Randomized Markovian approximation for rough volatility models

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Overview

- **Topic:** Markovian Approximation for Rough Volatility Models
- **Contribution:** Introduce stochastic nodes and weights.
  - Nodes and weights are chosen for each sample path.
  - $\mathbb{L}^2$-error minimization reduces to minimizing the time-integrated variance of a certain random variable.
  - This framework naturally incorporates variance reduction techniques.
- **Results:** When the number of nodes is small, this approach outperforms all other deterministic methods (except BL2).

Classical Stochastic Volatility Models

Volatility follows a stochastic differential equation (SDE):

$$V_t = V_0 + \int_0^t b(V_s)\,\mathrm{d}s + \int_0^t \sigma(V_s)\,\mathrm{d}W_s. \tag{1}$$

- **Markovian:** Future volatility depends only on its current state.
- **Example:** Heston model, SABR model

Rough Volatility Models (RVM)

Volatility follows a stochastic Volterra equation:

$$V_t = V_0 + \int_0^t K(t-s) b(V_s)\,\mathrm{d}s + \int_0^t K(t-s)\sigma(V_s)\,\mathrm{d}W_s, \qquad (2)$$

where the kernel $K(t)$ is typically given by the **fractional kernel**:

$$K(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}, \quad 0 < H < 1/2. \qquad (3)$$

- Reduces to a **fractional Brownian motion** with Hurst parameter H when $b = 0$ and $\sigma = 1$.
- **Non-Markovian:** Exhibits memory effects.
- **Highly irregular and rougher** than classical models.
- If $H = \frac{1}{2}$, the kernel reduces to $K(t) \equiv 1$.

Example: Rough Heston Model

The Rough Heston Model extends the classical Heston model:

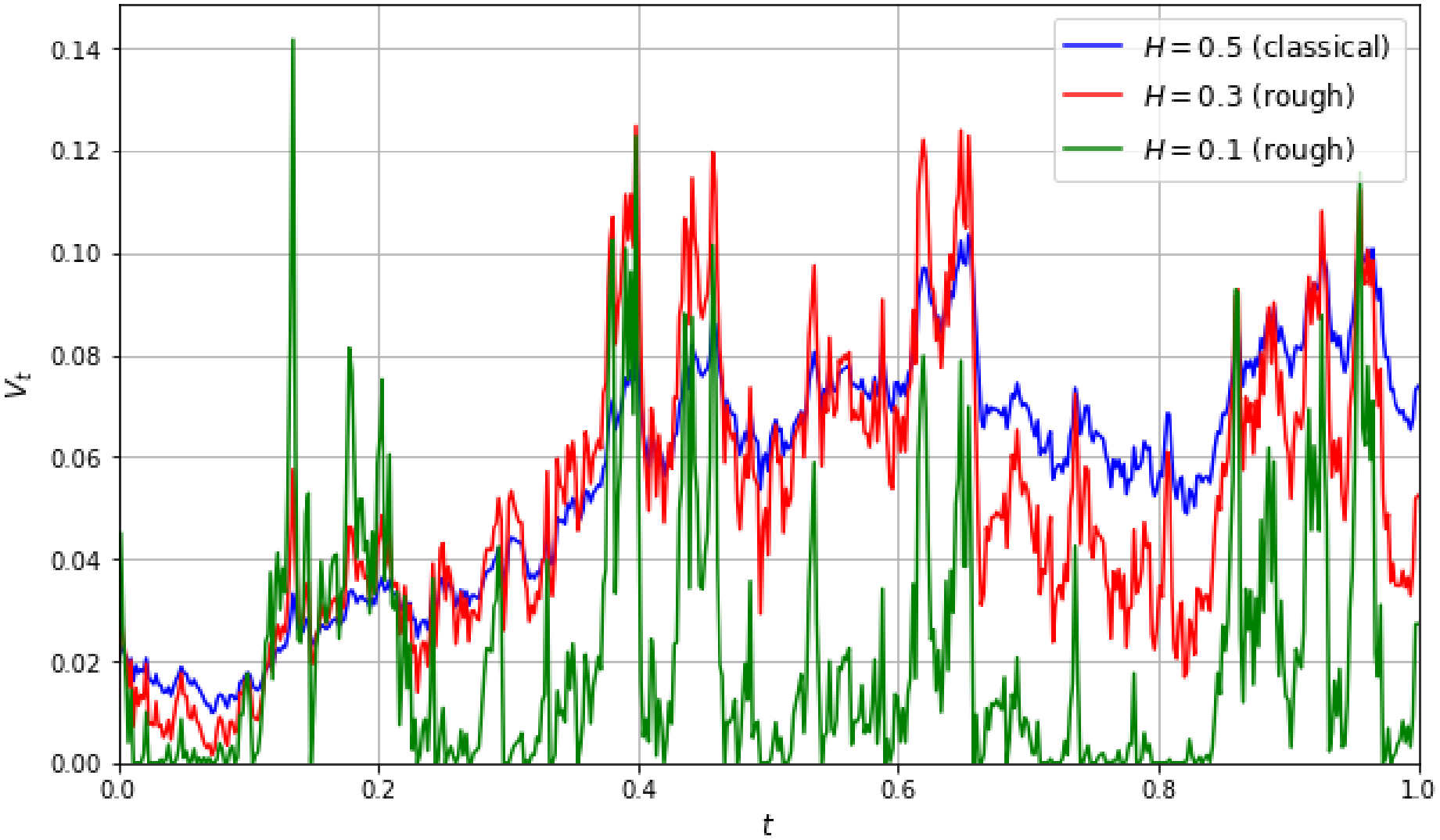
$$\mathrm{d}S_t = S_t\sqrt{V_t}\,\mathrm{d}B_t, \quad S_0 > 0, \tag{4}$$

$$V_t = V_0 + \int_0^t K(t-s)(\theta - \lambda V_s)\,\mathrm{d}s + \int_0^t K(t-s)\eta\sqrt{V_s}\,\mathrm{d}W_s, \tag{5}$$

where W_t and B_t are ρ-correlated Brownian motions.

- Admits a semi-analytical **characteristic function** [El Euch and Rosenbaum, 2019], allowing **Fourier pricing techniques**.

Volatility Path V_t in the Rough Heston Model



Why RVM?

- **Empirical Motivation:** Empirical studies [Gatheral et al., 2018] show that volatility exhibits **rough behavior**, resembling fractional Brownian motion.

- **Better Fit for IV Smiles:** RVM explains the **steep implied volatility (IV) smiles** observed in short-maturity options [Fukasawa, 2021].

Challenges in RVM

Non-Markovian Nature:

The fractional kernel $K(t)$ introduces memory effects, making the process V_t depend on its entire past.

Significantly increases the computational cost

Addressing the Non-Markovian Challenge

Key Idea:
Approximate the fractional kernel $K(t)$ using a sum of exponentials:

$$K(t) = \int_0^\infty c_H y^{-H-\frac{1}{2}} e^{-yt} \, \mathrm{d}y \approx \sum_{i=1}^{n} w_i e^{-y_i t} =: K^n(t). \qquad (6)$$

- This approximation, known as a **Markovian approximation** [Abi Jaber and El Euch, 2019], transforms the original model into an n-**dimensional Markov process**.
- Enable the use of **standard numerical methods** for efficient simulation.

Markovian Approximation

The **Markovian approximation** is defined as:

$$V_t^n = V_0 + \int_0^t K^n(t-s) b(V_t^n)\,\mathrm{d}t + \int_0^t K^n(t-s)\sigma(V_t^n)\,\mathrm{d}W_t \quad (7)$$

V_t^n is a n-dimensional Markovian process since:

- V_t^n can be expressed as $V_t^n = V_0 + \sum_{i=1}^n w_i V_t^{n,i}$, where $(V_t^{n,1}, \cdots, V_t^{n,n})$ is a solution to the n-dimensional SDE [Alfonsi and Kebaier, 2024]:

$$\begin{aligned} V_t^{n,i} = &- \int_0^t y_i V_s^{n,i}\,\mathrm{d}s + \int_0^t b(V_0 + \sum_{i=1}^n w_i V_s^{n,i})\,\mathrm{d}s \\ &+ \int_0^t \sigma(V_0 + \sum_{i=1}^n w_i V_s^{n,i})\,\mathrm{d}W_s. \end{aligned} \quad (8)$$

$\mathbb{L}^2$-error of Markovian Approximation

Proposition ([Alfonsi and Kebaier, 2024])

For every $T > 0$, there exists some $C > 0$ (depending on T, V_0, b, σ), such that

$$\mathbb{E}[|V_T - V_T^n|^2] \leq C \int_0^T |K(t) - K^n(t)|^2 dt. \tag{9}$$

- The choice of weights $(w_i)_{i=1}^n$ and nodes $(y_i)_{i=1}^n$ plays a crucial role in reducing the kernel approximation error $\int_0^T |K(t) - K^n(t)|^2 dt$.

Trade-off Accuracy and Efficiency

- **Increasing n improves approximation accuracy** by providing more weights and nodes for a better integral approximation.
- However, **a larger n increases the dimensionality of the SDE**, leading to higher computational cost and greater discretization error.

Achieving good accuracy with small n is crucial

Selection of Weights and Nodes

Deterministic Approaches:

- Use quadrature rules to approximate the kernel.
- Use same nodes and weights for all sample paths.
- Consequently, the bias introduced by the approximation is the same for every path.
- This effect is especially pronounced when n is small.

Introducing stochastic selection of nodes

Construction of Randomized Approximation Kernel

- Let Y be a random variable with density f_Y supported on $(0, \infty)$.
- The kernel $K(t)$ can be expressed as an **expectation**:

$$K(t) = \int_0^\infty c_H y^{-H-\frac{1}{2}} e^{-yt} \, \mathrm{d}y = \int_0^\infty c_H y^{-H-\frac{1}{2}} e^{-yt} \frac{1}{f_Y(y)} f_Y(y) \, \mathrm{d}y$$
$$= \mathbb{E}\left[\frac{c_H Y^{-H-\frac{1}{2}}}{f_Y(Y)} e^{-Yt}\right]. \tag{10}$$

- Let $Z_Y(t) := \frac{c_H Y^{-H-\frac{1}{2}}}{f_Y(Y)} e^{-Yt}$. By the law of large numbers, approximate this expectation by $K^{n,Y}(t)$ as a finite sum:

$$K^{n,Y}(t) := \frac{1}{n} \sum_{i=1}^{n} Z_{Y_i}(t), \tag{11}$$

where $Y_i \sim Y$ are i.i.d.

Randomized Markovian Approximation

A **randomized Markovian approximation** is defined as:

$$V_t^{n,Y} = V_0 + \int_0^t K^{n,Y}(t-s) b(V_t^{n,Y})\,\mathrm{d}t + \int_0^t K^{n,Y}(t-s)\sigma(V_t^{n,Y})\,\mathrm{d}W_t \tag{12}$$

- **As in the non-randomized case:**
  - $V_t^{n,Y}$ can be expressed using the solution of an n-dimensional SDE.
  - The $\mathbb{L}^2$-error is controlled by the kernel approximation error:

$$\begin{aligned}\mathbb{E}[|V_T - V_T^{n,Y}|^2] &\leq C\mathbb{E}\Big[\int_0^T |K(t) - K^{n,Y}(t)|^2\,\mathrm{d}t\Big] \\ &= \frac{C}{n}\int_0^T \mathrm{Var}[Z_Y(t)]\,\mathrm{d}t.\end{aligned} \tag{13}$$

Selection of density f_Y

- To minimize $\int_0^T \mathrm{Var}[Z_Y(t)]\,\mathrm{d}t$, the **optimal density** is given by:

$$f_Y(y) = \frac{y^{-H-1}\sqrt{1-e^{-2yT}}}{\int_0^\infty y^{-H-1}\sqrt{1-e^{-2yT}}\,\mathrm{d}y}. \tag{14}$$

- However, this optimal density is **impractical** for random number generation.
- Instead, use a **mimic density** f_Y as an alternative:

$$f_Y(y) = \begin{cases} C_1 y^{-\alpha}, & 0 < y \leq G, \\ C_2 y^{-\beta}, & G < y. \end{cases} \tag{15}$$

Variance Reduction Techniques

Objective:
Reduce $\int_0^T \mathrm{Var}[Z_Y(t)]\,\mathrm{d}t$ to minimize the kernel approximation error.

- **Control Variates (CV):**
  - Use an auxiliary variable with a known expectation to adjust the kernel approximation.
  - Minimize variance by **introducing an optimal correction term**.
- **Antithetic Variates (AV):**
  - In the case where $n = 2$, construct antithetic pairs (Y_1, Y_2).
  - Leverage **negative correlation** to reduce variance.

Effect of Large Node in Markovian Approximation

- In the Markovian approximation, the SDE (8) includes the **mean-reverting term:**

$$-\int_0^t y_i V_s^{n,i}\,\mathrm{d}s. \tag{16}$$

- Large y_i leads to **rapid decay**, increasing discretization error.
- This issue arises in both **quadrature-based** and **randomized** approximation:
  - **Quadrature:** Large nodes may be inevitable due to the integration rule.
  - **Randomized:** Since Y has support on $(0, \infty)$, large nodes can occasionally appear.

Strategy to Mitigate Large Node

Truncation of PDF:

For $L > 0$, let $c_L := \mathbb{P}(Y \leq L) = \int_0^L f_Y(y)\,\mathrm{d}y$.

- Define a density f_{Y^L} as:

$$f_{Y^L}(y) := \begin{cases} \frac{f_Y(y)}{c_L}, & y < L \\ 0, & L \leq y. \end{cases} \tag{17}$$

- The random variable $Y^L \sim f_{Y^L}$ takes values in **(0,L)**.

Kernel Approximation Error:

$$\mathbb{E}\Big[\int_0^T |K(t) - K^{n,Y^L}(t)|^2\,\mathrm{d}t\Big] = \frac{1}{n}\int_0^T \mathrm{Var}[Z_{Y^L}(t)]\,\mathrm{d}t + \int_0^T \Big(\int_L^\infty c_H y^{-H-\frac{1}{2}} e^{-yt}\,\mathrm{d}y\Big)^2\,\mathrm{d}t \text{ (truncation error)}. \tag{18}$$

Numerical Result

- **Model:** Rough Heston model
- **Option Type:** European Call Option
- **Parameters:**

$$\begin{aligned} H = 0.1, \quad \lambda = 0.3, \quad \nu = 0.3, \quad \theta = 0.02, \\ \rho = -0.7, \quad V_0 = 0.02, \quad S_0 = 1, \quad T = 1. \end{aligned} \tag{19}$$

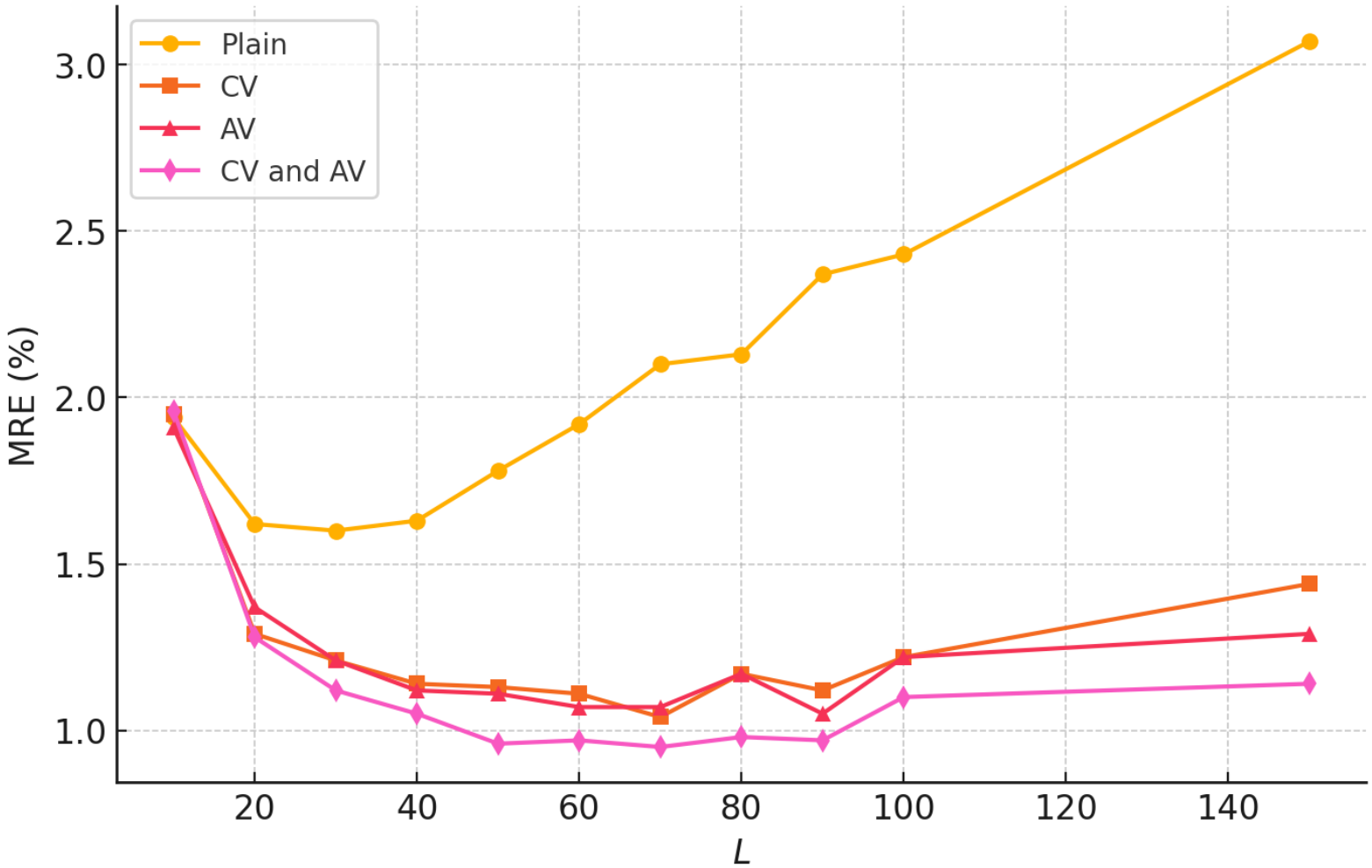
- **Monte Carlo Paths:** 10^7
- **Discretization:** Multi-Factor Euler Scheme [Alfonsi and Kebaier, 2024]
- **Time Steps:** 512
- **Number of Nodes:** $n = 2$
- **Benchmark:** Fourier-based Option Price

Maximum Relative Error (MRE)

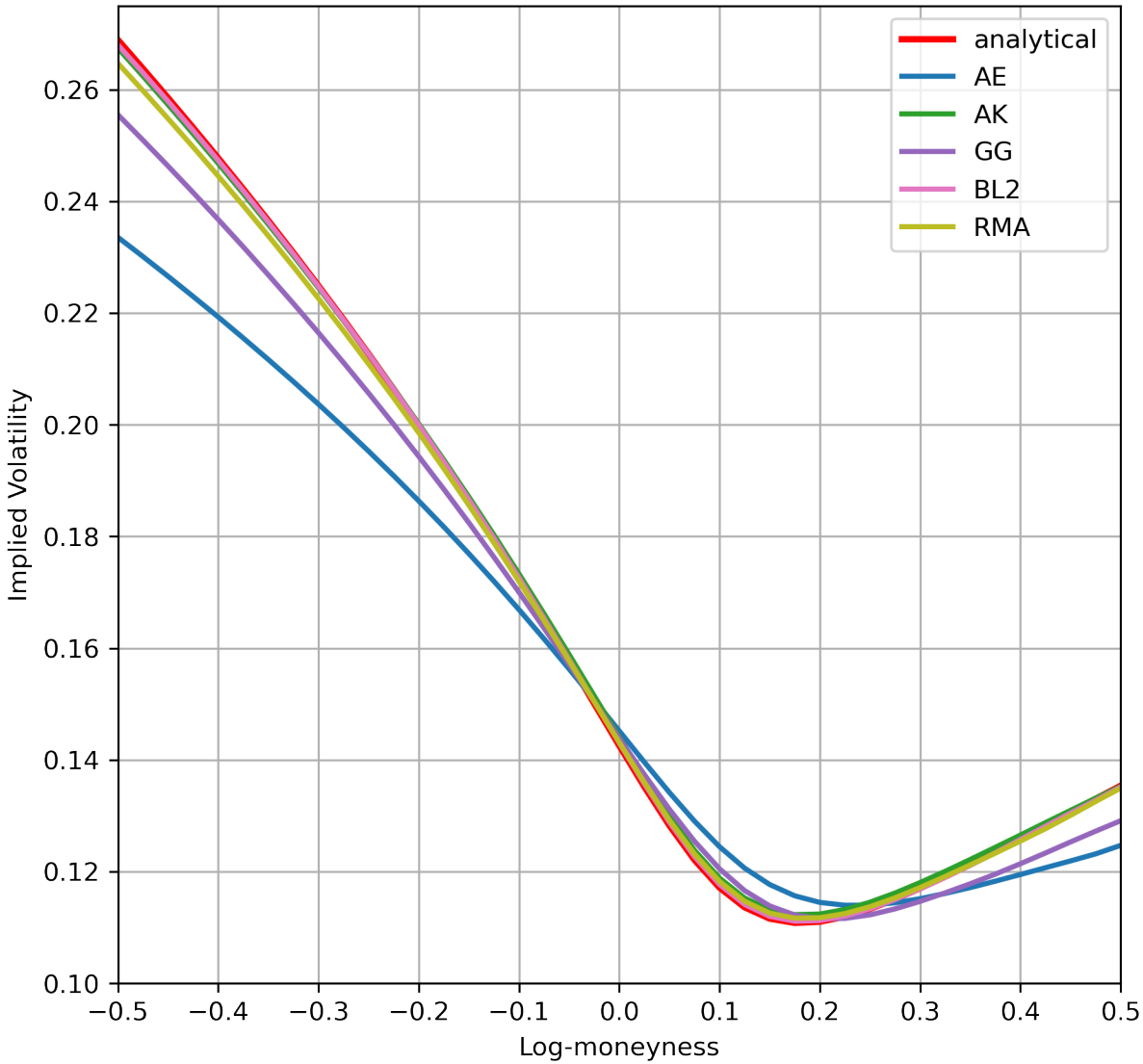
$$\mathrm{MRE} = \max_{k \in [-0.2, 0.2]} \frac{|\mathrm{IV}_{\mathrm{analytical}}(k) - \mathrm{IV}_{\mathrm{approximation}}(k)|}{\mathrm{IV}_{\mathrm{analytical}}(k)} \times 100 \qquad (20)$$

- $\mathrm{IV}_{\mathrm{analytical}}(k)$: IV obtained using the Fourier-based method
- $\mathrm{IV}_{\mathrm{approximation}}(k)$: IV obtained using the Markovian approximation and Monte Carlo simulation
- k: Log-moneyness
- The error is computed over the log-moneyness range $[-0.2, 0.2]$, using 17 equally spaced points.

MRE of IV for different values of L



IV: Deterministic vs Randomized



AE ([Abi Jaber and El Euch, 2019]), AK ([Alfonsi and Kebaier, 2024]), GG ([Bayer and Breneis, 2023a]), BL2 ([Bayer and Breneis, 2023b, Appendix F]), and RMA (Randomized Markovian

MRE: Deterministic vs Randomized

	AE	AK	GG	BL2	RMA
Error	6.77%	1.45%	2.87%	0.48%	0.96%

- RMA outperforms AE, AK, and GG.
- BL2 has the lowest error but relies on numerical optimization with weak theoretical justification.

Improvement

- The current approximation kernel only approximates the integral representation of $K(t)$ over $[0, L]$.
- As an improvement, the approximate kernel is scaled by a constant C to reduce the approximation error.
- Specifically, we seek to minimize the error

$$E \int_0^T \left| K(t) - C\, K^{n,Y}(t) \right|^2 \, \mathrm{d}t. \tag{21}$$

- The optimal constant C is given by

$$C = \frac{E \int_0^T K(t)\, K^{n,Y}(t)\, dt}{E \int_0^T \left(K^{n,Y}(t) \right)^2 \, \mathrm{d}t}. \tag{22}$$

Improvement Result

	BL2	RMA
Error	0.48%	0.82%

MRE for logmoneyness in $[-0.2, 0.2]$.

	BL2	RMA
Error	0.48%	0.41%

MRE for logmoneyness in $[-0.1, 0.1]$.

Summary

- Introduced the probabilistic approach to determining weights and nodes in the Markovian approximation.
- Provided the optimal density and the mimic density to minimize the kernel approximation error.
- Demonstrated the effectiveness of variance reduction techniques.
- Showed that the randomized Markovian approximation outperforms all quadrature methods except BL2.

Future Work

- Analyze the impact of specific nodes and weights on approximation accuracy and develop alternative strategies to $\mathbb{L}^2$ minimization.

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