3.2).

Theorem 4.1 *Suppose $\theta \geq 0$. Then, for all x such that $F(x) < 1/2$, we have $K_x(\theta) \geq 0$, i.e.*

$$\Phi[\Phi^{-1}(F(x)) + \theta] \geq E[\Phi(G^{-1}(F(x))Y + \theta)]$$

for all $Y > 0$, where $G(x)$ is the CDF of random variable U/Y.

Proof. Let $A = \{\alpha(x)Y \le \beta(x)\}$. Then, we have

$$\begin{aligned} K_x(\theta) &= E\left[\Phi(\beta(x)+\theta) - \Phi(\alpha(x)Y+\theta)\right] \\ &= E\left[1_A \int_{\alpha(x)Y+\theta}^{\beta(x)+\theta} \phi(s)\mathrm{d}s - 1_{A^c}\int_{\beta(x)+\theta}^{\alpha(x)Y+\theta} \phi(s)\mathrm{d}s\right], \end{aligned} \tag{4.2}$$

where $\phi(s)$ denotes the PDF of standard normal distributions. Since $\theta \ge 0$, we observe that $\phi(s+\theta)/\phi(s)$ is decreasing in s. It follows from (4.2) that

$$\begin{aligned} K_x(\theta) &= E\left[1_A \int_{\alpha(x)Y}^{\beta(x)} \frac{\phi(s+\theta)}{\phi(s)}\phi(s)\mathrm{d}s - 1_{A^c}\int_{\beta(x)}^{\alpha(x)Y} \frac{\phi(s+\theta)}{\phi(s)}\phi(s)\mathrm{d}s\right] \\ &\ge E\left[1_A \int_{\alpha(x)Y}^{\beta(x)} \frac{\phi(\beta(x)+\theta)}{\phi(\beta(x))}\phi(s)\mathrm{d}s - 1_{A^c}\int_{\beta(x)}^{\alpha(x)Y} \frac{\phi(\beta(x)+\theta)}{\phi(\beta(x))}\phi(s)\mathrm{d}s\right] \\ &= \frac{\phi(\beta(x)+\theta)}{\phi(\beta(x))} E\left[1_A \int_{\alpha(x)Y}^{\beta(x)} \phi(s)\mathrm{d}s - 1_{A^c}\int_{\beta(x)}^{\alpha(x)Y} \phi(s)\mathrm{d}s\right] \\ &= \frac{\phi(\beta(x)+\theta)}{\phi(\beta(x))} K_x(0). \end{aligned}$$

The theorem follows since $K_x(0) = 0$ from (4.1).

Theorem 4.1 provides the comparison of tail distributions between the cases of $Y > 0$ and $Y = 1$ (the Wang transform), but does not state any result on the comparison among positive random variables Y. In the rest of this section, we compare the tail parts of the transformed CDF $F^*(x)$ obtained by (3.4), i.e. the case of non-central t distributions, numerically. For this purpose, we invoke the result of Lenth [11] in order to calculate the tail parts of non-central t distributions.

Let T be a random variable that follows a non-central t distribution with ν degree of freedom and non-centrality parameter δ. According to Lenth [11], we have the following expansion:[4]

$$P\{0 < T < t\} = \sum_{j=0}^{\infty}\left[p_j I_x\left(j+\frac{1}{2}, \frac{\nu}{2}\right) + q_j I_x\left(j+1, \frac{\nu}{2}\right)\right] \tag{4.3}$$

where

$$\begin{aligned} I_x(a,b) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^x t^{a-1}(1-t)^{b-1}dt, \\ x &= \frac{t^2}{t^2+\nu}, \qquad p_j = \frac{1}{2}\mathrm{e}^{-\delta^2/2}\frac{\left(\frac{\delta^2}{2}\right)^j}{j!}, \qquad q_j = \frac{1}{2}\delta\mathrm{e}^{-\delta^2/2}\frac{\left(\frac{\delta^2}{2}\right)^j}{\sqrt{2}\Gamma(j+\frac{3}{2})}. \end{aligned}$$

Note that, once the first term in the right-hand side of (4.3) is obtained, the remaining terms are calculated recursively using the facts that $\Gamma(a+1) = a\Gamma(a)$ and

$$I_x(a+1,b) = I_x(a,b) - \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}x^a(1-x)^b.$$

[4] In the Lenth algorithm, 100 terms are enough to guarantee the accuracy within $\pm 10^{-6}$ errors for $-11.0 \le \delta \le 11.0$. It is reported that no errors larger than 10^{-6} were observed using his algorithm.

The CDF $P_{\nu:\delta}(x)$ can be evaluated from (4.3) and the relation

$$P_{\nu:\delta}(x) = P\{T \le 0\} + P\{0 < T < x\} = \Phi(-\delta) + P\{0 < T < x\}.$$

In the following numerical experiments, we consider the two cases: (1) $\theta = 0.2$ and (2) $\theta = 0.7$. Case (1) corresponds to the case of low risk premium, while Case (2) the case of a relatively high risk premium. For each case, we consider the degrees of freedom $\nu = 3, 5, 10$ and ∞, where $\nu = \infty$ corresponds to the case of normal distributions.

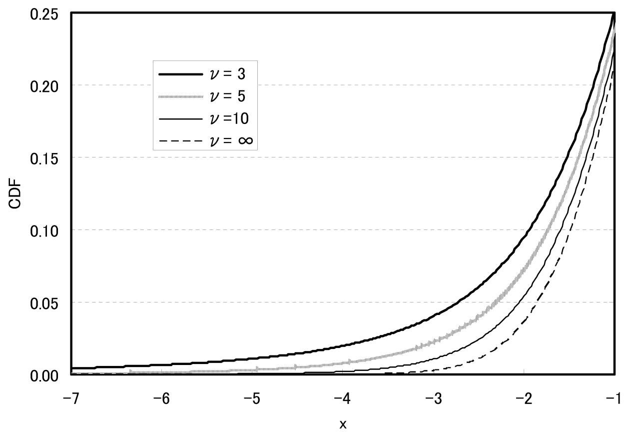

Figure 1: CDFs of non-central t distrinution for Case (1).

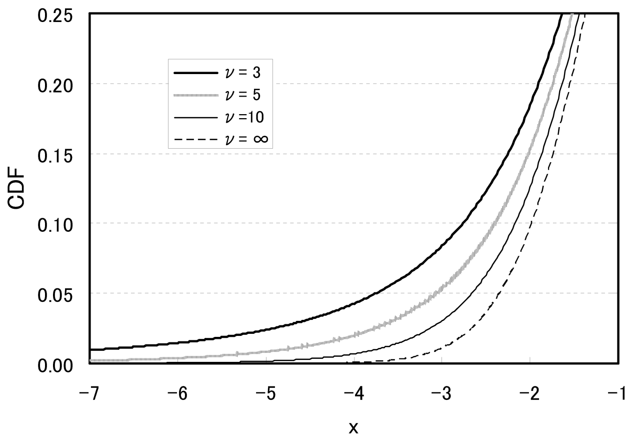

Figure 2: CDFs of non-central t distrinution for Case (2).

First, we check the fat-tailness of non-central t distributions $P_{\nu:-\theta}(x)$ using the Lenth algorithm. Fig. 1 and Fig. 2 depict the tail parts of the CDF's for Cases (1) and (2), respectively. Recall that we take $\theta > 0$, since we assume that the random variable X

represents the amount of a profit/loss. Hence, we are interested in values of large negative x. The other case can be treated similarly. From these figures, we observe that a non-central t distribution has fatter tails with a smaller degree ν of freedom.

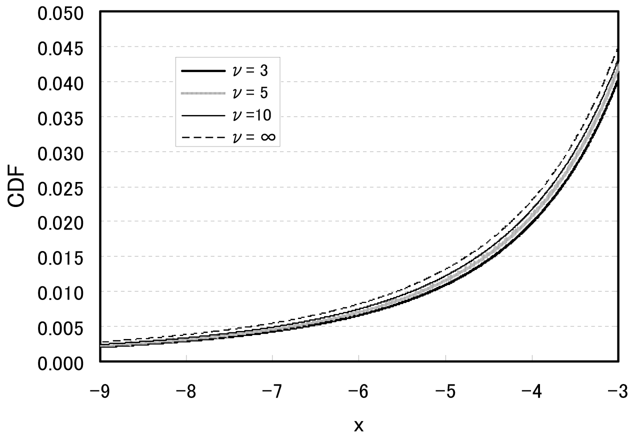

Figure 3: Risk adjusted CDFs when risk X follows T_3 for Case (1).

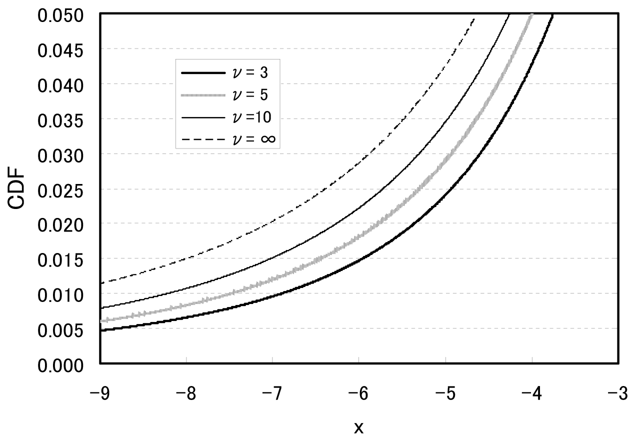

Figure 4: Risk adjusted CDFs when risk X follows T_3 for Case (2).

Now, we consider the risk-adjusted CDF $F^*(x)$ for a given CDF $F(x)$ of risk X. Fig. 3 depicts the risk-adjusted CDF $F^*(x)$ when the original risk follows a t distribution with $\nu = 3$ degree of freedom, i.e. $F(x) = T_3(x)$, for Case (1). Recall that empirical studies suggest to use such distributions for actual asset returns. The transformed CDF calculated by (3.4) is denoted by

$$F_\nu^t(x) = P_{\nu:-\theta}[T_\nu^{-1}(T_3(x))].$$

Note that, for $\nu = 3$, the transformation (3.4) produces a non-central t distribution, i.e. $F_3^t(x) = P_{3:-\theta}(x)$. For other ν, the transformed CDF $F_\nu^t(x)$ is not necessarily a non-central t distribution. Fig. 3 shows that the tail parts of the risk-adjusted distributions $F_\nu^t(x)$ are increasing in ν. That is, the transformation (3.4) produces fatter risk-adjusted CDF as Y becomes less variable. Note that this result is consistent with Theorem 4.1. For the high premium case, the situation remains the same. Fig. 4 depicts the risk-adjusted CDF $F_\nu^t(x)$ for Case (2).

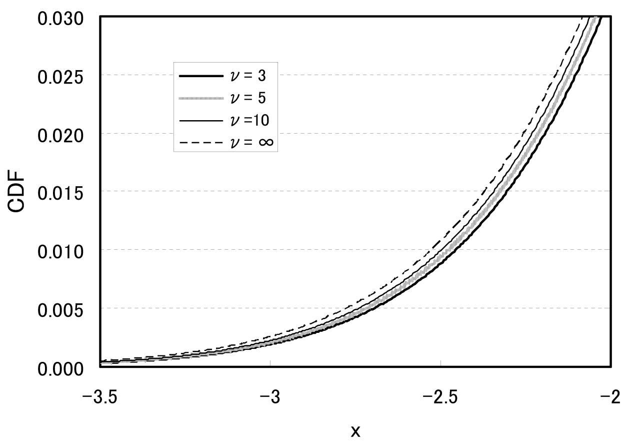

Figure 5: Risk adjusted CDFs when risk X follows Φ for Case (1).

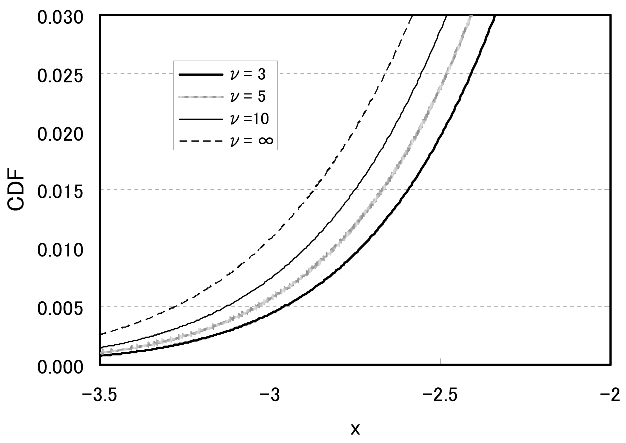

Figure 6: Risk adjusted CDFs when risk X follows Φ for Case (2).

We next consider the case that the original risk is normally distributed, i.e. $F(x) = \Phi(x)$. In this case, the Wang transform (1.1) becomes $F^*(x) = \Phi(x + \theta)$, while the transformation

(3.4) produces the risk-adjusted CDF

$$F_\nu^n(x) = P_{\nu:-\theta}[T_\nu^{-1}(\Phi(x))].$$

We expected that the transformation (3.4) produces a fatter tail than the normal counterpart, because it involves a non-central t distribution. Contrary to our intuition, but consistent with Theorem 4.1, the tail parts of the risk-adjusted distributions $F_\nu^n(x)$ are increasing in ν, as Fig. 5 (Case 1) and Fig. 6 (Case 2) reveal.

According to our numerical experiments, we conclude that the Wang transform (1.1) produces the fattest tail distributions among the class of t distributions. However, as Wang [15] reported, the one-parameter transform (1.1) is not flexible enough to match the actual market data. Thus, Wang [15] proposed the two-parameter transformation

$$F^*(x) = T_\nu[\Phi^{-1}(F(x)) + \theta], \tag{4.4}$$

and reported that (4.4) is much better to fit, although the two-parameter transform is not consistent with the economic premium principle (1.2), thereby lacking a sound economic interpretation.

Fig. 7 compares the tail distributions of the one-parameter and the two-parameter Wang transforms with $\nu = 5$ and $\theta = 0.2$. As expected, the two-parameter transformation (4.4) has fatter tails than the one-parameter counterpart.

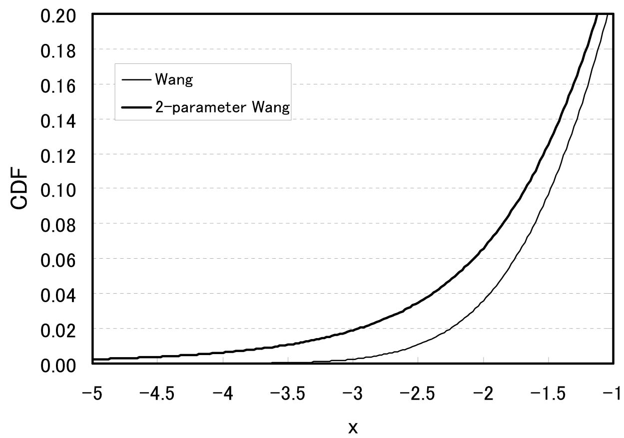

Figure 7: One-parameter and two-parameter Wang Transforms.

5 A multivariate extension

In the following two sections, we consider two extensions of the new transformation (3.2); one is a multivariate extension, and the other is a multiperiod case. In this section, we extend the transform (3.2) to the multivariate setting.

For this purpose, as in Kijima [6], suppose that the underlying risks are described by a multivariate random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and the σ-algebra is given by $\mathcal{F} = \sigma(X_1, X_2, \ldots, X_n)$. Note that a particular risk, Y say, is an $\mathcal{F}$-measurable random variable. That is, there exists an n-variate function $h(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, such that $Y = h(\boldsymbol{X})$ in general. Then, Bühlmann's economic premium principle is given by

$$\pi(Y) = \frac{E[h(\boldsymbol{X})\mathrm{e}^{-\lambda Z}]}{E[\mathrm{e}^{-\lambda Z}]}, \qquad Z = \sum_{j=1}^{n} X_j. \tag{5.1}$$

It is clear that this pricing functional $\pi(X)$ is linear.

The pricing formula (5.1) can be described in terms of the probability distortion (or the *change of measures* in other words). For the sake of simplicity, suppose that the joint PDF of $\boldsymbol{X}$ exists, which is denoted by $f(\boldsymbol{x})$. The PDF defined by

$$f^*(\boldsymbol{x}) = \frac{\mathrm{e}^{-\lambda z}}{E[\mathrm{e}^{-\lambda Z}]} f(\boldsymbol{x}), \qquad z = \sum_{j=1}^{n} x_j, \tag{5.2}$$

provides the risk-adjusted distortion.[5] That is, Bühlmann's equilibrium price is obtained as

$$\pi(Y) = \int_{R^n} \frac{h(\boldsymbol{x})\mathrm{e}^{-\lambda z}}{E[\mathrm{e}^{-\lambda Z}]} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{R^n} h(\boldsymbol{x}) f^*(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = E^*[h(\boldsymbol{X})],$$

where E^* denotes the expectation operator associated with $f^*(\boldsymbol{x})$.

5.1 The multivariate Wang transform

We first derive the multivariate Wang transform proposed by Kijima [6] using the idea described in Section 2.2. Consider a Gaussian copula[6] for the underlying risks $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$. That is, define

$$U_j \equiv \Phi^{-1}[F_j(X_j)], \qquad j = 1, 2, \ldots, n, \tag{5.3}$$

where $F_j(x)$ is the marginal CDF of X_j. For the sake of simplicity, it is assumed that $F_j(x)$ is strictly increasing in x for all j. A Gaussian copula assumes that $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)$ follows an n-variate standard normal distribution with correlation matrix $\boldsymbol{\Sigma}_\rho = (\rho_{ij})$.

Now, as in the univariate case, suppose that the standardized aggregate risk $Z_0 = (Z - \mu_Z)/\sigma_Z$ is expressed as (cf. (3.1))

$$Z_0 = \xi + \rho \sum_{j=1}^{n} w_j U_j, \qquad U_j = \Phi^{-1}[F_j(X_j)], \tag{5.4}$$

[5] Kijima [6] called the distortion (5.2) the *multivariate Esscher transform*.

[6] See Nelsen [12] for details of copulas.

where ξ is a normal random variable, independent of the other random variables. A justification of this assumption is found in Kijima [6]. It follows from (5.1) and (5.4) that

$$\pi(Y) = \frac{E[h(\boldsymbol{X})\mathrm{e}^{-\sum_{j=1}^n \theta_j U_j}]}{E[\mathrm{e}^{-\sum_{j=1}^n \theta_j U_j}]}, \qquad \theta_j = \lambda\sigma_Z\rho w_j.$$

Since $U \equiv \sum_{j=1}^n \theta_j U_j$ is normally distributed with mean 0 and variance $\sigma_U^2 = \sum_{i,j}\theta_i\theta_j\rho_{ij}$ by assumption, $\pi(Y)$ can be written as

$$\pi(Y) = \mathrm{e}^{-\sigma_U^2/2} E\left[h(\boldsymbol{X})\mathrm{e}^{-\sum_{j=1}^n \theta_j U_j}\right]. \tag{5.5}$$

It follows from (5.3) and (5.5) that the risk-adjusted PDF is given by

$$f^*(\boldsymbol{x}) = \mathrm{e}^{-\sigma_U^2/2}\mathrm{e}^{-\sum_{j=1}^n \theta_j \alpha_j(x_j)} f(\boldsymbol{x}), \qquad \alpha_j(x_j) = \Phi^{-1}[F_j(x_j)]. \tag{5.6}$$

As for the univariate case, let $I_{\boldsymbol{x}}(\boldsymbol{y}) = 1$ if $y_j \le x_j$ for all $j = 1, 2, \ldots, n$ and $I_{\boldsymbol{x}}(\boldsymbol{y}) = 0$ otherwise. Then, integrating (5.6) from $-\infty$ to x_j for each component j and using the function $I_{\boldsymbol{x}}(\boldsymbol{y})$, the risk-adjusted CDF $F^*(\boldsymbol{x})$ can be written as

$$\begin{aligned} F^*(\boldsymbol{x}) &= \mathrm{e}^{-\sigma_U^2/2} E\left[I_{\boldsymbol{x}}(\boldsymbol{X})\mathrm{e}^{-\sum_{j=1}^n \theta_j\alpha_j(X_j)}\right] \\ &= \mathrm{e}^{-\sigma_U^2/2} E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U}))\mathrm{e}^{-U}\right], \end{aligned} \tag{5.7}$$

where $U = \sum_{j=1}^n \theta_j U_j$ and $\beta(\boldsymbol{U}) = (F_1^{-1}(\Phi(U_1)), \ldots, F_n^{-1}(\Phi(U_n)))$. The second equality is derived from (5.3).

On the other hand, it is well known that the result in Lemma 2.1 can be extended to the multivariate setting.[7] That is, for any multivariate normal random vector $(\boldsymbol{X}, Y)$ and any function $h(\boldsymbol{x})$, we have

$$E[h(\boldsymbol{X})\mathrm{e}^{-Y}] = E[\mathrm{e}^{-Y}]E[h(\boldsymbol{X} - Cov[\boldsymbol{X}, Y])] \tag{5.8}$$

for which the expectations exist. Here, $Cov[\boldsymbol{X}, Y]$ means the vector with components $Cov[X_j, Y]$.

Applying this result to (5.7), we obtain

$$F^*(x) = E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U} - \boldsymbol{\theta}_\rho))\right], \qquad \boldsymbol{\theta}_\rho = (\theta_1^\rho,\, \theta_2^\rho, \ldots,\, \theta_n^\rho),$$

since $E[\mathrm{e}^{-\theta U}] = \mathrm{e}^{\sigma_U^2/2}$ and $\theta_j^\rho \equiv Cov[U_j, U] = \sum_{i=1}^n \theta_i\rho_{ij}$. It follows from the definition of $I_{\boldsymbol{x}}(\boldsymbol{y})$ that the risk-adjusted CDF $F^*(\boldsymbol{x})$ is obtained as

$$\begin{aligned} F^*(\boldsymbol{x}) &= P\left\{U_1 \le \Phi^{-1}[F_1(x_1)] + \theta_1^\rho,\, \ldots,\, U_n \le \Phi^{-1}[F_n(x_n)] + \theta_n^\rho\right\} \\ &= \Phi_n\left(\Phi^{-1}[F_1(x_1)] + \theta_1^\rho,\, \ldots, \Phi^{-1}[F_n(x_n)] + \theta_n^\rho\right), \qquad \theta_j^\rho = \sum_{i=1}^n \theta_i\rho_{ij}. \end{aligned} \tag{5.9}$$

[7] See Kijima and Miyake [7] for the trivariate case. Their proof can be extended to the multivariate case with ease.

Note that, when $n = 1$, (5.9) coincides with the Wang transform (1.1) since $\rho_{11} = 1$. Thus, Kijima [6] called (5.9) the *multivariate Wang transform*.

5.2 An extension of the multivariate Wang transform

The idea employed in this section is the same as the univariate case. That is, given the underlying risks $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, suppose that there exist a Gaussian copula $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)$ with correlation matrix $\boldsymbol{\Sigma}_\rho = (\rho_{ij})$ and a positive random variable Y, independent of $\boldsymbol{U}$, such that

$$F_j(X_j) = G_j(U_j/Y), \qquad j = 1, 2, \ldots, n, \tag{5.10}$$

where $F_j(x_j)$ is the marginal CDF of risk X_j and $G_j(x)$ denotes the CDF of random variable U_j/Y. It is assumed that each $G_j(x)$ is strictly increasing in x for the sake of simplicity. Note that, when $Y = 1$ almost surely, the assumption (5.10) agrees with (5.3).

Now, as in (5.4), suppose that the standardized aggregate risk $Z_0 = (Z - \mu_Z)/\sigma_Z$ is expressed as

$$Z_0 = \xi + \rho \sum_{j=1}^{n} w_j U_j, \qquad U_j = Y \alpha_j(X_j),$$

where $\alpha_j(x) = G_j^{-1}(F_j(x))$ and ξ is a normal random variable, independent of the other random variables. Since nothing has been changed except the definition of U_j, the risk-adjusted CDF $F^*(\boldsymbol{x})$ is expressed as in (5.7). That is, we have

$$F^*(\boldsymbol{x}) = \mathrm{e}^{-\sigma_U^2/2} E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U}, Y))\mathrm{e}^{-U}\right], \tag{5.11}$$

where $U = \sum_{j=1}^{n} \theta_j U_j$ and, in turn, $\beta(\boldsymbol{U}, Y) \equiv (\alpha_1^{-1}(U_1/Y), \ldots, \alpha_n^{-1}(U_n/Y))$ for this case.

At this point, we apply the result (5.8) to (5.11). Since Y and $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)$ are independent of each other by assumption, we obtain from (5.8) that

$$E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U}, Y)\mathrm{e}^{-U}\right] \;=\; \mathrm{e}^{\sigma_U^2/2} E\left[E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U} - \boldsymbol{\theta}_\rho, Y))\middle| Y\right]\right], \qquad \boldsymbol{\theta}_\rho = (\theta_1^\rho, \theta_2^\rho, \ldots, \theta_n^\rho).$$

It follows from the definition of $I_{\boldsymbol{x}}(\boldsymbol{y})$ that

$$\begin{aligned} F^*(\boldsymbol{x}) &= E\left[E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U} - \boldsymbol{\theta}_\rho, Y))\middle| Y\right]\right] \\ &= E\left[P\left\{\beta(\boldsymbol{U} - \boldsymbol{\theta}_\rho, Y) \le \boldsymbol{x}\middle| Y\right\}\right] \\ &= E\left[P\left\{U_1 \le \alpha_1(x_1)Y + \theta_1^\rho, \ldots, U_n \le \alpha_n(x_n)Y + \theta_n^\rho\middle| Y\right\}\right]. \end{aligned}$$

Therefore, we obtain the multivariate extension of the generalized transform as

$$F^*(\boldsymbol{x}) = E\left[\Phi_n\left(\alpha_1(x_1)Y + \theta_1^\rho, \ldots, \alpha_n(x_n)Y + \theta_n^\rho\right)\right], \tag{5.12}$$

where $\alpha_j(x_j) \equiv G_j^{-1}(F_j(x_j))$ and $\theta_j^\rho = \sum_{i=1}^n \theta_i \rho_{ij}$. Note that, when $Y = 1$, we obtain (5.9).

Special cases of the generalized transform (5.12) can be considered as for the univariate case. Among them, when $Y = \sqrt{\chi_\nu^2/\nu}$ as in Kijima and Muromachi [9], we obtain the following multivariate transform with t copula:

$$F^*(\boldsymbol{x}) = P(\boldsymbol{\beta}; n, \nu, -\boldsymbol{\theta}, \boldsymbol{\Sigma}_\rho) = \int^{\boldsymbol{\beta}} p(\boldsymbol{u}; n, \nu, -\boldsymbol{\theta}, \boldsymbol{\Sigma}_\rho) \mathrm{d}\boldsymbol{u},$$

where $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)$ with $\beta_j = T_\nu^{-1}(F_j(x_j))$, $j = 1, \cdots, n$. Here,

$$\begin{aligned} p(\boldsymbol{u}; n, \nu, \boldsymbol{\delta}, \boldsymbol{\Sigma}_\rho) \quad &\equiv \quad \left(\frac{1}{2}\right)^{\frac{\nu+n}{2}-1} \frac{\exp\left\{-\frac{1}{2}\boldsymbol{\delta}^\top \boldsymbol{\Sigma}_\rho \boldsymbol{\delta}\right\}}{(\pi\nu)^{n/2}\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{|\boldsymbol{\Sigma}_\rho|}} \\ &\quad \times \int_0^\infty z^{\nu+n-1} \exp\left\{-\frac{1}{2}\left[\left(1 + \frac{\boldsymbol{u}^\top \boldsymbol{\Sigma}_\rho^{-1} \boldsymbol{u}}{\nu}\right) z^2 - 2\frac{\boldsymbol{u}^\top \boldsymbol{\delta}}{\sqrt{\nu}} z\right]\right\} \mathrm{d}z, \end{aligned}$$

where $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_n)$ and $\boldsymbol{\Sigma}_\rho = (\rho_{ij})$ is a correlation matrix.

6 A multiperiod extension

The Wang transform and its variants can be extended to the multiperiod setting based on the results in Iwaki et al. [3].

Iwaki et al. [3] considered a multiperiod economy to derive the multiperiod extension of the economic premium principle of Bühlmann [1]. They constructed a consumption/portfolio model in which each agent is characterized by his/her utility function and income and invests his/her wealth in both insurance and financial markets so as to maximize the expected, discounted total utility from consumption. The state price density in equilibrium is obtained in terms of the Arrow–Pratt index of absolute risk aversion for the representative agent in the general setting. As a special case of exponential utilities, it is shown that the state price density $\phi(t)$ at time t coincides with that of Bühlmann [1]. Namely,

$$\phi(t) = \exp\left\{-\lambda(Z(t) - Z(0))\right\}, \qquad t = 0, 1, \ldots, T,$$

where $\lambda^{-1} = \sum_{j=1}^n \lambda_j^{-1}$ and $Z(t) = \sum_{j=1}^n X_j(t)$ is the aggregate risk at time t. In particular, when the risk-free interest rates are deterministic, the time t equilibrium price of the risk $X(s)$, $s > t$, is given by

$$\pi_t(X(s)) = \frac{E_t\left[X^*(s)\mathrm{e}^{-\lambda Z(s)}\right]}{E_t\left[\mathrm{e}^{-\lambda Z(s)}\right]}, \qquad t < s,$$

where $X^*(s) = X(s)/B(s)$ with $B(s)$ being the money-market account at time s. Here, E_t denotes the conditional expectation operator given the information available at time t.

Suppose for the sake of simplicity that $B(t) = 1$, i.e., interest rates are zero. Let $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$ and define the σ-algebra by $\mathcal{F}_t = \sigma(\boldsymbol{X}(u),\, u \leq t)$. Given the filtration $\mathcal{F}_t$, all the information necessary to calculate the multivariate transform (5.12) for the risk $\boldsymbol{X}(t+1)$ can be determined. For example, the marginal CDF $F_j^t(x)$ of $X_j(t+1)$ as well as the risk premium $\theta_j^\rho(t)$ at time t determines the transformation (5.12). Summarizing, provided that the stochastic process $\{\boldsymbol{X}(t); t = 0, 1, \ldots\}$ for the underlying risks is specified, we can use the Wang transform and/or its variants in the multiperiod setting for the pricing of financial and insurance risks.

7 Concluding remarks

In this paper, we derive a class of probability transform (1.3) that are consistent with Bühlmann's economic premium principle (1.2), thereby extending the result of Wang [15]. The transform is further extended to the multivariate setting by using a Gaussian copula, in order to preserve the linearity for the pricing functional, and to the multiperiod setting within the equilibrium pricing framework.

We compare the tail parts of the risk-adjusted distributions and conclude that the original Wang transform (1.1) produces the fattest tail in the class of the probability transform (1.3). However, as Wang [15] reported, the one-parameter transform (1.1) is not flexible enough to match the actual market data. Although Wang [15] proposed the two-parameter transformation (4.4) to fit the market data, it is not consistent with the economic premium principle (1.2). It is therefore important to derive probability transforms that can produce fatter tail distributions and, at the same time, has a sound economic interpretation. This will be our future research.

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